THE ABSOLUTE EULER SUMMABILITY OF FOURIER SERIES

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1. Introduction

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|E_\alpha|$ $(0 < \alpha < 1)$ if

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} s_\nu,$$

where $s_\nu = a_0 + a_1 + \cdots + a_\nu$, and

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{1}$$

Since

$$\tau_n = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu a_\nu = n(t_n - t_{n-1})$$

(see [2]), (1) is equivalent to

$$\sum_{n=1}^{\infty} \left| \frac{\tau_n}{n} \right| < \infty. \tag{2}$$

We suppose throughout that $f(x)$ is a periodic function with period 2π, integrable in the Lebesgue sense. Let

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \sum_{n=0}^{\infty} A_n(x). \end{aligned} \tag{3}$$

The series conjugate to (3) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x), \tag{4}$$

and the differentiated series of (3) is

$$\sum_{n=1}^{\infty} n B_n(x). \tag{5}$$

We write

$$\phi(t) = \phi_x(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$
$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}.$$

N. Tripathy [3] has shown the condition that $\phi(t)$ is of bounded variation in $(0, \pi)$ does not ensure the summability of (3) by $|E_\alpha|$.

In this paper we shall prove

THEOREM 1. *If $g(t) = \phi(t) \log 1/t$ is of bounded variation in $0 \leqq t \leqq \delta < 1$, then $\sum_{n=0}^{\infty} A_n(x)$ is summable $|E_\alpha|$. $g(t)$ cannot be replaced by $g_\eta(t) = \phi(t)(\log 1/t)^\eta$ for $0 < \eta < 1$.*

THEOREM 2. *If*

$$\int_0^\delta \log \frac{1}{u} \, |d\phi(u)| < \infty,$$

then $\sum_{n=0}^{\infty} A_n(x)$ is summable $|E_\alpha|$.

J. M. Whittaker [4] proved that, if $\phi(t)/t \in L(0, \delta)$, then the Fourier series (3) is summable $|A|$. We shall prove

THEOREM 3. *The condition $\phi(t)/t^\eta \in L(0, \pi)$, where $\eta < 2$, does not ensure that (3) is summable $|E_\alpha|$.*

However we have

THEOREM 4. *If $\phi(t)/t^2 \in L(0, \delta)$, then (3) is summable $|E_\alpha|$.*

For the conjugate series we have

THEOREM 5. *If $\psi(+0) = 0$ and*

$$\int_0^\delta \log \frac{1}{t} \, |d\psi(t)| < \infty$$

then the conjugate series (4) is summable $|E_\alpha|$.

Finally we shall prove the following theorem on the differentiated series.

THEOREM 6. *If $\psi(+0) = 0$ and*

$$\int_0^\delta \frac{1}{u^2} \, |d\psi(u)| < \infty, \tag{6}$$

then (5) is summable $|E_\alpha|$. (6) cannot be replaced by

$$\int_0^\delta \frac{1}{u^\eta} \, |d\psi(u)| < \infty \tag{7}$$

for any $\eta < 2$.

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2. Proof of Theorem 1

Let

$$G_n(u) = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^{\nu}(1-\alpha)^{n-\nu} \nu \cos \nu u.$$

Then

$$\tau_n = \frac{2}{\pi}\int_0^{\pi} \phi(u) G_n(u)\, du$$

(8)
$$= \frac{2}{\pi}\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) \phi(u) G_n(u)\, du$$

$$= \frac{2}{\pi}(I_n' + I_n'').$$

Now

$$G_n(u) = \frac{d}{du}\left(\sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^{\nu}(1-\alpha) \sin \nu u\right)$$

(9)
$$= \operatorname{Im} \frac{d}{du}(1-\alpha+\alpha e^{iu})^n$$

$$= \operatorname{Im}\{n\alpha i e^{iu}(1-\alpha+\alpha e^{iu})^{n-1}\}$$

$$= n\alpha\rho^{n-1}(u) \operatorname{Im}\{i e^{iu+i\overline{n-1}\,\theta(u)}\}$$

$$= n\alpha\rho^{n-1}(u) \cos(u+\overline{n-1}\,\theta(u)),$$

where

$$\rho(u) = \sqrt{1-4\alpha(1-\alpha)\sin^2\frac{u}{2}},$$

$$\theta(u) = \tan^{-1}\frac{\alpha \sin u}{1-\alpha+\alpha\cos u}.$$

It is clear that $\rho(u) \leqq e^{-cu^2}$ $(0 \leqq u \leqq \pi)$, where c is a positive constant. Hence

$$\sum_{n=1}^{\infty} \frac{|I_n''|}{n} = 0\left(\int_{\delta}^{\pi} |\phi(u)| \left(\sum_{n=1}^{\infty} \rho^{n-1}(u)\right) du\right)$$

