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K-COHERENCE IS CYCLICLY EXTENSIBLE AND REDUCIBLE

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Introduction. K-coherence (K an integer $\geqq -1$), has been defined by W. R. R. Transue [**3**] in such a way that 0-coherence is connectedness and 1-coherence is unicoherence plus local connectedness. It is well-known (see, for instance, [**5**, p. 82]), that for metric spaces, unicoherence is cyclicly extensible and reducible; furthermore, this result has been generalized by Minear to locally connected spaces, [**2**, Theorems 4.1 and 4.3]. In this paper we show that for a $(k - 1)$-coherent and locally $(k - 1)$-coherent Hausdorff space M, k-coherence is cyclicly extensible and reducible.

1. Preliminaries. Throughout this paper, M denotes a nondegenerate connected and locally connected Hausdorff space. A point p of M is a *cut point* of M if $M - p$ is disconnected; p is an *end point* of M if p has a neighborhood base of open sets having singleton boundaries. If $E \subset M$, then E is an E_0-*set* of M if E is nondegenerate, connected, contains no cut point of itself, and is maximal with respect to these properties. A *cyclic element* of M is a subset of M which is an E_0-set of M or is a singleton cut point or end point of M. A property is said to be *cyclicly reducible* if whenever a space has the property then every cyclic element has the property and *cyclicly extensible* if whenever every cyclic element of the space has the property, then the space does also. A nonempty closed subset A of M is an A-*set* of M if every component of $M - A$ has singleton boundary. If $a, b \in M$, $C(a, b) = \bigcap \{A: A \text{ is an } A\text{-set of } M \text{ and } a, b \in A\}$ is an A-set and is called the *cyclic chain in M from a to b* (between a and b). A subset H of M is an *H-set of M* if either H consists of a single cut point or end point of M, or contains $C(a, b)$ for every pair of points in H. It is immediate that every A-set is an H-set. Further, every E_0-set and thus every cyclic element of M is an A-set of M [**6**, Theorem 6.2].

We will have occasion to refer to the following results, which are to be found in [**1**] or [**6**].

a) If A is an A-set of M and Z is a connected subset of M such that $A \cap Z \neq \emptyset$ then $A \cap Z$ is connected; thus every A-set of M is locally connected [**6**, Theorem 5.3].

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b) If A is an A-set of M and C is a component of $M - A$, then C is an A-set of M [**1**, 2.5].

c) Every E_0-set of an H-set of M is an E_0-set of M [**1**, 3.3].

d) Every H-set in M is a connected and locally connected Hausdorff space [**1**, 3.9].

e) If H is an H-set of M and Z is a locally connected subset of M, then $H \cap Z$ is locally connected [**1**, 3.10].

f) If H is an H-set of M, then $\bar{H}$ is an A-set of M [**1**, 3.11].

g) If H is an H-set of M, and Z is any connected and locally connected subset of M such that $H \cap Z$ is nondegenerate, then $H \cap Z$ is an H-set of Z [**1**, 3.25].

The notation we adopt is standard with the exception that if $K \subset M$ and $E \subset K$, we use $\mathrm{Int}_K(E)$, $\mathrm{Ext}_K(E)$ and $\partial_K(E)$ to denote respectively the interior, exterior, and boundary of E in K. Also, if A is an A-set of M and $S \subset A$, S' denotes the union of all components C of $M - A$ such that $\partial(C) \subset S$ and S^* denotes $S \cup S'$.

The following definition is due to Transue, [**3**, p. 2].

Definition. If S is a topological space

(1) S is (-1)-*coherent* if S is nonempty.

(2) S is *locally k-coherent at $p \in S$* if p has a neighbourhood base of k-coherent open sets.

(3) S is *locally k-coherent* if S is locally k-coherent at each point.

(4) S is *k-coherent*, $k \geqq 0$, if S is $(k - 1)$-coherent and locally $(k - 1)$-coherent and if whenever $S = A \cup B$, A, B closed and $(k - 1)$-coherent, then $A \cap B$ is $(k - 1)$-coherent.

Our first lemma is due to Minear ([**2**], p. 19) and the theorem following generalizes 3.2, p. 67 of [**5**].

