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QUASI-DUALITY, LINEAR COMPACTNESS AND MORITA DUALITY FOR POWER SERIES RINGS

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ABSTRACT. As a generalization of Morita duality, Kraemer introduced the notion of quasi-duality and showed that each left linearly compact ring has a quasi-duality. Let R be an associative ring with identity and $R[[x]]$ the power series ring. We prove that (1) $R[[x]]$ has a quasi-duality if and only if R has a quasi-duality; (2) $R[[x]]$ is left linearly compact if and only if R is left linearly compact and left noetherian; and (3) $R[[x]]$ has a Morita duality if and only if R is left noetherian and has a Morita duality induced by a bimodule ${}_RU_S$ such that S is right noetherian.

0. **Introduction.** Let R be a ring and $R[[x]]$ be the ring of all formal power series in x with coefficients in R. If ${}_RU$ is a left R-module, we let $U[x^{-1}]$ consist of all polynomials in x^{-1} with coefficients in U. Thus a typical element of $U[x^{-1}]$ is an expression

$$u_0 + u_1x^{-1} + u_2x^{-2} + \cdots + u_nx^{-n}$$

where $u_i \in U$. Now $U[x^{-1}]$ can be turned into a left $R[[x]]$-module. The addition in $U[x^{-1}]$ is componentwise and the scalar multiplication is defined as follows

$$(\Sigma_{i\geq 0}r_ix^i)(\Sigma_{j\geq 0}u_jx^{-j}) = \Sigma_{j\geq 0}(\Sigma_{i\geq 0}r_iu_{i+j})x^{-j}$$

where $\Sigma_{i\geq 0}r_ix^i \in R[[x]]$ and $\Sigma_{j\geq 0}u_jx^{-j} \in U[x^{-1}]$. Note that, in particular,

$$(rx^m)(ux^{-n}) = \begin{cases} 0 & \text{when } m > n, \\ rux^{-(n-m)} & \text{when } m \leq n. \end{cases}$$

Then $U[x^{-1}]$ becomes a left $R[[x]]$-module. Similarly, if U_S is a right S-module for some ring S, then $U[x^{-1}]$ is a right $S[[x]]$-module. If ${}_RU_S$ is an R-S-bimodule, according to the above construction, $U[x^{-1}]$ becomes a left $R[[x]]$- and right $S[[x]]$-bimodule.

In this paper, rings are associative with identity and modules are unitary. We always let R and S be rings and freely use the terminologies and notations of [1].

Recall that a bimodule ${}_RU_S$ defines a *Morita duality* if the bimodule ${}_RU_S$ is faithfully balanced and both ${}_RU$ and U_S are injective cogenerators (see [1, Theorem 24.1] or [13, Theorem 2.4]), and in this case, R has a *Morita duality*. Morita duality was established by Azumaya [3] and Morita [8], and a presentation of this duality can be found in Anderson and Fuller [1, § 23, § 24] and the author's book Xue [13].

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As a generalization of Morita duality, Kraemer [5] said that a bimodule ${}_RU_S$ defines a *quasi-duality* in case ${}_RU_S$ is faithfully balanced and both ${}_RU$ and U_S are quasi-injective and finitely cogenerated, and in this case, R has a *quasi-duality*.

This paper consists of two sections. The main result in Section 1 is Theorem 1.5 which states that a bimodule ${}_RU_S$ defines a quasi-duality if and only if the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ defines a quasi-duality. It follows that R has a quasi-duality if and only if $R[[x]]$ has a quasi-duality.

In Section 2, we consider when the power series ring $R[[x]]$ is left linearly compact or has a Morita duality. We prove (Theorem 2.3) that $R[[x]]$ is left linearly compact if and only if R is left linearly compact and left noetherian. This is a generalization of a result of Anh and Menini (informed to us by Anh), and Herbera (informed to us by Faith) who proved this for commutative rings. In [14, Theorem 1.3] we proved that if ${}_RU_S$ defines a Morita duality, R is left noetherian and S is right noetherian, then the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ defines a Morita duality. We shall establish the converse (Theorem 2.4). Consequently, $R[[x]]$ has a Morita duality if and only if R is left noetherian and has a Morita duality induced by a bimodule ${}_RU_S$ such that S is right noetherian.

1. **Quasi-duality for power series rings.** Let ${}_RU$ be an R-module. McKerrow [6] proved that if the $R[[x]]$-module $U[x^{-1}]$ is injective then ${}_RU$ must be injective [6, Proposition 1], and the converse is true if R is left noetherian [6, Theorem 1]. We shall see that the noetherian condition is essential (Example 2.6). However, we have the following result for quasi-injectivity.

