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Part 3. Biological applications

ASYMPTOTIC HITTING PROBABILITIES FOR THE BOLTHAUSEN–SZNITMAN COALESCENT

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ASYMPTOTIC HITTING PROBABILITIES FOR THE BOLTHAUSEN–SZNITMAN COALESCENT

BY MARTIN MÖHLE

Abstract

The probability $h(n, m)$ that the block counting process of the Bolthausen–Sznitman n-coalescent ever visits the state m is analyzed. It is shown that the asymptotic hitting probabilities $h(m) = \lim_{n\to\infty} h(n, m)$, $m \in \mathbb{N}$, exist and an integral formula for $h(m)$ is provided. The proof is based on generating functions and exploits a certain convolution property of the Bolthausen–Sznitman coalescent. It follows that $h(m) \sim 1/\log m$ as $m \to \infty$. An application to linear recursions is indicated.

Keywords: Asymptotic hitting probability; Bolthausen–Sznitman coalescent; generating function

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1. Introduction and main results

Let $X = (X_k)_{k\in\mathbb{N}_0:=\{0,1,2,\dots\}}$ be a Markov chain with state space $S := \mathbb{N} := \{1, 2, \dots\}$. For given states $n, m \in S$, we are interested in the hitting probability

$$h(n, m) \;:=\; \mathbb{P}(\text{the chain } X \text{ ever visits the state } m \mid X_0 = n). \tag{1.1}$$

Clearly, $h(n, n) = 1$ for all $n \in S$. It is known (see, for example, [12, Theorem 1.3.2]) that the vector of hitting probabilities $(h(n, m) : n \in S)$ is the minimal nonnegative solution to the system of equations

$$h(n, m) = \begin{cases} 1 & \text{for } m = n, \\ \sum_{k\in S} p_{nk} h(k, m) & \text{otherwise,} \end{cases}$$

where $P = (p_{nk})_{n,k\in S}$ denotes the transition matrix of X. Minimality means that if $x = (x_n : n \in S)$ is another solution with $x_n \ge 0$ for all $n \in S$ then $x_n \ge h(n, m)$ for all $n \in S$. Explicit solutions for $h(n, m)$ are known only for particular Markov chains, e.g. for birth-and-death chains [12, Example 1.3.4]. For some chains X, the so-called asymptotic hitting probabilities

$$h(m) \;:=\; \lim_{n\to\infty} h(n, m), \qquad m \in S, \tag{1.2}$$

exist and formulae for $h(m)$ can be provided.

We mention a concrete nontrivial example. Fix a parameter $u \in (0, 1)$, and consider the chain X on state space $\mathbb{N}$ with transition probabilities $p_{11} := 1$ and

$$p_{nk} \;:=\; \frac{\binom{n-1}{k-1} u^{n-k}(1-u)^{k-1}}{1-(1-u)^{n-1}}, \qquad 1 \le k < n.$$

In this case the same renewal argument as in [7, pp. 85–86] shows that $h(1) = 1$ and $h(m) = [1 - (1-u)^{m-1}]/[(m-1)\mu]$ for all $m \geq 2$, where $\mu := -\log(1-u)$.

In this article we are interested in the particular Markov chain X that has transition probabilities $p_{11} := 1$ and

$$p_{nk} := \frac{n}{(n-1)(n-k)(n-k+1)}, \qquad 1 \leq k < n. \tag{1.3}$$

This chain occurs for example when successively cutting edges at random of a random recursive tree and recording after each cut of an edge the size of the remaining tree containing the root. See Meir and Moon [9, 10] and Panholzer [13] for more details.

This Markov chain also arises in coalescent theory. It is known (see, for example, [4, Equation (9)]) that p_{nk} in (1.3) is the probability that the jump chain of the block counting process of the Bolthausen–Sznitman coalescent [3] jumps from state n to state k. For fundamental information on the class of coalescent processes with multiple collisions, we refer the reader to [14] and [16].

Our main result, Theorem 1.1 below, shows that the probability $h(n, m)$ that the block counting process of the Bolthausen–Sznitman n-coalescent ever visits the state m converges as $n \to \infty$ and provides an integral representation for the asymptotic hitting probability $h(m) := \lim_{n\to\infty} h(n, m)$.

