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A REMARK ON THE TRACIAL ROKHLIN PROPERTY

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Abstract

To explore the difficulties of classifying actions with the tracial Rokhlin property using K-theoretic data, we construct two $\mathbb{Z}_2$ actions α_1, α_2 on a simple unital AF algebra A such that α_1 has the tracial Rokhlin property and α_2 does not, while $(\alpha_1)_* = (\alpha_2)_*$, where $(\alpha_i)_*$ is the induced map by α_i acting on $K_0(A)$ for $i = 1, 2$.

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1. Introduction and preliminaries

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes [6]. Following this approach to Rokhlin actions, Herman and Jones [13] considered the Rokhlin property for $\mathbb{Z}_2$ in the setting of UHF algebras. In the remarkable papers [14, 15], Izumi investigated the Rokhlin property for finite actions on unital C^*-algebras. The Rokhlin property has been used for classification of actions on C^*-algebras (see [13–15]). However, finite group actions with the Rokhlin property are rare. For example, even for the CAR algebra, A, there is no action of $\mathbb{Z}_3$ on A which has the Rokhlin property [21, Example 13.22].

For actions of finite groups on C^*-algebras, the tracial Rokhlin property is a weak version of the Rokhlin property which is much more common and very useful in the classification theory for simple C^*-algebras. Indeed, if G is a finite group, A is a simple unital C^*-algebra with tracial rank zero and α is a G-action of A with the tracial Rokhlin property, then $C^*(G, A, \alpha)$ also has tracial rank zero [22, Theorem 2.6]. This result can be used for classification purposes by Lin's classification theorem [19]. Phillips' seminal papers [22, 23] contain more information about the tracial Rokhlin property.

We will use the following useful criterion for an action of $\mathbb{Z}_2$ to have the tracial Rokhlin property.

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LEMMA 1.1 [23, Lemma 1.8]. *Let A be a finite infinite-dimensional simple unital C^*-algebra with tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$ satisfy $\alpha^2 = \mathrm{id}_A$. Suppose that for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections e_0, e_1 such that:*

(1) $\|\alpha(e_0) - e_1\| < \varepsilon$;
(2) $\|e_j a - a e_j\| < \varepsilon$ *for all* $a \in F$ *and* $j = 0, 1$;
(3) $\tau(1 - e_0 - e_1) < \varepsilon$ *for each tracial state* τ *on* A.

Then the action of $\mathbb{Z}_2$ *generated by* α *has the tracial Rokhlin property.*

It is an important problem to find and classify all order 2 automorphisms of an AF algebra. Partly because of their intrinsic interest and partly because of their applications in C^*-dynamical systems, this problem has attracted considerable attention (see, for example, [3, 5, 10, 12, 13, 23]). Amongst these results is Blackadar's construction [3] of an action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra such that the crossed product has nontrivial K_1-group. This gave a negative answer to one of the two questions about AF algebras which he posed in [2, 10.11.3]. For the other question, Blackadar considered the following AF algebra A and K-theory structure [2, 10.11.3, Question 2]. We use this structure detailed in Notation 1.2 throughout the paper.

NOTATION 1.2. Let A be a simple unital AF algebra, whose ordered K_0-group is isomorphic to

$$(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{D} \oplus \mathbb{Z}, \mathbb{D}_{>0} \oplus \mathbb{Z} \cup \{(0,0)\}, (1,0)).$$

Let σ be the order 2 scaled ordered automorphism of $K_0(A)$ which is defined by $\sigma(a,b) = (a, -b)$, where $(a,b) \in \mathbb{D} \oplus \mathbb{Z}$.