LEMMA 1.1. *An R-module ${}_RU$ is quasi-injective if and only if the $R[[x]]$-module ${}_{R[[x]]}U[x^{-1}]$ is quasi-injective.*

PROOF. (⇒). Let W be an $R[[x]]$-submodule of $U[x^{-1}]$ and $h\colon W \longrightarrow U[x^{-1}]$ be an $R[[x]]$-homomorphism. Let

$$F = \{f\colon L \longrightarrow U[x^{-1}] \mid W \leq L \leq U[x^{-1}] \text{ and } f|_W = h\}$$

be a set of $R[[x]]$-homomorphisms. If $f_i\colon L_i \longrightarrow U[x^{-1}]$ are two elements in W $(i = 1,2)$, we define $f_1 \leq f_2$ in case $L_1 \leq L_2$ and $f_2|_{L_1} = f_1$. By Zorn's Lemma, the partial ordered set $(F, \leq)$ has a maximal element, say $\bar{h}\colon M \longrightarrow U[x^{-1}]$. To show $M = U[x^{-1}]$, we need only to prove that each $\Sigma_{i=0}^n Ux^{-i} \subseteq M$ $(n = 0, 1, \ldots)$. Let $W_n = M \cap (\Sigma_{i=0}^n Ux^{-i})$ and $p_j\colon U[x^{-1}] \longrightarrow U$ be the *j-th* projections $(n, j = 0, 1, \ldots)$. Since ${}_RU$ is quasi-injective and $p_j\bar{h}|_{W_n}\colon W_n \longrightarrow U$ is an R-homomorphism, there are elements $s_{0j}, s_{1j}, \ldots, s_{nj} \in S = \mathrm{End}({}_RU)$ such that for each $\Sigma_{i=0}^n u_ix^{-i} \in W_n$,

$$p_j\bar{h}(\Sigma_{i=0}^n u_ix^{-i}) = \Sigma_{i=0}^n u_is_{ij} \qquad (j = 0, 1, \ldots),$$

where we view ${}_RU_S$ as a left R- and right S-bimodule. Let $f\colon M + (\Sigma_{i=0}^n Ux^{-i}) \longrightarrow U[x^{-1}]$ via $m + \Sigma_{i=0}^n u_ix^{-i} \longmapsto \bar{h}(m) + \Sigma_{j=0}^n(\Sigma_{i=0}^n u_is_{ij})x^{-j}$. If $m = -(\Sigma_{i=0}^n u_ix^{-i}) \in W_n$ then $0 = \bar{h}(x^jm) = x^j\bar{h}(m)$ for each $j > n$. Hence $\bar{h}(m) = \Sigma_{j=0}^n v_jx^{-j} \in \Sigma_{i=0}^n Ux^{-i}$, and $v_j = p_j\bar{h}(m) = p_j\bar{h}(-\Sigma_{i=0}^n u_ix^{-i}) = -\Sigma_{i=0}^n u_is_{ij}$ and $\bar{h}(m) = -\Sigma_{j=0}^n(\Sigma_{i=0}^n u_is_{ij})x^{-j}$. So f is well-defined and it is routine to check that f is an $R[[x]]$-homomorphism. Since $f|_M = \bar{h}$, by the maximality of $\bar{h}$, we have $\Sigma_{i=0}^{n} Ux^{-i} \subseteq M$.

($\Leftarrow$). Let $V \leq {}_RU$ and $h: V \longrightarrow U$ an R-homomorphism. Then $V[x^{-1}]$ is an $R[[x]]$-submodule of $U[x^{-1}]$ and $H: V[x^{-1}] \longrightarrow U[x^{-1}]$ via $\Sigma_i v_i x^{-i} \longmapsto \Sigma_i h(v_i)x^{-i}$ is an $R[[x]]$-homomorphism. By the quasi-injectivity of ${}_{R[[x]]}U[x^{-1}]$, we can find an $R[[x]]$-homomorphism $\bar{H}: U[x^{-1}] \longrightarrow U[x^{-1}]$ such that $\bar{H}|_{V[x^{-1}]} = H$. We view U as an $R[[x]]$-submodule of $U[x^{-1}]$ and $xU = 0$; hence $x\bar{H}(U) = 0$ and $\bar{H}(U) \subseteq U$. Therefore, $\bar{h} = \bar{H}|_U: U \longrightarrow U$ is an R-homomorphism and $\bar{h}|_V = h$.