Theorem 1.1. *For the Bolthausen–Sznitman coalescent, for every $m \in \mathbb{N} \setminus \{1\}$, the hitting probabilities $h(n, m)$ are strictly decreasing in $n \in \{m, m+1, m+2, \dots\}$ and the asymptotic hitting probabilities (1.2) are given by $h(1) = 1$ and*

$$h(m) = (m-1) \int_0^1 \frac{t^{m-1}}{-\log(1-t)} \, dt, \qquad m \geq 2. \tag{1.4}$$

Remark 1.1. Our proof of Theorem 1.1 in Section 3 is based on generating functions and exploits a certain convolution property of the Bolthausen–Sznitman coalescent. Essentially the same convolution property has been used by Panholzer [13] and Drmota *et al.* [5] to study the number of cuts needed to isolate the root of a random recursive tree, and by Drmota *et al.* [4] to study the number of collisions and the total branch length of the Bolthausen–Sznitman coalescent.

Remark 1.2. The proof of Theorem 1.1 gives more information than stated in the theorem, namely, it provides a formula for the hitting probability $h(n, m)$ in terms of the Bernoulli numbers of the second kind (see (3.4) below). Note that, for $n \geq 2$, $p_{nn} = 0$, so, for $n \geq 2$, the hitting probability $h(n, m)$ coincides with the Green matrix entry $g(n, m) := \mathbb{E}(\sum_{k=0}^{\infty} \mathbf{1}_{\{X_k = m\}} \mid X_0 = n)$ (see, for example, [12, p. 145]).

Remark 1.3. Our method of proof of Theorem 1.1 is adapted specifically to the Bolthausen–Sznitman coalescent; it does not seem to work directly for other coalescent processes. Clearly, for the Kingman coalescent, $h(m) = 1$ for all $m \in \mathbb{N}$, since the block counting process of the Kingman coalescent has all jumps of size 1 and, hence, visits every state m almost surely. The other extreme is the star-shaped coalescent, for which we have $h(m) = 0$ for all $m \geq 2$. Verifying the existence of the limits $h(m) := \lim_{n\to\infty} h(n, m)$ and finding expressions for the asymptotic hitting probabilities $h(m)$ for other exchangeable coalescents, e.g. for beta coalescents different from the Bolthausen–Sznitman coalescent, seems to be an open problem.

Alternative integral formulae for $h(m)$, $m \ge 2$, are obtained from (1.4) via the substitutions $t = \mathrm{e}^{-u}$ and $t = 1 - \mathrm{e}^{-u}$, respectively, namely,

$$h(m) = (m-1)\int_0^\infty \frac{\mathrm{e}^{-mu}}{-\log(1-\mathrm{e}^{-u})}\,\mathrm{d}u = (m-1)\int_0^\infty (1-\mathrm{e}^{-u})^{m-1}\frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u. \quad (1.5)$$

Based on (1.5), further properties of $h(m)$ can be derived. For example (see (3.5) below), we can derive further integral representations for $h(m)$ by partial integration. The following corollary shows that $h(m)$ is strictly decreasing in m and clarifies the asymptotic behavior of $h(m)$ as m tends to ∞.

Corollary 1.1. *For the Bolthausen–Sznitman coalescent, $h(m)$ is strictly decreasing in m and $h(m) \sim 1/\log m$ as $m \to \infty$.*

The next corollary provides an alternative formula for the asymptotic hitting probability $h(m)$; it is particularly useful for computing $h(m)$ for small values of m.

Corollary 1.2. *For the Bolthausen–Sznitman coalescent,*

$$h(m) = (m-1)\sum_{i=1}^{m-1}\binom{m-1}{i}(-1)^{i-1}\log(i+1), \qquad m \ge 2. \quad (1.6)$$

For instance, $h(2) = \log 2 \approx 0.693\,147$ and $h(3) = 4\log 2 - 2\log 3 \approx 0.575\,364$.

Remark 1.4. For $n \in \mathbb{N}$ and a given subset A of the state space $\mathbb{N}$, there is some interest (see, for example, [12]) in more general hitting probabilities of the form

$$h(n, A) := \mathbb{P}(\text{the chain } X \text{ ever visits a state in } A \mid X_0 = n).$$

For $A = \{m\}$, we recover the hitting probability $h(n, m) = h(n, \{m\})$ in (1.1). Depending on the choice of A, the analysis of $h(n, A)$ can be simple or complicated. For example, for the Bolthausen–Sznitman coalescent, if $A = A_m := \{m, m+1, m+2, \ldots\}$ for some fixed $m \in \mathbb{N}$ then $h(n, A_m)$ is equal to the probability that after the first jump the chain X is still in a state larger than or equal to m. We therefore obtain the simple expression

$$h(n, A_m) = \sum_{k=m}^{n-1} p_{nk} = \frac{n(n-m)}{(n-1)(n-m+1)}, \qquad 1 \le m < n.$$

In particular, $h(A_m) := \lim_{n\to\infty} h(n, A_m) = 1$ for all $m \in \mathbb{N}$. We leave the analysis of $h(n, A)$ for general subsets $A \subseteq \mathbb{N}$ to future work.