2. Lemma. *Let A be an A-set of M and $S \subset A$. If S is closed in A, then S^* is closed in M. If S is open in A, then S^* is open in M.*

3. Theorem. *If A is an A-set in M and $A = S \cup T$ is a division of A into sets, (closed sets), (connected sets), then there is a division $M = L \cup N$ of M into sets, (closed sets), (connected sets) such that $L \cap N = S \cap T$, $S \subset L$, $T \subset N$. Further, if M is a continuum and S and T are continua, then L and N are continua.*

Proof. Let $L = S^*$, $N = T \cup (A - S)^*$. Then $M = L \cup N$, $S \subset L$, and $T \subset N$, so $S \cap T \subset L \cap N$. If $x \in L \cap N$, then if $x \notin A$ there is a component C of $M - A$ such that $x \in C$. Since $x \in L$, $\partial(C) \in S$. But since $x \in N$, $\partial(C) \in A - S$. It follows that $x \in A$. Since $x \in L$, $x \in S$, and since $x \in N$, $x \in T$. Thus $L \cap N = S \cap T$.

Clearly, if S and T are connected, then L and N are connected.

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Now if S and T are closed, then by 2, L is closed. Further,

$$M - N = (A - T)^* \cup (S \cap T)',$$

and by 2 and the definition of $(S \cap T)'$, each of $(A - T)^*$ and $(S \cap T)'$ is open in M and it follows that N is closed.

It is now immediate that if M, S, and T are continua, then L and N are continua.

4. THEOREM. *Let A be an A-set of M, $S \subset A$, and $\mathscr{C}$ a collection of components of $M - A$ such that if $C \in \mathscr{C}$, then $\partial(C) \in S$. If S is locally connected, then $S \cup \mathscr{U}\mathscr{C}$ is locally connected.*

Proof. Suppose S is locally connected and let $K = S \cup \mathscr{U}\mathscr{C}$. Let $p \in K$ and let $O^* = O \cap K$ be an open set in K such that $p \in O^*$ and O is open in M. If $p \in \mathrm{Int}_K(S)$ or $p \in C$ for $C \in \mathscr{C}$, then since $\mathrm{Int}_K(S)$ and C are locally connected, K is locally connected at p. Assume then that $p \in \partial_K(S)$. Since $O \cap S$ is open in S, there is an open set V of M such that $p \in V$, $V \cap S \subset O \cap S \subset O^*$ and $V \cap S$ is connected.

Now $V \cap S = V \cap (\mathrm{Int}_K(S)) \cup (V \cap \partial_K(S))$. For each $x \in V \cap \partial_K(S)$, let G_x be a connected open subset of M such that $x \in G_x \subset V \cap O$. Then for each x in $V \cap \partial_K(S)$,

$$G_x \cap K \subset V \cap O \cap K \subset O^*.$$

Let

$$G = (V \cap S) \cup \mathscr{U}_{x \in V \cap \partial_K(S)} (G_x \cap K).$$

Clearly, G is open in K, $G \subset O^*$, and $p \in G$. Further, since each G_x is connected and open in M and

$$G = (V \cap S) \cup \mathscr{U}\{G_x \cap \bar{C} | x \in \partial_K(S), c \in \mathscr{C}\},$$

a straightforward argument shows that G is connected.

5. COROLLARY. *If A is an A-set of M and S is a locally connected subset of A, then S^* is locally connected.*

6. COROLLARY. *If A is an A-set of M, $A = S \cup T$ a division of A into locally connected sets and $L = S^*$, $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into locally connected sets L, N such that $L \cap N = S \cap T$.*

7. THEOREM. *If M is unicoherent and H is an H-set of M, then H is unicoherent.*

Proof. By 1-d, H is a connected and locally connected Hausdorff space. Suppose E is a cyclic element of H. If E is degenerate, E is unicoherent; and if E is an E_0-set of H, then since E is an E_0-set of M, (1-c), and unicoherence is cyclicly reducible, E is unicoherent. Thus every cyclic

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element of H is unicoherent. Since unicoherence is cyclicly extensible, H is unicoherent.

8. THEOREM. *If H is an H-set of M and Z is a locally connected and unicoherent subset of M such that $H \cap Z \neq \emptyset$, then $H \cap Z$ is locally connected and unicoherent.*

Proof. If $H \cap Z$ is degenerate, there is nothing to prove. If $H \cap Z$ is nondegenerate, then it is locally connected by 1-e; further, since $H \cap Z$ is an H-set in Z, (1-g), it follows from Theorem 6 that $H \cap Z$ is unicoherent.

9. THEOREM. *Let A be an A-set of M. If M is unicoherent and $A = S \cup T$ is a division of A into closed unicoherent sets, $L = S^*$ and $N = T \cup (A - S)^*$, then L and N are unicoherent.*

Proof. Suppose $L = B \cup C$, where B and C are closed and connected, and let

$$\mathscr{D} = \{D | D \text{ is a component of } M - A \text{ such that } \partial(D) \in S\}.$$

Since L is connected, $B \cap C \neq \emptyset$. If B or C is contained either in S or in some member D of $\mathscr{D}$, then since S and $\bar{D}$ are each unicoherent, $B \cap C$ is connected.