LEMMA 1.2. *An R-module ${}_RU$ is finitely cogenerated if and only if the $R[[x]]$-module ${}_{R[[x]]}U[x^{-1}]$ is finitely cogenerated.*

PROOF. ($\Rightarrow$). We note that $\mathrm{Soc}({}_RU)$ is a finitely generated semisimple $R[[x]]$-submodule of $U[x^{-1}]$. If W is a non-zero $R[[x]]$-submodule of $U[x^{-1}]$, it is easy to see that $W \cap U \neq 0$. Since ${}_RU$ is finitely cogenerated, $W \cap \mathrm{Soc}({}_RU) = (W \cap U) \cap \mathrm{Soc}({}_RU) \neq 0$. Hence $U[x^{-1}]$ is finitely cogenerated as an $R[[x]]$-module.

($\Leftarrow$). If ${}_{R[[x]]}U[x^{-1}]$ is finitely cogenerated, its $R[[x]]$-submodule U is also finitely cogenerated. Since $xU = 0$, ${}_RU$ is finitely cogenerated.

LEMMA 1.3. *An R-module ${}_RU$ is faithful if and only if the $R[[x]]$-module ${}_{R[[x]]}U[x^{-1}]$ is faithful.*

PROOF. Straightforward.

LEMMA 1.4. *An R-S-bimodule ${}_RU_S$ is balanced if and only if the $R[[x]]$-$S[[x]]$-bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ is balanced.*

PROOF. ($\Rightarrow$). This is [12, Lemma 1.1].

($\Leftarrow$). Use the proof of Lemma 1.1 ($\Leftarrow$).

Kraemer [5, p. 11] said that a bimodule ${}_RU_S$ defines a *quasi-duality* in case ${}_RU_S$ is faithfully balanced and both ${}_RU$ and U_S are quasi-injective and finitely cogenerated, and in this case R is said to have a quasi-duality. The following result follows from the above four lemmas and their right symmetric versions.

THEOREM 1.5. *A bimodule ${}_RU_S$ defines a quasi-duality if and only if the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ defines a quasi-duality.*

It is not known whether or not a factor ring of a ring with a quasi-duality has a quasi-duality. However, if R has a quasi-duality and I is an ideal which is finitely generated as a left R-module then R/I has a quasi-duality by [5, Lemma 2.3(3)(4)]. Hence we have

COROLLARY 1.6. *A ring R has a quasi-duality if and only if $R[[x]]$ has a quasi-duality.*

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2. **Linear compactness and Morita duality for power series rings.** The following interesting result, due to Kraemer [5], will be often used throughout the rest of this paper. The reader is referred to [11, § 3, § 4] for linearly compact modules.

KRAEMER'S THEOREM. *Let ${}_RU_S$ define a quasi-duality. Then*

(1) *The following are equivalent: (i) R is left linearly compact; (ii) ${}_RU$ is an injective cogenerator; (iii) U_S is linearly compact.*

(2) *The following are equivalent: (i) S is right linearly compact; (ii) U_S is an injective cogenerator; (iii) ${}_RU$ is linearly compact.*

(3) *The following are equivalent: (i) R has a Morita duality; (ii) S has a right Morita duality; (iii) ${}_RU_S$ defines a Morita duality; (iv) the equivalent conditions of both (1) and (2) hold.*

(4) *R is left noetherian if and only if U_S is artinian; consequently, R is left linearly compact.*

PROOF. (1), (2) and (3) are the contents of [5, Theorem 2.6]. Using [5, Lemma 2.3(2)(3)], we can prove that R is left noetherian if and only if U_S is artinian. Since an artinian module is linearly compact, R must be left linearly compact by (1).

In this section we shall use Theorem 1.5 and Kraemer's Theorem to determine when $R[[x]]$ is left linearly compact and when it has a Morita duality.

Let U_S be a right S-module. Then we have a right $S[[x]]$-module $U[x^{-1}]$. If $f = u_0 + u_1x^{-1} + \cdots + u_ix^{-i} \in U[x^{-1}]$ and $u_i \neq 0$, we say that f has degree i. Let F be an $S[[x]]$-submodule of $U[x^{-1}]$. For each $i \geq 0$, we let $L_i(F) = \{0\} \cup \{$leading coefficients of elements of degree i in $F\}$, which is an S-submodule of U. Moreover, it is easy to see that $L_i(F) \supseteq L_{i+1}(F)$ for each $i \geq 0$.