2. An application: linear recursions

For each $n \in \mathbb{N}$, let p_{nm}, $m \in \{1, \ldots, n\}$, be a probability distribution with $p_{nn} = 0$ for all $n \ge 2$. Note that $p_{11} = 1$. Define the sequence $(a_n)_{n\in\mathbb{N}}$ as the unique solution to the recursion

$$a_n = r_n + \sum_{m=1}^{n-1} p_{nm} a_m, \qquad n \ge 2, \quad (2.1)$$

for given $r_2, r_3, \ldots \in \mathbb{R}$ and given initial value $a_1 \in \mathbb{R}$. Linear recursions of this form occur in many fields in applied mathematics, in particular in the analysis of algorithms and

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in probability. For example, in [8, Lemma A.1] recursions of the form (2.1) (and more general linear recursions) are considered and a result on the O-behavior of the sequence $(a_n)_{n\in\mathbb{N}}$ is established. In particular cases Lemma A.1 of [8] leads to $a_n = O(1)$. In this situation it is natural to ask whether the limit $a := \lim_{n\to\infty} a_n$ exists or not. Even if a exists, recursion (2.1) usually does not provide direct information on a, since, for $n \to \infty$, (2.1) usually degenerates to the uninformative equation $a = a$.

Here is another criterion which yields the convergence of the sequence $(a_n)_{n\in\mathbb{N}}$ and provides a formula for the limit. As before, we interpret the p_{nm} as the transition probabilities of a Markov chain X and let $h(n, m)$ denote the hitting probability that the Markov chain X ever visits state m conditional on the chain starting in state n.

Proposition 2.1. *Suppose that* $\sum_{m=2}^{\infty} |r_m| < \infty$. *If the asymptotic hitting probabilities* $h(m) := \lim_{n\to\infty} h(n, m)$ *exist for all* $m \in \mathbb{N}$ *then*

$$\lim_{n\to\infty} a_n = a_1 + \sum_{m=2}^{\infty} h(m) r_m.$$

Example. Suppose that $a_1 = 0$ and that $r_m = 1/[m(m-1)]$ for all $m \geq 2$. For the Bolthausen–Sznitman coalescent, i.e. for the Markov chain with transition probabilities (1.3), a combination of Theorem 1.1 and Proposition 2.1 shows that sequence (2.1) converges and has limit

$$\lim_{n\to\infty} a_n = \sum_{m=2}^{\infty} r_m h(m) = \sum_{m=2}^{\infty} \frac{1}{m} \int_0^1 \frac{t^{m-1}}{-\log(1-t)}\,\mathrm{d}t = \int_0^1 \left(\frac{1}{t} + \frac{1}{\log(1-t)}\right) \mathrm{d}t = \gamma,$$

where $\gamma := -\Gamma'(1) \approx 0.577\,216$ denotes Euler's constant.

3. Proofs

Throughout the proofs, $D := \{z \in \mathbb{C} \colon |z| < 1\}$ denotes the open unit disc and we write $L(z) := -\log(1-z)$, $z \in D$. Furthermore, for $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we use the notation $(x)_n := x(x-1)\cdots(x-n+1)$ for the descending factorials, with the convention that $(x)_0 := 1$. We start with the following auxiliary lemma.