Suppose then that each of B and C meets both S and $L - S$. Since S is unicoherent, $S \cap B \cap C$ is connected. Let $D \in \mathscr{D}$. If $\bar{D} \cap B \cap C \neq \emptyset$, then since $\bar{D}$ is an A-set, (1-b), and therefore unicoherent, $\bar{D} \cap B \cap C$ is connected. Since B and C are each connected and meet both D and $M - D$, $\partial(D) \in \bar{D} \cap B \cap C$. Finally, since

$$B \cap C = (S \cap B \cap C) \cup \mathscr{U}\{\bar{D} \cap B \cap C | D \in \mathscr{D}\},$$

it follows that $B \cap C$ is connected. Thus L is unicoherent. Similarly, N is unicoherent.

Since every cyclic element of M is locally connected and unicoherence is cyclicly extensible and reducible the next theorem follows easily.

10. THEOREM. *M is 1-coherent if and only if every cyclic element of M is 1-coherent.*

11. THEOREM. *For each non-negative interger k, if M is $(k - 1)$-coherent and locally $(k - 1)$-coherent, then M is k-coherent if and only if every cyclic element of M is k-coherent.*

Proof. We proceed by induction on k. Let $P(k)$ be the following statement:

a. if M is k-coherent, then every cyclic element of M is k-coherent;

b. If M is $(k - 1)$-coherent and locally $(k - 1)$-coherent and every cyclic element of M is k-coherent, then M is k-coherent;

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c. if M is k-coherent, then every A-set of M is k-coherent, and if A is an A-set of M and Z is any k-coherent subset of M such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is k-coherent;

d. if M is k-coherent, A an A-set of M, $A = S \cup T$ a division of A into closed k-coherent sets and $L = S^*$, $N = T \cup (A - S)^*$, then L and N are each k-coherent.

$P(0)$-a and b are trivial. $P(0)$-c is 1-a and $P(0)$-d was established in Theorem 3. Also, $P(1)$-a and $P(1)$-b are Theorem 10 and $P(1)$-c and $P(1)$-d follow immediately from Corollary 6 and Theorems 7, 8 and 9.

Suppose then that $P(k - 1)$ has been proved for $k \geqq 2$. Suppose further that M is k-coherent and E is an E_0-set of M. Since M is k-coherent, M is $(k - 1)$-coherent, so by $P(k - 1)$, E is $(k - 1)$-coherent. Further, since M is locally $(k - 1)$-coherent, $P(k - 1)$-c implies that E is locally $(k - 1)$-coherent. If $E = S \cup T$ is a division of E into closed $(k - 1)$-coherent sets, let $L = S^*$, $N = T \cup (E - S)^*$. Then by $P(k - 1)$-d, $M = L \cup N$ is a division of M into closed $(k - 1)$-coherent sets. Since M is k-coherent, $L \cap N$ is $(k - 1)$-coherent, and we have established in Theorem 3 that $L \cap N = S \cap T$. Thus E is k-coherent, and it follows that every cyclic element of M is k-coherent. Thus $P(k - 1)$ implies $P(k)$-a.

Assume now that M is $(k - 1)$-coherent and locally $(k - 1)$-coherent, that every cyclic element of M is k-coherent, and that $M = S \cup T$ is a division of M into closed $(k - 1)$-coherent sets. Since $k \geqq 2$, S and T are each connected and locally connected and $S \cap T$ is connected; further it follows from Theorem 2 of [**3**] that $S \cap T$ is locally connected.

Let E be an E_0-set of $S \cap T$. Then $E \subset E^*$ for some E_0-set E^* of M. Since M is $(k - 1)$-coherent, $P(k - 1)$ implies that each of E^*, $S \cap E^*$, and $T \cap E^*$ is $(k - 1)$-coherent. Since E^* is k-coherent, $E^* \cap S \cap T$ is $(k - 1)$-coherent. Now $E \subset E^* \cap S \cap T$ and is an E_0-set of $E^* \cap S \cap T$, so by $P(k - 1)$-a, E is $(k - 1)$-coherent. It follows that every cyclic element of $S \cap T$ is $(k - 1)$-coherent, so again by the induction hypothesis, $S \cap T$ is $(k - 1)$-coherent. Thus M is k-coherent and $P(k)$-b is established.