(10)
$$= 0\left(\sum_{n=1}^{\infty} e^{-cn\delta^2}\right)$$

$$= 0(1).$$

Let

$$E_n(u) = \int_0^u \left(\log\frac{1}{t}\right)^{-1} G_n(t)\, dt.$$

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Then

$$I'_n = \int_0^\delta g(u)\left(\log\frac{1}{u}\right)^{-1} G_n(u)\,du$$
$$= g(\delta)E_n(\delta) - \int_0^\delta E_n(u)\,dg(u).$$

Hence

$$\sum_{n=1}^{\infty}\frac{|I'_n|}{n} < \infty \tag{11}$$

if

$$\sum_{n=1}^{\infty}\frac{|E_n(u)|}{n} < \infty \tag{12}$$

uniformly for $0 \leqq u \leqq \delta$. Let

$$H_n(u) = \int_0^u G_n(t)\,dt$$
$$= \sum_{\nu=1}^{n}\binom{n}{\nu}\alpha^\nu(1-\alpha)^{n-\nu}\sin \nu u$$
$$= \rho^n(u)\sin n\theta(u).$$

Then

$$E_n(u) = \left(\log\frac{1}{u}\right)^{-1} H_n(u) - \int_0^u \frac{1}{t}\left(\log\frac{1}{t}\right)^{-2} H_n(t)\,dt. \tag{13}$$

Let N be the smallest positive integer such that $Nu^4 > 1$. Then

$$\left(\log\frac{1}{u}\right)^{-1}\sum_{n=1}^{\infty}\frac{H_n(u)}{n} \geqq \left(\log\frac{1}{u}\right)^{-1}\left(\sum_{n=1}^{N}\left|\frac{\rho^n(u)\sin n\theta(u)}{n}\right| + \sum_{n=N+1}^{\infty}\frac{e^{-cnu^2}}{n}\right)$$
$$= 0\left(\left(\log\frac{1}{u}\right)^{-1}\sum_{n=1}^{N}\frac{1}{n}\right) + 0\left(\sum_{n=1}^{\infty}\frac{e^{-c\sqrt{n}}}{n}\right) \tag{14}$$
$$= 0(1),$$

uniformly for $0 < u \leqq \delta$. Now write

$$\int_0^u \frac{1}{t}\left(\log\frac{1}{t}\right)^{-2} H_n(t)\,dt = J'_n + J''_n, \tag{15}$$

where

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$$J_n' = \begin{cases} \displaystyle\int_0^u \frac{1}{t}\left(\log\frac{1}{t}\right)^{-2} H_n(t)\,dt & \left(u \leqq \dfrac{1}{n}\right), \\ \displaystyle\int_0^{1/n} \frac{1}{t}\left(\log\frac{1}{t}\right)^{-2} H_n(t)\,dt & \left(u > \dfrac{1}{n}\right), \end{cases}$$

$$J_n'' = \begin{cases} 0 & \left(u \leqq \dfrac{1}{n}\right), \\ \displaystyle\int_{1/n}^u \frac{1}{t}\left(\log\frac{1}{t}\right)^{-2} H_n(t)\,dt & \left(u > \dfrac{1}{n}\right). \end{cases}$$

Since $\sin \nu t = 0(\nu t)$, we have

$$\begin{aligned} H_n(t) &= 0\left(t \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu\right) \\ &= 0(nt). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|J_n'|}{n} &= 0\left(\sum_{n=1}^{\infty} \int_0^{1/n} \left(\log\frac{1}{t}\right)^{-2} dt\right) \\ &= 0\left(\sum_{n=2}^{\infty} \frac{1}{n\log^2 n}\right) \qquad (16) \\ &= 0(1). \end{aligned}$$

It is clear that

$$\sum_{\nu=1}^{n} \frac{1}{\nu}\binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} = 0\left(\frac{1}{n}\right),$$

and hence, for $u > 1/n$,

$$\begin{aligned} J_n'' &= n(\log n)^{-2} \int_{1/n}^{\xi_n} \left(\sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \sin \nu t\right) dt \\ &= 0\left(\frac{1}{\log^2 n}\right), \end{aligned}$$

where $1/n \leqq \xi_n \leqq u$. It follows that

$$\sum_{n=1}^{\infty} \frac{|J_n''|}{n} = 0(1). \qquad (17)$$

From (13), (14), (15), (16) and (17), we see that (12), and hence (11) holds. The first part of Theorem 1 follows from (8), (10) and (11).