In this note, for the tuple (A, σ), we construct two $\mathbb{Z}_2$ actions α_1, α_2 on A such that α_1 has the tracial Rokhlin property and α_2 does not have the tracial Rokhlin property, while $(\alpha_1)_* = (\alpha_2)_* = \sigma$, where $(\alpha_i)_*$ is the induced map by α_i acting on $K_0(A)$ for $i = 1, 2$. In fact, $C^*(\mathbb{Z}_2, A, \alpha_1)$ shares many common properties with $C^*(\mathbb{Z}_2, A, \alpha_2)$. The examples demonstrate the differences between an action with the tracial Rokhlin property and an action without the tracial Rokhlin property and may shed some light on the classification of all $\mathbb{Z}_2$-actions on general AF algebras.

2. Main result

We are now in a position to prove the main result. The structure of the proof of Theorem 2.1 is similar to that of [23, Proposition 4.6].

THEOREM 2.1. *There exist order 2 automorphisms* α_i *of* A, $i = 1, 2$, *such that:*

(1) $(\alpha_i)_* = \sigma$ *for* $i = 1, 2$;
(2) $K_1(C^*(\mathbb{Z}_2, A, \alpha_i)) \cong \mathbb{Z}$ *for* $i = 1, 2$;
(3) α_1 *has the tracial Rokhlin property and* $C^*(\mathbb{Z}_2, A, \alpha_1)$ *is a unital simple* $A\mathbb{T}$-*algebra with tracial rank zero and a unique tracial state;*

(4) *α_2 does not have the tracial Rokhlin property and $C^*(\mathbb{Z}_2, A, \alpha_2)$ is a unital simple $A\mathbb{T}$-algebra with two extreme tracial states;*

(5) *the CAR algebra is a unital subalgebra of A^{α_i} for $i = 1, 2$. That is, for each α_i, there exists a unital subalgebra C_i of A such that $\alpha_i|_{C_i} = \mathrm{id}_{C_i}$ and C_i is isomorphic to the CAR algebra for $i = 1, 2$.*

PROOF. For $k \in \mathbb{N}$, choose positive integers m_k, n_k such that $2n_k + 2 \leq m_{k+1}/m_k$ and $\log_4(m_{k+1}/m_k) \in \mathbb{N}$. Let $A_k = M_{m_k}(C(S^2))$ and let λ be the reflection map defined by

$$\lambda(u, v, w) = (-u, v, w), \quad \text{where } (u, v, w) \in S^2.$$

Suppose that $x_1, x_2, \ldots \in S^2$ are such that $\{x_k : k \geq n\}$ is dense in S^2 for each $n \in \mathbb{N}$, and let z^* be a fixed point of λ.

Define $\Phi_k : A_k \to A_k$ by

$$\Phi_k(f) = \mathrm{diag}(f; \overbrace{f(z^*), \ldots, f(z^*)}^{(m_{k+1}/m_k)-2n_k-1}; f(x_1), f(\lambda(x_1)), \ldots, f(x_{n_k}), f(\lambda(x_{n_k}))).$$

Set

$$\mathfrak{A} = \lim_{k\to\infty}(A_k, \Phi_k).$$

Then $\mathfrak{A}$ is a simple Goodearl algebra with $\mathrm{RR}(\mathfrak{A}) = 0$ (see [11] or [24, Proposition 3.1.8]). Moreover, it is routine to check that

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}]) = (K_0(A), K_0(A)_+, [1_A]).$$

By [8], we have $\mathfrak{A} \cong A$.

For $k \in \mathbb{N}$, define $\rho_k = \lambda^* \otimes \mathrm{id}_{M_{m_k}}$. Set $u_1 = \mathrm{id}_{A_1}$ and define

$$u_{k+1} = \mathrm{diag}\left\{\overbrace{1, \ldots, 1}^{(m_{k+1}/m_k)-2n_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{n_k}\right\} \otimes \mathrm{id}_{M_{m_k}} \quad \text{for } k \in \mathbb{N}.$$