Suppose that M is k-coherent and A is an A-set of M. Then by $P(k - 1)$, A is $(k - 1)$-coherent. That A is locally $(k - 1)$-coherent also follows from $P(k - 1)$ and the fact that M is locally $(k - 1)$-coherent. If E is an E_0-set of A, then E is an E_0-set of M and we have shown that $P(k - 1)$ implies $P(k)$-a. Thus E is k-coherent. It follows that every cyclic element of A is k-coherent and since $P(k - 1)$ implies $P(k)$-b, A is k-coherent. Now let Z be any k-coherent subset of M such that $A \cap Z \neq \emptyset$. From 1-g, $A \cap Z$ is an A-set of Z, and it follows from what we have just proved that $A \cap Z$ is k-coherent.

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Finally, suppose that M is k-coherent, A is an A-set of M, $A = S \cup T$ is a division of A into closed, k-coherent sets, and that $L = S^*$, $N = T \cup (A - S)^*$. By $P(k-1)$, L and N are $(k-1)$-coherent. We show that L is locally $(k-1)$-coherent.

Let $x \in L$. If $x \in C$, for C a component of $M - A$, or $x \in \text{Int}_L(S)$, then since M and A are each locally $(k-1)$-coherent, L is locally $(k-1)$-coherent at x. Suppose that $x \in \partial_L(S)$ and O^* is an open set of L such that $x \in O^*$. Let O be open in M such that $O^* = O \cap L$. Since S is locally $(k-1)$-coherent and $x \in O \cap S$, there is an open set V of M such that $V \cap S \subset O \cap S$, $x \in V$ and $V \cap S$ is $(k-1)$-coherent. For each $y \in V \cap \partial_L(S)$, let W_y be an open $(k-1)$-coherent subset of M such that $y \in W_y \subset V \cap O$ and let

$$W^* = (V \cap S) \cup \mathscr{U}_{y \in v \cap \partial_L(S)} (W_y \cap L).$$

Then W^* is an open connected set in L. Since L is locally connected, W^* is also locally connected. Let E be an E_0-set of W^* and let E^* be the E_0-set of M such that $E \subset E^*$. It follows from what has already been proved, that E^* is k-coherent. If $E \subset S$, then since $V \cap S$ is $(k-1)$-coherent, $P(k-1)$-c implies that $E^* \cap V \cap S$ is $(k-1)$-coherent. Further, E is an E_0-set of $E^* \cap V \cap S$, so by $P(k-1)$-a, E is $(k-1)$-coherent. If $E \subset \bar{C}$ for some component C of $M - A$, $C \subset L$, then for some $y \in V \cap \partial_L(S)$, $E \subset W_y \cap \bar{C}$. Since $\bar{C}$ is an A-set, $W_y \cap \bar{C}$ is $(k-1)$-coherent by $P(k-1)$-c and so, similarly, is $E^* \cap W_y \cap \bar{C}$. But E is an E_0-set of $E^* \cap W_y \cap \bar{C}$, so $P(k-1)$-a implies that E is $(k-1)$-coherent. It follows that every cyclic element of W^* is $(k-1)$-coherent, so by $P(k-1)$-b, W^* is $(k-1)$-coherent. Thus L is locally $(k-1)$-coherent. The proof that N is locally $(k-1)$-coherent is similar.

Now let E be an E_0-set of L. If $E \subset \bar{C}$ for some component C of $M - A$, then E is an E_0-set of $\bar{C}$ and therefore of M, and it follows from what has already been proved that E is k-coherent. If $E \subset A$, let E^* be the E_0-set of M such that $E \subset E^*$. Then E^* is k-coherent. Further, since S is a k-coherent subset of M, it again follows from what has already been proved that $S \cap E^*$ is k-coherent. Now since E is an E_0-set of $S \cap E^*$, E is k-coherent. Thus every cyclic element of L is k-coherent, so L is k-coherent. Since the case for N is similar, the theorem is proved.

12. Corollary. *If M is k-coherent for some non-negative integer k, then* (1) *every A-set of M is k-coherent and* (2) *if A is an A-set of M and Z is a k-coherent subset of M such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is k-coherent.*

13. Corollary. *If M is k-coherent for some non-negative integer k, and A is an A-set of M; then if $A = S \cup T$ is a division of A into closed sets and $L = S^*$, $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into closed k-coherent sets and $L \cap N = S \cap T$.*

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The following theorem is easily proved:

14. THEOREM. *Every dendron is k-coherent for every non-negative integer k.*

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