LEMMA 2.1. *Let U_S be an S-module. If $F \supseteq G$ are $S[[x]]$-submodules of $U[x^{-1}]$ satisfying $L_i(F) = L_i(G)$ for all $i \geq 0$, then $F = G$.*

PROOF. Modify the proof of [12, Lemma 2.2].

To characterize the linear compactness of $R[[x]]$, we need the following result which has its own interest.

PROPOSITION 2.2. *The following are equivalent for a right S-module U_S:*

(1) *U_S is artinian;*

(2) *$U[x^{-1}]_{S[[x]]}$ is artinian;*

(3) *$U[x^{-1}]_{S[[x]]}$ is linearly compact.*

PROOF. (1) ⇒ (2). We modify the proof of [12, Theorem A (a) ⇒ (b)]. Let

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$$

be a descending chain of $S[[x]]$-submodules of $U[x^{-1}]$. From the comments preceeding Lemma 2.1, we get $L_i(F_j) \supseteq L_{i+1}(F_j)$ for each $i \geq 0$ and $j \geq 0$. Also $F_j \supseteq F_{j+1}$ implies $L_i(F_j) \supseteq L_i(F_{j+1})$ for each $i \geq 0$ and $j \geq 0$. Since U_S is artinian, $\{L_i(F_j)\}_{i\geq 0, j\geq 0}$ has a minimal element, say $L_k(F_n)$. Then $L_i(F_j) = L_k(F_n)$ whenever $i \geq k$ and $j \geq n$. For each fixed $i < k$, because M_S is artinian, we can find an integer $t(i)$ with $L_i(F_j) = L_i(F_{t(i)})$ for $j \geq t(i)$. Let $t = \max\{t(0), t(1), \ldots, t(k-1), n\}$. Then $L_i(F_j) = L_i(F_t)$ for $j \geq t$ and all $i \geq 0$. From Lemma 2.1, we see that $F_j = F_t$ for $j \geq t$. Hence $U[x^{-1}]_{S[[x]]}$ is artinian.

(2) $\Rightarrow$ (3). Each artinian module is linearly compact.

(3) $\Rightarrow$ (1). Suppose U_S is not artinian, then U_S has a strictly infinite chain of S-submodules:

$$U_0 > U_1 > U_2 > \cdots.$$

We view each $U_i[x^{-1}]$ as an $S[[x]]$-submodule of $U[x^{-1}]$. Let $u_i \in U_i \backslash U_{i+1}$ for each i. Then the $S[[x]]$-module $U[x^{-1}]$ has a finitely solvable family

$$\{(\Sigma_{j=0}^{i-1} u_j x^{-j}),\ U_i[x^{-1}]\}_{i \geq 1}$$

which is not solvable. Hence the $S[[x]]$-module $U[x^{-1}]$ is not linearly compact, a contradiction.

The next result is a characterization of the linear compactness of $R[[x]]$, where the equivalence (1) $\Leftrightarrow$ (3) was proved by Anh and Menini, and Herbera for commutative rings.

THEOREM 2.3. *The following are equivalent for a ring R:*

(1) R is left linearly compact and left noetherian;

(2) R has a quasi-duality and is left noetherian;

(3) $R[[x]]$ is left linearly compact;

(4) $R[[x_1, \ldots, x_n]]$ is left linearly compact for any finitely many variables $x_1, \ldots, x_n$.

PROOF. (1) $\Rightarrow$ (2) Kraemer [5, Proposition 2.4] proved that each left linearly compact ring has a quasi-duality.

(2) $\Rightarrow$ (4). Since R is a left noetherian ring with a quasi-duality, the left noetherian ring $R[[x_1, \ldots, x_n]]$ has a quasi-duality by Corollary 1.6 and it is left linearly compact by Kraemer's Theorem.

(4) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). We see that R is left linearly compact by (3), since R is a factor ring of $R[[x]]$. By [5, Proposition 2.4], R has a quasi-duality induced by a bimodule ${}_RU_S$. Then the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ defines a quasi-duality by Theorem 1.5. Since $R[[x]]$ is left linearly compact, $U[x^{-1}]_{S[[x]]}$ is linearly compact by Kraemer's Theorem. Hence U_S is artinian by Proposition 2.2 and then R is left noetherian by Kraemer's Theorem again.

Vámos [11] mentioned as a slightly modified version of Müller [9, Theorem 1] that a ring R has a Morita duality induced by ${}_RU_{\mathrm{End}({}_RU)}$ if and only if R is left linearly compact and ${}_RU$ is a linearly compact and finitely cogenerated injective cogenerator. (See [13, Theorem 4.5]). Anh [2] proved that each commutative linearly compact ring has a Morita duality.