It is straightforward to check that $\Phi_k \circ (\rho_k) = \mathrm{Ad}u_{k+1} \circ (\rho_{k+1}) \circ \Phi_k$ and $u_{k+1}^2 = \mathrm{id}_{A_{k+1}}$ for $k \in \mathbb{N}$. Set $v_1 = \mathrm{id}_{A_1}$ and define $v_{k+1} = \Phi_k(v_k)u_{k+1}$ inductively. Then $v_k = v_k^*$ and $v_k^2 = \mathrm{id}_{A_k}$ for $k = 1, 2, \ldots$. Define $\gamma_k = \mathrm{Ad}v_k \circ \rho_k$, $k \in \mathbb{N}$. One can easily construct the following commutative diagram:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots A \\ \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} & & \\ A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots A \end{array}$$

Hence, the automorphisms γ_k define $\gamma : A \to A$, which is a symmetry with $\gamma_* = \sigma$.

Let $B = C^*(\mathbb{Z}_2, S^2, \lambda)$. Then, exactly as in the proof of [23, Proposition 4.6]:

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(a) $K_0(B) = K_0(C) = \mathbb{Z}[e] \oplus \mathbb{Z}[1-e] \cong \mathbb{Z}^2$, $K_0(B)_+ = \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}$; and
(b) $K_1(B) \cong \mathbb{Z}$ and $B_k = C^*(\mathbb{Z}_2, A_k, \beta_k) \cong M_{n_k} \otimes B$,

where

$$C = \{f \in C(L, M_2) : f(x) \text{ is diagonal for } x \in E\}, \quad e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1 - e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$E = \{(x_0, x_1, x_2) \in S^2 : x_0 = 0\}, \quad L = \{(x_0, x_1, x_2) \in S^2 : x_0 \geq 0\}.$$

Let $\tilde{\Phi}_k : B_k \to B_{k+1}$ be the map induced by Φ_k and $\tilde{\gamma}_k : B_k \to B_k$ the map induced by γ_k for $k \in \mathbb{N}$. Then it is not hard to check that the induced map $\tilde{\gamma}_{k_*} : K_0(B_k) = \mathbb{Z} \oplus \mathbb{Z} \to K_0(B_k) = \mathbb{Z} \oplus \mathbb{Z}$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $\tilde{\gamma}_{k+1} \circ \tilde{\Phi}_k = \tilde{\Phi}_k \circ \tilde{\gamma}_k$ for $k \in \mathbb{N}$; hence, $\tilde{\Phi}_{k_*}$ must have the form

$$\begin{bmatrix} a_k & b_k \\ b_k & a_k \end{bmatrix}.$$

Then, following the discussion in the proof of [23, Proposition 4.2], it is not hard to check that $\tilde{\Phi}_{k_*} : K_0(B_k) = \mathbb{Z} \oplus \mathbb{Z} \to K_0(B_{k+1}) = \mathbb{Z} \oplus \mathbb{Z}$ is given by

$$\tilde{\Phi}_{k_*} = \begin{bmatrix} (m_{k+1}/m_k) - 2n_k & 2n_k \\ 2n_k & (m_{k+1}/m_k) - 2n_k \end{bmatrix}.$$

Exactly as in the proof of [23, Proposition 4.6], for each $k \in \mathbb{N}$, $\tilde{\Phi}_k$ is unitarily equivalent to the direct sum of the identity map and a map which factors through a finite-dimensional C^*-algebra. So, $K_1(B_k) \to K_1(B_{k+1})$ is an isomorphism for all $k \in \mathbb{N}$ and it follows that $K_1(C^*(\mathbb{Z}_2, A, \gamma)) \cong \mathbb{Z}$.

Choose $m_k = 2^{(k+2)(k-1)/2}$ and $n_k = 2^k - 1$ and denote the corresponding action γ by α_1. Similarly, for $m_k = 2^{(k+2)(k-1)/2}$ and $n_k = 1$, denote the corresponding action γ by α_2. Notice that we have finished the proof of (1) and (2).