To prove the second part, we required the following lemma due to L. S. Bosanquet and H. Kestleman [1].

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LEMMA 1. *Suppose that $f_n(x)$ is measurable in (a, b), where $b-a \leqq \infty$, for $n = 1, 2, \cdots$. Then a necessary and sufficient condition that, for every function $h(x)$ summable over (a, b), the functions $f_n(x)h(x)$ should be summable over (a, b) and*

$$\sum_{n=1}^{\infty} \left| \int_a^b h(x) f_n(x)\, dx \right| < \infty$$

is that $\sum_{n=1}^{\infty} |f_n(x)|$ should be essentially bounded in (a, b).

(10) is unaffected when $g(t)$ is replaced by $g_\eta(t)$. Let

$$E_n^\eta(u) = \int_0^u \left(\log \frac{1}{t}\right)^{-\eta} G_n(t)\, dt.$$

Then

$$I_n' = g_\eta(\delta) E_n^\eta(\delta) - \int_0^\delta E_n^\eta(u)\, dg_\eta(u). \tag{18}$$

We have

$$E_n^\eta(\delta) = \left(\log \frac{1}{\delta}\right)^{-\eta} H_n(\delta) - \eta \int_0^\delta \frac{1}{t} \left(\log \frac{1}{t}\right)^{-\eta-1} H_n(t)\, dt.$$

Since

$$\sum_{n=1}^{\infty} \frac{H_n(\delta)}{n} = 0 \left(\sum_{n=1}^{\infty} \frac{e^{-cn\delta^2}}{n} \right)$$
$$= 0(1),$$

and (16), (17) remain valid when $(\log 1/t)^{-2}$ is replaced by $(\log 1/t)^{-\eta-1}$,

$$\sum_{n=1}^{\infty} \frac{|E_n^\eta(\delta)|}{n} < \infty. \tag{19}$$

It follows from Lemma 1, (18) and (19) that a necessary condition for (11) to hold is that

$$\sum_{n=1}^{\infty} \frac{|E_n^\eta(u)|}{n} \tag{20}$$

should be essentially bounded for $0 \leqq u \leqq \delta$. Now

$$E_\eta^\eta(u) = \left(\log \frac{1}{u}\right)^{-\eta} H_n(u) - \eta \int_0^u \frac{1}{t} \left(\log \frac{1}{t}\right)^{-\eta-1} H_n(t)\, dt,$$

and from (15), (16) and (17) with $(\log 1/t)^{-2}$ replaced by $(\log 1/t)^{-\eta-1}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^u \frac{1}{t} \left(\log \frac{1}{t}\right)^{-\eta-1} H_n(t)\, dt \right| < \infty$$

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uniformly for $0 \leqq u \leqq \delta$. Since (see [3], page 24)

$$\left(\log \frac{1}{u}\right)^{-\eta} \sum_{n=1}^{\infty} \frac{|H_n(u)|}{n} \to \infty$$

as $u \to 0$, (20) is not essentially bounded. This proves the second part of the theorem.

3. Proof of Theorem 2

We shall deduce Theorem 2 from Theorem 1.

Suppose that the conditions of Theorem 2 are satisfied. Then $\phi(u)$ is of bounded variation in $(0, \delta)$, and hence it must tend to a limit as $t \to 0$. By altering the value of s if necessary, we may suppose that this limit is 0. (Note that the hypothesis of Theorem 2 is unaffected by a change of the value of s.) Now

$$\int_0^\delta |dg(u)| \leqq \int_0^\delta \log \frac{1}{u} \, |d\phi(u)| + \int_0^\delta \frac{1}{u} \, |\phi(u)| \, du. \tag{21}$$

The first term on the right of (21) is finite by hypothesis. Since $\phi(u) \to 0$ as $u \to 0$,

$$|\phi(u)| \leqq \int_0^u |d\phi(t)|$$

so that the second term on the right of (21) does not exceed

$$\begin{aligned} \int_0^\delta \frac{1}{u} \int_0^u |d\phi(t)| \, du &= \int_0^\delta |d\phi(t)| \int_u^\delta \frac{dt}{t} \\ &= \int_0^\delta \log \frac{\delta}{u} \, |d\phi(u)| \\ &< \infty. \end{aligned}$$

Hence the result.