Let R be a commutative linearly compact ring which is not noetherian (*e.g.*, the ring R in [13, Example 10.9]). Then $R[[x]]$ is not linearly compact by Theorem 2.3. Since R has a Morita duality, this gives negative answers to both [13, Question 3.7] and [13, Question 4.16]. Professor P. Vámos has also informed us that the answers to these two questions are "No". Let U be the minimal injective cogenerator in the category of R-modules. By [2], R has a Morita duality induced by ${}_RU_R$ which is not an artinian module, since R is not noetherian. If $R[x]$ denotes the polynomial ring, we see that each $R[x]$-submodule of $U[x^{-1}]$ is automatically an $R[[x]]$-submodule. Hence the $R[x]$-module $U[x^{-1}]$ is finitely cogenerated by Lemma 1.2 but not linearly compact by Proposition 2.2. This shows that $R[x]$ is not a Vámos ring, answering a question of Professor C. Faith (private communication) in the negative, where a commutative ring is called Vámos if each finitely cogenerated module is linearly compact.

The next two results give conditions for the power series ring $R[[x]]$ to have a Morita duality.

THEOREM 2.4. *The following two statements are equivalent for a bimodule* ${}_RU_S$*:*

(1) ${}_RU_S$ *defines a Morita duality, R is left noetherian and S is right noetherian;*

(2) the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ *defines a Morita duality.*

PROOF. ($\Rightarrow$). This is [14, Theorem 1.3].

($\Leftarrow$). Since $R[[x]]$ is left linearly compact, R is left noetherian and left linearly compact by Theorem 2.3. Similarly, S is right noetherian and right linearly compact. By Theorem 1.5, ${}_RU_S$ defines a quasi-duality which is a Morita duality by Kraemer's Theorem.

COROLLARY 2.5. *The following are equivalent for a ring R:*

(1) R is a left noetherian ring with a Morita duality induced by a bimodule ${}_RU_S$ *such that S is right noetherian;*

(2) $R[[x]]$ *has a Morita duality;*

PROOF. (1) $\Rightarrow$ (2). By Theorem 2.4.

(2) $\Rightarrow$ (1). Since a factor ring of a ring with a Morita duality has a Morita duality [13, Corollary 2.5], R is a left noetherian ring with a Morita duality by Theorem 2.3. Let ${}_RU_S$ define a Morita duality. Then by Theorem 1.5, the bimodule ${}_{R[[x]]}U[x^{-1}]_{S[[x]]}$ defines a quasi-duality, which is a Morita duality by Kraemer's Theorem. Hence S is right noetherian by Theorem 2.4.

We conclude this paper with an example to illustrate our results.

EXAMPLE 2.6. Let F be a field and $F((y))$ the quotient field of $F[[y]]$. By Menini [7, Example 2.6.1] or Müller [10, p. 73],

$$R = \begin{bmatrix} F((y)) & F((y)) \\ 0 & F[[y]] \end{bmatrix}$$

has a Morita self-duality defined by an R-bimodule ${}_RU_R$. We note that R is left noetherian but not right noetherian. Hence $R[[x]]$ does not have a Morita duality by Corollary 2.5. By Theorem 1.5, the bimodule ${}_{R[[x]]}U[x^{-1}]_{R[[x]]}$ defines a quasi-duality which is not a Morita duality. Since $R[[x]]$ is left linearly compact but not right linearly compact by Theorem 2.3, it follows from Kraemer's Theorem that (1) ${}_{R[[x]]}U[x^{-1}]$ is an injective cogenerator which is not linearly compact, and (2) $U[x^{-1}]_{R[[x]]}$ is a linearly compact module which is not an injective cogenerator. Since R and $R[[x]]$ have the same simple right modules, each simple right $R[[x]]$-module embedes into $U[x^{-1}]$, hence $U[x^{-1}]_{R[[x]]}$ is not an injective module. Since ${}_RU_R$ defines a Morita duality, U_R is an injective cogenerator. This shows that the noetherian condition in [6, Theorem 1] can not be dropped as we promised at the beginning of Section 1. Let

$$A = R[[x]] \propto U[x^{-1}]$$

be the trivial extension. Since the $R[[x]]$-bimodule $U[x^{-1}]$ is faithfully balanced, we see from [13, Theorem 10.7] that A is a left PF-ring which is not right PF, *i.e.*, ${}_AA$ is an injective cogenerator but A_A is not injective. The first example (different from ours) of one-sided PF-rings was given by Dischinger and Müller in [4].

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