Lemma 3.1. *The function* $g \colon D \to \mathbb{C}$, *defined by* $g(0) := 1$ *and* $g(z) := z/L(z)$ *for* $z \in D \setminus \{0\}$, *has Taylor expansion* $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in D$, *with coefficients*

$$a_n := \frac{(-1)^n}{n!} \int_0^1 (x)_n \,\mathrm{d}x, \qquad n \geq 0.$$

Remark 3.1. We provide here more information on the coefficients a_n, $n \in \mathbb{N}_0$. Note first that $a_0 = 1$ while all the other coefficients $a_1, a_2, \ldots$ are strictly negative. The coefficients $b_n := \int_0^1 (x)_n \,\mathrm{d}x$, $n \in \mathbb{N}_0$, are the Bernoulli numbers of the second kind (see, e.g. [15, p. 114]). From $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} z^n/(n+1)) = 1$, it follows that $\sum_{j=0}^{n} a_j/(n+1-j) = 0$, $n \in \mathbb{N}$. Replacing n by $n+1$ we conclude that the coefficients a_n, $n \in \mathbb{N}_0$, satisfy the recursion

$$a_{n+1} = -\sum_{j=0}^{n} \frac{a_j}{n+2-j}, \qquad n \in \mathbb{N}_0.$$

It is also known (see, for example, [6, p. 387]) that $a_n \sim -1/(n \log^2 n)$ as $n \to \infty$.

Proof of Lemma 3.1. Clearly, for $z = 0$, we have $\sum_{n=0}^{\infty} a_n z^n = a_0 = 1 = g(0)$. Assume now that $z \in D \setminus \{0\}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \int_0^1 (x)_n \,\mathrm{d}x \\ &= \int_0^1 \sum_{n=0}^{\infty} \binom{x}{n} (-z)^n \,\mathrm{d}x \\ &= \int_0^1 (1-z)^x \,\mathrm{d}x \\ &= \left[\frac{(1-z)^x}{\log(1-z)}\right]_0^1 \\ &= \frac{z}{L(z)} \\ &= g(z),\end{aligned}$$

where, for the second equality in the chain of equations above, dominated convergence justifies the interchange of the infinite sum and the integral.

Proof of Theorem 1.1. For $m \in \mathbb{N}$ and $z \in D$, define the generating function $\phi_m(z) := \sum_{n=m}^{\infty} h(n, m) z^n$. Note that $\phi_m(0) = 0$ for all $m \in \mathbb{N}$. In the following we use a particular convolution property of the Bolthausen–Sznitman coalescent in order to determine $h(m)$. We have (see [12, Theorem 1.3.2]) $h(m, m) = 1$ and, for $n > m$, $h(n, m) = \sum_{k=1}^{n} p_{nk} h(k, m) = \sum_{k=m}^{n-1} p_{nk} h(k, m)$. Hence,

$$\begin{aligned}\sum_{n=m+1}^{\infty} h(n, m) \frac{n-1}{n} z^n &= \sum_{n=m+1}^{\infty} \sum_{k=m}^{n-1} p_{nk} h(k, m) \frac{n-1}{n} z^n \\ &= \sum_{k=m}^{\infty} h(k, m) \sum_{n=k+1}^{\infty} p_{nk} \frac{n-1}{n} z^n \\ &= \sum_{k=m}^{\infty} h(k, m) z^k \sum_{n=k+1}^{\infty} \frac{1}{(n-k)(n-k+1)} z^{n-k},\end{aligned}$$

$$\sum_{n=m}^{\infty} h(n, m) \frac{n-1}{n} z^n = \frac{m-1}{m} z^m + \sum_{k=m}^{\infty} h(k, m) z^k \sum_{i=1}^{\infty} \frac{z^i}{i(i+1)} = \frac{m-1}{m} z^m + \phi_m(z) a(z),$$

where

$$a(z) := \sum_{i=1}^{\infty} \frac{z^i}{i(i+1)} = 1 - \frac{(1-z)L(z)}{z}, \qquad z \in D.$$

On the other hand,

$$\begin{aligned}\sum_{n=m}^{\infty} h(n, m) \frac{n-1}{n} z^n &= \sum_{n=m}^{\infty} h(n, m) z^n - \sum_{n=m}^{\infty} h(n, m) \frac{z^n}{n} \\ &= \phi_m(z) - \int_0^z \sum_{n=m}^{\infty} h(n, m) t^{n-1} \,\mathrm{d}t \\ &= \phi_m(z) - \int_0^z \frac{\phi_m(t)}{t} \,\mathrm{d}t.\end{aligned}$$

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Thus,

$$\phi_m(z) - \int_0^z \frac{\phi_m(t)}{t}\,\mathrm{d}t \;=\; \frac{m-1}{m}z^m + \phi_m(z)a(z),$$

or, equivalently,

$$\int_0^z \frac{\phi_m(t)}{t}\,\mathrm{d}t \;=\; [1-a(z)]\phi_m(z) - \frac{m-1}{m}z^m.$$