4. Proof of Theorem 3

Since

$$\tau_n = \frac{2}{\pi} \int_0^\pi \phi(u) G_n(u) \, du, \tag{22}$$

it follows from Lemma 1 that (3) is summable $|E_\alpha|$ if and only if $u^\eta \sum_{n=1}^\infty |G_n(u)|/n$, is essentially bounded for $0 \leqq u \leqq \pi$, or, from (9),

$$u^\eta \sum_{n=1}^{\infty} \rho^{n-1}(u) | \cos (u + \overline{n-1}\theta(u))| \tag{23}$$

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is essentially bounded for $0 \leqq u \leqq \pi$. Let $M = [1/u]$ and $N = [1/u^2]$. Then (23) is greater than

$$\begin{aligned} u^\eta \sum_{n=M}^{N} \rho^{n-1}(u)|\cos(u+\overline{n-1}\theta(u))| &\geqq u^\eta \sum_{n=M}^{N} \rho^{n-1}(u)\cos^2(u+\overline{n-1}\theta(u)) \\ &= \frac{u^\eta}{2}\sum_{n=M}^{N} \rho^{n-1}(u) + \frac{u^\eta}{2}\sum_{n=M}^{N} \rho^{n-1}(u)\cos(2u+2\overline{n-1}\theta(u)) \\ &= S_1 + S_2. \end{aligned}$$

Without loss of generality, we assume that $1 < \eta < 2$. Then

$$\begin{aligned} S_2 &= 0\{u^\eta \rho^{M-1}(u)|\max_{M \leqq m \leqq N} \sum_{n=M}^{m} \cos(2u+2\overline{n-1}\theta(u))|\} \\ &= 0\left(\frac{u^\eta e^{-cMu^2}}{\sin\theta(u)}\right) \\ &= 0(1). \end{aligned}$$

There exists a positive constant c_1 such that $\rho^2(u) \geqq e^{-c_1 u^2}$. Hence, for $n \leqq 1/u^2$, $\rho^{n-1}(u) \geqq c_2$ for some constant $c_2 > 0$. Therefore

$$S_1 \geqq \frac{c_2(N-M)u^\eta}{2} \to \infty$$

as $u \to 0+$. Hence (23) is not essentially bounded.

5. Proof of Theorem 4

It follows from (22) that (3) is summable $|E_\alpha|$ if

$$u^2 \sum_{n=1}^{\infty} \frac{1}{n} |G_n(u)| < \infty \tag{24}$$

uniformly for $0 \leqq u \leqq \pi$. Now the left hand side of (24) is equal to

$$\begin{aligned} \alpha u^2 \sum_{n=1}^{\infty} \rho^{n-1}(u)|\cos(u+\overline{n-1}\theta(u))| &= 0(u^2 \sum_{n=1}^{\infty} e^{-cnu^2}) \\ &= 0\left(u^2 \int_1^\infty e^{-cu^2 y} dy\right) \\ &= 0(1) \end{aligned}$$

uniformly for $0 < u < \pi$. Hence (24) is satisfied.

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6. Proof of Theorem 5

Let

$$F_n(u) = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu \sin \nu u.$$

Then

$$\tau_n = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu B_\nu(x)$$

(25)
$$= \frac{2}{\pi} \int_0^\pi \psi(u) F_n(u)\, du$$

$$= \frac{2}{\pi} \left(\int_0^\delta + \int_\delta^\pi \right) \psi(u) F_n(u)\, du$$

$$= \frac{2}{\pi} (X_n' + X_n'').$$

We have

$$F_n(u) = n\alpha\rho^{n-1}(u) \sin (u + \overline{n-1}\theta(u)).$$

Hence

(26)
$$\sum_{n=1}^{\infty} \frac{|X_n''|}{n} = 0(\sum_{n=1}^{\infty} e^{-(n-1)c\delta^2})$$

$$= 0(1).$$

Now

$$X_n' = \left\{ -\psi(u) \int_u^\delta F_n(t)\, dt \right\}_0^\delta + \int_0^\delta \left(\int_u^\delta F_n(t)\, dt \right) d\psi(u)$$

$$= \int_0^\delta \left(\int_u^\delta F_n(t)\, dt \right) d\psi(u)$$

so that

(27)
$$\sum_{n=1}^{\infty} \frac{|X_n'|}{n} < \infty$$

if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_u^\delta F_n(t)\, dt \right| = 0\left(\frac{1}{u}\right).$$

Since

$$\int_u^\delta F_n(t)\, dt = \rho^n(u) \cos n\theta(u) - \rho^n(\delta) \cos n\theta(\delta),$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_u^\delta F_n(t)\,dt \right| = 0\left(\sum_{n=1}^{\infty} \frac{e^{-ncu^2}}{n} \right)$$

$$= 0\left(\int_1^\infty \frac{e^{-cu^2 y}}{y}\,dy \right)$$

$$= 0\left(\int_1^{1/u} \frac{e^{-cu^2 y}}{y}\,dy \right) + 0\left(\int_{1/u}^\infty \frac{e^{-cu^2 y}}{y}\,dy \right)$$

$$= 0\left(u \int_1^{1/u} \frac{dy}{y} \right) + 0\left(\int_{1/u}^\infty e^{-cu^2 y}\,dy \right)$$

$$= 0\left(\frac{1}{u} \right).$$

Hence (27) holds and the theorem follows from (25), (26) and (27).