We are now in a position to show that the action of $\mathbb{Z}_2$ generated by α_1 has the tracial Rokhlin property. To this end, we fix a finite set $F \subset \{a : a \in A, \|a\| \leq 1\}$ and $\varepsilon > 0$. Then there exists a positive integer k such that $F \subset_{\varepsilon/2} A_k$ and

$$\frac{1}{2^k} < \frac{\varepsilon}{100}.$$

Define

$$e = \operatorname{diag}\Big\{0, 0; \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{2^k - 1}\Big\} \otimes \mathrm{id}_{M_{m_k}}.$$

Then $\alpha_{k+1}(e)e = 0$ and e commutes with $\Phi_k(A_k)$. Moreover, for any tracial state τ on A_{k+1},

$$\tau(1 - e - \alpha_{k+1}(e)) < \frac{\varepsilon}{2}.$$

By Lemma 1.1, α has the tracial Rokhlin property.

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Hence, by [22, Corollary 1.6 and Theorem 2.6], $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple, separable C^*-algebra with tracial rank zero. Since A_k is nuclear and $\mathbb{Z}_2$ is amenable and compact, it follows from [27, Corollary 7.18] and [7, Proposition 6.1] that $C^*(\mathbb{Z}_2, A_k, \alpha_k)$ is nuclear and satisfies the universal coefficient theorem. Hence, by [24, Proposition 2.4.7(ii)],

$$C^*(\mathbb{Z}_2, A, \alpha) = \lim_{k\to\infty}(C^*(\mathbb{Z}_2, A_k, \alpha_k), \Phi_k)$$

is also nuclear and satisfies the universal coefficient theorem. Therefore, by [18, Theorems 6.4.11 and 4.7.5], $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple $A\mathbb{T}$-algebra. This finishes the proof of (3).

Define $p_k = \frac{1}{2}(m_{k+1}/m_k) = 2^{k+1}$, $q_k = \frac{1}{2}((m_{k+1}/m_k) - 4)$ for $k \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \prod_{k=3}^{\infty} \frac{p_k}{q_k} \le \lim_{n\to\infty}\Big(1 + \frac{1}{n}\Big)^n < \infty.$$

Hence, $K_0(\mathfrak{B}) = \lim_{k\to\infty}(\mathbb{Z} \oplus \mathbb{Z}, (\Theta_k)_*)$ is a 2-symmetric group of type (4) in the sense of [2, 7.7.4 (4)], $K_0(\mathfrak{B})$ has two different states by [4, Corollary 3.4] and

$$|\mathrm{ext}(T(C^*(\mathbb{Z}_2, A, \beta)))| \ge 2.$$

In fact, by [20, Proposition 2.4], $|\mathrm{ext}(T(C^*(\mathbb{Z}_2, A, \beta)))| = 2$. Recall that A has a unique tracial state; hence, by [7, Proposition 5.7], β does not have the tracial Rokhlin property.

Evidently, α_2 is outer, that is, there is no unitary $u \in A$ such that $uau^* = \alpha_2(a)$ for all $a \in A$. Then, by [17, Theorem 3.1], $C^*(\mathbb{Z}_2, A, \alpha_2)$ is simple. Note that by [28, Theorem 1.6], the decomposition rank $\mathrm{dr}(B)$ of B is $\mathrm{dr}(B) = 2$. Therefore, by elementary properties of the decomposition rank (see [16]),

$$\mathrm{dr}\, C^*(\mathbb{Z}_2, A, \alpha_2) \le \liminf_{k\to\infty}(\mathrm{dr}\, B_k) = \mathrm{dr}\, B = 2.$$

Then it follows from [29, Theorem 5.1] that $C^*(\mathbb{Z}_2, A, \alpha_2)$ is Jiang–Su stable. Therefore, by [9, Theorem 5.9], $C^*(\mathbb{Z}_2, A, \alpha_2)$ is classifiable (by means of the naive Elliott invariant). Notice that $T(C^*(\mathbb{Z}_2, A, \alpha_2))$ and $S(K_0(C^*(\mathbb{Z}_2, A, \alpha_2)))$ both have two extreme points; hence, by [26, Theorem 4.2], $C^*(\mathbb{Z}_2, A, \alpha_2)$ is a unital simple $A\mathbb{T}$-algebra with two extreme tracial states.