Taking the derivative with respect to z yields

$$\frac{\phi_m(z)}{z} \;=\; -a'(z)\phi_m(z) + [1-a(z)]\phi_m'(z) - (m-1)z^{m-1},$$

or, equivalently,

$$[1-a(z)]\phi_m'(z) \;=\; \left(\frac{1}{z} + a'(z)\right)\phi_m(z) + (m-1)z^{m-1}.$$

Since $1-a(z) = (1-z)L(z)/z$ and $a'(z) = -1/z + L(z)/z^2$, it follows that ϕ_m satisfies the differential equation

$$\phi_m'(z) \;=\; \frac{\phi_m(z)}{z(1-z)} + r_m(z), \tag{3.1}$$

where

$$r_m(z) \;:=\; \frac{(m-1)z^m}{(1-z)L(z)}.$$

For $m=1$, the solution of the (homogeneous) differential equation (3.1) with initial conditions $\phi_1(0) = 0$ and $\phi_1'(0) = 1$ is $\phi_1(z) = z/(1-z)$, in agreement with $h(n,1) = 1$ for all $n \in \mathbb{N}$. Assume now that $m \geq 2$. Then the solution of the (inhomogeneous) differential equation (3.1) with initial conditions $\phi_m^{(j)}(0) = 0$ for all $j \in \{0,\ldots,m-1\}$ and $\phi_m^{(m)}(0) = m!$ is $\phi_m(z) = c_m(z)z/(1-z)$, where

$$c_m(z) := \int_0^z \frac{1-t}{t} r_m(t)\,\mathrm{d}t \;=\; (m-1)\int_0^z \frac{t^{m-1}}{L(t)}\,\mathrm{d}t, \qquad m \geq 2. \tag{3.2}$$

For a power series $f(z) = \sum_{n=0}^\infty f_n z^n$, denote the coefficient f_n of z^n by $[z^n]f(z)$. In this notation we obtain

$$\begin{aligned} h(n,m) &= [z^n]\phi_m(z) \\ &= [z^n]\left(c_m(z)\frac{z}{1-z}\right) \\ &= \sum_{k=m-1}^{n-1} ([z^k]c_m(z))\left([z^{n-k}]\frac{z}{1-z}\right) \\ &= \sum_{k=m-1}^{n-1} [z^k]c_m(z), \qquad 2 \leq m \leq n. \end{aligned} \tag{3.3}$$

From (3.2) and Lemma 3.1, it follows that

$$\begin{aligned} c_m(z) &= (m-1)\int_0^z t^{m-2}\frac{t}{L(t)}\,\mathrm{d}t \\ &= (m-1)\int_0^z t^{m-2}\sum_{j=0}^{\infty} a_j t^j\,\mathrm{d}t \\ &= (m-1)\sum_{j=0}^{\infty} a_j \int_0^z t^{j+m-2}\,\mathrm{d}t \\ &= (m-1)\sum_{j=0}^{\infty}\frac{a_j}{j+m-1}z^{j+m-1}, \qquad m \ge 2. \end{aligned}$$

Thus, $[z^k]c_m(z) = (m-1)a_{k-(m-1)}/k$ for all $k \ge m-1$. Substitution into (3.3) gives the explicit expression

$$h(n,m) = (m-1)\sum_{k=m-1}^{n-1}\frac{a_{k-(m-1)}}{k} = (m-1)\sum_{j=0}^{n-m}\frac{a_j}{j+m-1}, \qquad 2 \le m \le n, \tag{3.4}$$

for the hitting probabilities. From Remark 3.1, the coefficients $a_1, a_2, \ldots$ are all strictly negative. Consequently, by (3.4), for each $m \ge 2$, the sequence $(h(n,m)\colon n = m, m+1, \ldots)$ is strictly decreasing. In particular, the limit $h(m) := \lim_{n\to\infty} h(n,m)$ exists and we obtain for $m \ge 2$ the solution

$$h(m) = \lim_{n\to\infty} h(n,m) = \sum_{k=m-1}^{\infty}[z^k]c_m(z) = c_m(1) = (m-1)\int_0^1 \frac{t^{m-1}}{L(t)}\,\mathrm{d}t,$$

which is (1.4).