7. Proof of Theorem 6

We have

$$\tau_n = \sum_{\nu=1}^{n} \binom{n}{\nu} \alpha^\nu (1-\alpha)^{n-\nu} \nu^2 B_n(x)$$

$$= -\frac{2}{\pi} \int_0^\pi \psi(u) \frac{d}{du} G_n(u)\,du$$

$$= -\frac{2}{\pi} \left(\int_0^\delta + \int_\delta^\pi \right) \psi(u) \frac{d}{du} G_n(u)\,du$$

$$= -\frac{2}{\pi} (Y_n' + Y_n'').$$

Since

$$\frac{d}{du} G_n(u) = -n(n-1)\alpha^2 \rho^{n-2}(u) \sin(2u + \overline{n-2}\theta(u)) - n\alpha\rho^{n-1}(u) \sin(u + \overline{n-1}\theta(u))$$

$$= 0(n^2 e^{-ncu^2}),$$

$$\sum_{n=1}^{\infty} \frac{|Y_n''|}{n} = 0\left(\int_\delta^\pi |\psi(u)| \left(\sum_{n=1}^{\infty} n e^{-ncu^2} \right) du \right)$$

(28) $$= 0\left(\sum_{n=1}^{\infty} n e^{-nc\delta^2} \right)$$

$$= 0(1).$$

We have

$$Y_n' = \psi(\delta)G_n(\delta) - \int_0^\delta G_n(u)\,d\psi(u).$$

From (23),

$$\sum_{n=1}^{\infty} \frac{|G_n(\delta)|}{n} = 0\left(\sum_{n=1}^{\infty} \frac{e^{-nc\delta^2}}{n}\right) \tag{29}$$

$$= 0(1),$$

and

$$\sum_{n=1}^{\infty} \frac{|G_n(u)|}{n} = \alpha \sum_{n=1}^{\infty} \rho^{n-1}(u)|\cos(u + \overline{n-1}\theta(u))|$$

$$= 0\left(\sum_{n=1}^{\infty} e^{-ncu^2}\right)$$

$$= 0\left(\int_1^\infty e^{-cu^2 y}\,dy\right)$$

$$= 0\left(\frac{1}{u^2}\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{|Y_n'|}{n} = 0(1). \tag{30}$$

It follows from (28) and (30) that (2) holds, and hence the first part of Theorem 6 is true.

When (6) is replaced by (7), (28) and (29) are not affected. Since, from the proof of Theorem 3, $u^n \sum_{n=1}^{\infty} |G_n(u)|/n$ is not essentially bounded, there exists a summable function $a(x)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^\delta u^n a(u) G_n(u)\,du \right| = \infty.$$

Let $\psi(u) = \int_0^u u^n a(u)du$. Then $\psi(+0) = 0$. Since

$$|\tau_n| \geqq \frac{2}{\pi}|Y_n'| - \frac{2}{\pi}|Y_n''|$$

$$\geqq \frac{2}{\pi}\left|\int_0^\delta G_n(u)\,d\psi(u)\right| - \frac{2}{\pi}|\psi(\delta)G_n(\delta)| - \frac{2}{\pi}|Y_n''|,$$

we have

$$\sum_{n=1}^{\infty} \frac{|\tau_n|}{n} = \infty,$$

which proves the second part of the theorem.

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References

[1] L. S. Bosanquet and H. Kestleman, 'The absolute convergence of a series of integrals', *Proc. London Math. Soc.* (2), 45 (1939) 88–97.

[2] K. Knopp and G. G. Lorentz, 'Beiträge zur absoluten Limitierung', *Arch. Math.* 2 (1949–50), 10–16.

[3] N. Tripathy, 'On the absolute Hausdorff summability of Fourier series', *Jour. London Math. Soc.* 44 (1969), 15–25.

[4] J. M. Whittaker, 'The absolute summability of Fourier series', *Proc. Edinburgh Math. Soc.* 2 (1930), 1–5.

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