It remains to prove (5). Since $C^*(\mathbb{Z}_2, A, \alpha_i)$ is simple, by [25] or [22, Proposition 4.7], A^{α_i} is simple and isomorphic to a full hereditary subalgebra, and is strongly Morita equivalent to $C^*(\mathbb{Z}_2, A, \alpha_i)$ for $i = 1, 2$. Then $K_0(C^*(\mathbb{Z}_2, A, \alpha_i))$ and $K_0(A^{\alpha_i})$ are order isomorphic and, by [24, Proposition 3.2.5], A^{α_i} is an $A\mathbb{T}$-algebra. In particular, $\mathrm{tsr}(A^{\alpha_i}) = 1$ for $i = 1, 2$. Notice that $K_0(C^*(C^*(\mathbb{Z}_2, A, \alpha_i)))$ is a 2-divisible scaled ordered group. Hence, by the existence theorem [18, Theorem 3.4.6], there exists an embedding $h_i : M_{2^\infty} \to A^{\alpha_i}$ for $i = 1, 2$, where M_{2^∞} is the CAR algebra. This completes the proof. □

REMARK 2.2. Let α be any order 2 automorphism of A such that $\alpha_* = \sigma$. As in the proof outlined in [2, 10.11.3], $C^*(\mathbb{Z}_2, A, \alpha)$ is not an AF algebra. Hence, by [22, Theorem 2.2], α does not have the Rokhlin property.

Lastly, we suggest two avenues for further study.

QUESTION 2.3. Is $K_0(C^*(\mathbb{Z}_2, A, \alpha))$ always torsion free, where α is any order 2 automorphism of A such that $\alpha_* = \sigma$?

REMARK 2.4. According to the discussion in [2, 10.11.3], either $K_0(C^*(\mathbb{Z}_2, A, \alpha))$ has torsion or $K_1(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$. Note that in Theorem 2.1, $K_0(C^*(\mathbb{Z}_2, A, \alpha_i))$ is torsion free for $i = 1, 2$.

As the tuple (A, σ) is fixed, the following conjecture about the range is more ambitious.

CONJECTURE 2.5. For any simple dimension group G, there exists an order 2 automorphism α of A such that $\alpha_* = \sigma$. Moreover, $K_0(C^*(\mathbb{Z}_2, A, \alpha)) = G$ and $K_1(C^*(\mathbb{Z}_2, A, \alpha)) = \mathbb{Z}$.

REMARK 2.6. Results of this kind can be found in [5, Theorems 1.1 and 1.2] and [1, Example 4.2]. For simplicity, perhaps, we should first consider the special case in which G is a 2-symmetric group in the sense of [2, 7.7.4].

References

[1] S. Barlak and G. Szabó, 'Rokhlin actions of finite groups on UHF-absorbing C^*-algebras', *Trans. Amer. Math. Soc.* **369**(2) (2017), 833–859.

[2] B. Blackadar, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, 5 (Springer, New York, 1986).

[3] B. Blackadar, 'Symmetries of the CAR algebra', *Ann. of Math. (2)* **131**(3) (1990), 589–623.

[4] B. Blackadar and M. Rørdam, 'Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras', *J. Algebra* **152**(1) (1992), 240–247.

[5] O. Bratteli, G. A. Elliott, D. E. Evans and A. Kishimoto, 'Finite group actions on AF algebras obtained by folding the interval', *K-Theory* **8**(5) (1994), 443–464.

[6] A. Connes, 'Outer conjugacy classes of automorphisms of factors', *Ann. Sci. Éc. Norm. Supér. (4)* **8**(3) (1975), 383–419.

[7] S. Echterhoff, W. Lück, N. C. Phillips and S. Walters, 'The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$', *J. reine angew. Math.* **639** (2010), 173–221.