Proof of Corollary 1.1. Define the functions $g\colon (0,\infty) \to \mathbb{R}$ and $h_m\colon (0,\infty) \to \mathbb{R}$, $m \in \mathbb{N}$, via $g(u) := (1-\mathrm{e}^{-u})/u$ and $h_m(u) := (1-\mathrm{e}^{-u})^{m-1}$, $u \in (0,\infty)$. Note that $g'(u) = (u\mathrm{e}^{-u} + \mathrm{e}^{-u} - 1)/u^2$ and that $h'_m(u) = (m-1)(1-\mathrm{e}^{-u})^{m-2}\mathrm{e}^{-u}$, $u \in (0,\infty)$. By (1.5), for all $m \ge 2$, $h(m) = \int_0^\infty g(u)h'_m(u)\,\mathrm{d}u$. Partial integration yields

$$h(m) = -\int_0^\infty g'(u)h_m(u)\,\mathrm{d}u = \int_0^\infty \frac{1-\mathrm{e}^{-u}-u\mathrm{e}^{-u}}{u^2}(1-\mathrm{e}^{-u})^{m-1}\,\mathrm{d}u. \tag{3.5}$$

Note that (3.5) also holds for $m = 1$. The integral on the right-hand side of (3.5) is strictly decreasing in $m \ge 1$, since $1-\mathrm{e}^{-u}-u\mathrm{e}^{-u} > 0$ and $1-\mathrm{e}^{-u} \in (0,1)$ for all $u \in (0,\infty)$. Also, (3.5) implies that $h(m) = \mathbb{E}[(1-\mathrm{e}^{-U})^{m-1}]$ for all $m \in \mathbb{N}$, where U is a positive random variable with density $u \mapsto (1-\mathrm{e}^{-u}-u\mathrm{e}^{-u})/u^2$, $u \in (0,\infty)$.

To verify that $h(m) \sim 1/\log m$ as $m \to \infty$, we proceed much as in the proof of Corollary 4.1 of [11]. Consider the kernel $k\colon (0,\infty) \to \mathbb{R}$, $k(t) := t\mathrm{e}^{-t}$, and its Mellin transform $\check{k}(z) := \int_{(0,\infty)} t^{-z-1}k(t)\,\mathrm{d}t = \int_{(0,\infty)} t^{-z}\mathrm{e}^{-t}\,\mathrm{d}t = \Gamma(1-z)$, which converges at least for all $z \in \mathbb{C}$ with $-\infty < \mathrm{Re}(z) < 1$. Define the function $f\colon (0,\infty) \to \mathbb{R}$ via

$$f(t) := \frac{\mathrm{e}^{-1/t}}{-\log(1-\mathrm{e}^{-1/t})}, \qquad t \in (0,\infty),$$

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such that the Mellin convolution, $k *_{\mathrm{M}} f$, of k and f satisfies

$$(k *_{\mathrm{M}} f)(x) := \int_0^\infty k\left(\frac{x}{t}\right) f(t)\,\frac{\mathrm{d}t}{t} = \int_0^\infty k(xu) f\left(\frac{1}{u}\right)\frac{\mathrm{d}u}{u} = x\int_0^\infty \frac{\mathrm{e}^{-(x+1)u}}{-\log(1-\mathrm{e}^{-u})}\,\mathrm{d}u. \tag{3.6}$$

Obviously, $t^2 f(t)$ is bounded on every interval $(0, a]$ and $f(t) \sim 1/\log t$ as $t \to \infty$. Thus we can apply Theorem 4.1.6 of [2] (with $\sigma = -2$, $\tau = \frac{1}{2}$, $\rho = 0$, and $l(x) = 1/\log x$) to conclude that $(k *_{\mathrm{M}} f)(x) \sim \check{k}(0)l(x) = \Gamma(1)/\log x = 1/\log x$ as $x \to \infty$. Replacing x by $m-1$ and noting that (compare (3.6) with the first equation in (1.5)) $(k *_{\mathrm{M}} f)(m-1) = h(m)$ it follows that $h(m) \sim 1/\log m$ as $m \to \infty$.

Proof of Corollary 1.2. For $m \in \mathbb{N}$ and $x, u \in (0,\infty)$, define $f_m(x,u) := (1-\mathrm{e}^{-xu})^m \mathrm{e}^{-u}/u$. We have $(\partial/\partial x) f_m(x,u) = m(1-\mathrm{e}^{-xu})^{m-1}\mathrm{e}^{-xu}\mathrm{e}^{-u} \le m\mathrm{e}^{-u} =: d_m(u)$ for all $x, u \in (0,\infty)$ and the dominating function d_m is integrable with respect to Lebesgue measure on $(0,\infty)$. Hence, we can differentiate $\int_0^\infty f_m(x,u)\,\mathrm{d}u$ with respect to x under the integral. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x}\int_0^\infty f_m(x,u)\,\mathrm{d}u &= m\int_0^\infty (1-\mathrm{e}^{-xu})^{m-1}\mathrm{e}^{-(x+1)u}\,\mathrm{d}u\\ &= m\int_0^\infty \sum_{i=1}^m \binom{m-1}{i-1}(-\mathrm{e}^{-xu})^{i-1}\mathrm{e}^{-(x+1)u}\,\mathrm{d}u\\ &= m\sum_{i=1}^m \binom{m-1}{i-1}(-1)^{i-1}\int_0^\infty \mathrm{e}^{-(ix+1)u}\,\mathrm{d}u\\ &= m\sum_{i=1}^m \binom{m-1}{i-1}(-1)^{i-1}\frac{1}{ix+1}.\end{aligned}$$

Integration yields

$$\int_0^\infty f_m(x,u)\,\mathrm{d}u = m\sum_{i=1}^m \binom{m-1}{i-1}(-1)^{i-1}\frac{\log(ix+1)}{i} = \sum_{i=1}^m \binom{m}{i}(-1)^{i-1}\log(ix+1).$$

It remains to note that, by the second equation in (1.5), $h(m) = (m-1)\int_0^\infty f_{m-1}(1,u)\,\mathrm{d}u$, and (1.6) follows immediately.

Remark 3.2. It is readily checked that the hitting probabilities satisfy

$$h(n,m) = \delta_{nm} + \sum_{k=m+1}^n h(n,k)p_{km}, \qquad 1 \le m \le n, \tag{3.7}$$

where δ_{nm} denotes the Kronecker symbol. For the Bolthausen–Sznitman, $\sum_{k=m+1}^\infty p_{km} = \sum_{k=m+1}^\infty k/((k-1)(k-m)(k-m+1)) \le 2\sum_{k=m+1}^\infty 1/((k-m)(k-m+1)) = 2 < \infty$ for all $m \in \mathbb{N}$, so the measure μ_m, defined via $\mu_m(k) := p_{km}$ for all $k \in \{m+1, m+2, \dots\}$, is finite. Moreover, the hitting probabilities $h(n,k)$ are dominated by 1 and converge as $n \to \infty$ to $h(k)$ by Theorem 1.1. Letting $n \to \infty$ on both sides of (3.7), it follows by dominated convergence that the asymptotic hitting probabilities $h(m)$, $m \in \mathbb{N}$, satisfy the system of equations

$$h(m) = \sum_{k=m+1}^\infty h(k)p_{km}, \qquad m \in \mathbb{N}. \tag{3.8}$$

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Iteration of (3.8) leads to

$$h(m) = \sum_{k=m+r}^{\infty} h(k) p_{km}^{(r)}, \qquad m, r \in \mathbb{N},$$

where the $p_{km}^{(r)}$ denote the r-step transition probabilities of the chain X. Note that $\sum_{m=1}^{\infty} h(m) = \infty$, because otherwise one would obtain

$$\infty > \sum_{m=1}^{\infty} h(m) = \sum_{m=1}^{\infty} \sum_{k=m+1}^{\infty} h(k) p_{km} = \sum_{k=2}^{\infty} h(k) \sum_{m=1}^{k-1} p_{km} = \sum_{k=2}^{\infty} h(k),$$

and, hence, $h(1) = 0$, in contradiction to $h(1) = 1$.

It is unclear whether system (3.8) has only one bounded solution $h = (h(m) \colon m \in \mathbb{N})$ satisfying $h(1) = 1$. In other words, we do not know whether the vector space of all bounded sequences $h = (h(m) \colon m \in \mathbb{N})$ satisfying (3.8) has dimension 1 or larger. In order to give a more functional analytic description of this dimension problem, let ℓ^∞ denote the Banach space of all bounded sequences $x = (x(n) \colon n \in \mathbb{N})$ equipped with the norm $\|x\| := \sup_{n \in \mathbb{N}} |x(n)|$. The operator $T \colon \ell^\infty \to \ell^\infty$, defined via

$$(Tx)(m) := \sum_{k=m+1}^{\infty} x(k) p_{km}$$

for all $x = (x(n) \colon n \in \mathbb{N}) \in \ell^\infty$ and all $m \in \mathbb{N}$, is clearly linear and also continuous, since $\|Tx\| \le \|x\| \sup_{m \in \mathbb{N}} \sum_{k=m+1}^{\infty} p_{km} \le 2\|x\|$ for all $x \in \ell^\infty$.