[8] G. A. Elliott and G. Gong, 'On the classification of C^*-algebras of real rank zero. II', *Ann. of Math. (2)* **144**(3) (1996), 497–610.

[9] G. A. Elliott, G. Gong, H. Lin and Z. Niu, 'The classification of simple separable unital $\mathcal{Z}$-stable locally ASH algebras', *J. Funct. Anal.* **272**(12) (2017), 5307–5359.

[10] G. A. Elliott and H. Su, 'K-theoretic classification for inductive limit Z_2 actions on AF algebras', *Canad. J. Math.* **48**(5) (1996), 946–958.

[11] K. R. Goodearl, 'Notes on a class of simple C^*-algebras with real rank zero', *Publ. Mat.* **36**(2A) (1992), 637–654.

[12] D. Handelman and W. Rossmann, 'Actions of compact groups on AF C^*-algebras', *Illinois J. Math.* **29**(1) (1985), 51–95.

[13] R. H. Herman and V. F. R. Jones, 'Period two automorphisms of UHF C^*-algebras', *J. Funct. Anal.* **45**(2) (1982), 169–176.
[14] M. Izumi, 'Finite group actions on C^*-algebras with the Rohlin property. I', *Duke Math. J.* **122**(2) (2004), 233–280.
[15] M. Izumi, 'Finite group actions on C^*-algebras with the Rohlin property. II', *Adv. Math.* **184**(1) (2004), 119–160.
[16] E. Kirchberg and W. Winter, 'Covering dimension and quasidiagonality', *Int. J. Math.* **15**(1) (2004), 63–85.
[17] A. Kishimoto, 'Outer automorphisms and reduced crossed products of simple C^*-algebras', *Comm. Math. Phys.* **81**(3) (1981), 429–435.
[18] H. Lin, *An Introduction to the Classification of Amenable* C*-Algebras* (World Scientific, River Edge, NJ, 2001).
[19] H. Lin, 'Classification of simple C^*-algebras of tracial topological rank zero', *Duke Math. J.* **125**(1) (2004), 91–119.
[20] N. Nawata, 'Finite group actions on certain stably projectionless C^*-algebras with the Rohlin property', *Trans. Amer. Math. Soc.* **368**(1) (2016), 471–493.
[21] N. C. Phillips, 'An introduction to crossed product C^*-algebras and minimal dynamics', http://pages.uoregon.edu/ncp/Courses/CRMCrPrdMinDyn/CRMCrPrdMinDyn.html.
[22] N. C. Phillips, 'The tracial Rokhlin property for actions of finite groups on C^*-algebras', *Amer. J. Math.* **133**(3) (2011), 581–636.
[23] N. C. Phillips, 'Finite cyclic group actions with the tracial Rokhlin property', *Trans. Amer. Math. Soc.* **367**(8) (2015), 5271–5300.
[24] M. Rørdam, 'Classification of nuclear, simple C^*-algebras', in: *Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras*, Encyclopaedia of Mathematical Sciences, 126 (Springer, Berlin, 2002), 1–145.
[25] J. Rosenberg, 'Appendix to O. Bratteli's paper on "Crossed products of UHF algebras by product type actions"', *Duke Math. J.* **46**(1) (1979), 25–26; *Duke Math. J.* **46**(1) (1979), 1–23.
[26] J. Villadsen, 'The range of the Elliott invariant', *J. reine angew. Math.* **462** (1995), 31–55.
[27] D. P. Williams, *Crossed Products of C^*-Algebras*, Mathematical Surveys and Monographs, 134 (American Mathematical Society, Providence, RI, 2007).
[28] W. Winter, 'Decomposition rank of subhomogeneous C^*-algebras', *Proc. Lond. Math. Soc. (3)* **89**(2) (2004), 427–456.
[29] W. Winter, 'Decomposition rank and $\mathcal{Z}$-stability', *Invent. Math.* **179**(2) (2010), 229–301.

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