We verify that T is not compact. For $i \in \mathbb{N}$, let e_i denote the ith unit vector in ℓ^∞. For all $i, j \in \mathbb{N}$ with $i < j$, we have

$$\|Te_j - Te_i\| \ge |(Te_j)(j-1) - (Te_i)(j-1)| = |p_{j,j-1} - 0| = p_{j,j-1} = \frac{j}{2(j-1)} \ge \frac{1}{2}.$$

Thus, the sequence $(Te_i)_{i \in \mathbb{N}}$ does not contain any Cauchy subsequences, implying that T is not compact. We therefore cannot apply functional analytic results for compact operators (such as the Riesz–Schauder theorem) in order to obtain further information on the kernel $\ker(\mathrm{Id} - T) = \{h \in \ell^\infty \colon Th = h\}$ consisting of all $h = (h(m) \colon m \in \mathbb{N}) \in \ell^\infty$ satisfying (3.8).

Finally, we show that T is not a Krein operator. It is known (see, e.g. [1, p. 170]) that ℓ^∞ is a Krein space with closed cone $K := \{x \in \ell^\infty \colon x \ge 0\}$. Note that $x = (x(n) \colon n \in \mathbb{N}) \in K$ is an order unit (internal point) if and only if $\inf\{x(n) \colon n \in \mathbb{N}\} > 0$. By Corollary 1.1, $h(n) \sim 1/\log n \to 0$ as $n \to \infty$, so h cannot be an order unit. Thus, T cannot be a Krein operator in the sense of [1, Definition 4.1], since $h > 0$ but $T^n h = h$ is not an order unit for all $n \in \mathbb{N}$. We therefore cannot apply results for Krein operators (such as [1, Lemma 4.10]) in order to conclude that $\ker(\mathrm{Id} - T)$ is one dimensional. The fact that T is neither compact nor a Krein operator illustrates the complexity of the operator T.

Proof of Proposition 2.1. From (2.1), it readily follows by induction on $N \in \mathbb{N}$ that

$$a_n = a_1 p_{n1}^{(N)} + \sum_{m=2}^{n} \sum_{j=0}^{N-1} p_{nm}^{(j)} r_m + \sum_{m=2}^{n} p_{nm}^{(N)} a_m, \qquad n \ge 2, \tag{3.9}$$

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where the $p_{nm}^{(j)}$ denote the j-step transition probabilities of the Markov chain X. For $N = 1$, (3.9) reduces to (2.1) since $p_{nn} = 0$ for all $n \geq 2$. The induction step from N to $N + 1$ is performed by making use of the Chapman–Kolmogorov equations.

Now note that $p_{nm}^{(N)} \to \delta_{m1}$ as $N \to \infty$, since the state 1 is absorbing and all other states are transient. Thus, taking the limit $N \to \infty$ on both sides of (3.9) yields

$$a_n = a_1 + \sum_{m=2}^{n} \sum_{j=0}^{\infty} p_{nm}^{(j)} r_m = a_1 + \sum_{m=2}^{n} h(n, m) r_m, \qquad n \geq 2, \tag{3.10}$$

since $p_{nn} = 0$ for all $n \geq 2$ and, hence,

$$\begin{aligned} h(n, m) &= \mathbb{P}(\text{the chain } X \text{ ever visits } m \mid X_0 = n) \\ &= \mathbb{P}\bigg(\bigcup_{j=0}^{\infty} \{X_j = m\} \,\bigg|\, X_0 = n\bigg) \\ &= \sum_{j=0}^{\infty} \mathbb{P}(X_j = m \mid X_0 = n) \\ &= \sum_{j=0}^{\infty} p_{nm}^{(j)}, \qquad 2 \leq m \leq n. \end{aligned}$$

By assumption, $\sum_{m=2}^{\infty} |r_m| < \infty$ and $h(n, m) \to h(m)$ as $n \to \infty$ for all $m \geq 2$. Since $0 \leq h(n, m) \leq 1$, the last sum in (3.10) converges to $\sum_{m=2}^{\infty} h(m) r_m$ as $n \to \infty$ by dominated convergence.

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