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EXT-FINITE MODULES FOR WEAKLY SYMMETRIC ALGEBRAS WITH RADICAL CUBE ZERO

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Abstract

Assume that A is a finite-dimensional algebra over some field, and assume that A is weakly symmetric and indecomposable, with radical cube zero and radical square nonzero. We show that such an algebra of wild representation type does not have a nonprojective module M whose ext-algebra is finite dimensional. This gives a complete classification of weakly symmetric indecomposable algebras which have a nonprojective module whose ext-algebra is finite dimensional. This shows in particular that existence of ext-finite nonprojective modules is not equivalent with the failure of the finite generation condition (Fg), which ensures that modules have support varieties.

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1. Introduction

Assume that A is a finite-dimensional algebra over a field K. We say that an A-module M is ext-finite if there is some $n \geq 0$ such that $\mathrm{Ext}_A^k(M, M) = 0$ for $k > n$.

If $A = KG$, the group algebra of a finite group, then any ext-finite module is projective (this may be found in [4, Ch. 5]). On the other hand, there is a four-dimensional selfinjective algebra which has nonprojective ext-finite modules, first described in [15]. This algebra is known as a q-exterior algebra; see Section 4. If a selfinjective algebra A has a nonprojective ext-finite module, there is no support variety theory for A-modules via Hochschild cohomology. This follows from [10, Corollary 2.3]; it shows that the finite generation conditions [10, (Fg1) and (Fg2)] (and [16, (Fg)]) must fail. That is, existence of ext-finite nonprojective modules gives information about the action of the Hochschild cohomology on ext-algebras of modules.

There is also the 'generalised Auslander–Reiten condition', GARC, which has been introduced in [2] in the context of homological conjectures, and which has attracted a lot of interest; see for example [6–9, 13]. The condition GARC for a ring R is stated as follows.

If M is an R-module and there is some $n \geq 0$ such that $\mathrm{Ext}^k_R(M, M \oplus R) = 0$ for $k > n$, then M has projective dimension at most n.

The four-dimensional local algebra mentioned above does not satisfy GARC, there are even counterexamples with $n = 1$; see Section 4. It is not known whether there is a ring R which has a counterexample with $n = 0$.

If $R = A$ and A is a selfinjective finite-dimensional algebra, then GARC states that *any ext-finite module is projective*.

The four-dimensional algebras which have nonprojective ext-finite modules belong to the class of weakly symmetric algebras with radical cube zero. These algebras have been studied in [5, 12]. In particular, it is understood when such an algebra does not satisfy the (Fg) condition, as follows.

Assume that A is weakly symmetric with $J^3 = 0$ and $J^2 \neq 0$, where J is the radical of A. Assume also that A is indecomposable. Let E be the matrix with entries $\dim \mathrm{Ext}^1(S_i, S_j)$, where $S_1, S_2, \ldots, S_r$ are the simple A-modules. Then E is a symmetric matrix, so it has real eigenvalues. The largest eigenvalue λ, say, occurs with multiplicity one, and has a positive eigenvector; this is the Perron–Frobenius theorem. It is proved in [12] that A does not satisfy (Fg) if and only if either $\lambda > 2$, or else A is Morita equivalent to either a four-dimensional local algebra as above or to a 'double Nakayama algebra' (see Section 4), where in both cases there is a deformation parameter which is not a root of unity.

These double Nakayama algebras also have ext-finite nonprojective modules; this is probably known: we will give a proof in Section 4.

Our main result shows that a weakly symmetric algebra with radical cube zero and $\lambda > 2$ does not have ext-finite nonprojective modules. With this, we get the following result.

THEOREM 1.1. *Assume that A is a weakly symmetric indecomposable algebra over an algebraically closed field, with $J^3 = 0 \neq J^2$. Then A has an ext-finite nonprojective module if and only if $\lambda = 2$ and A is Morita equivalent to either a four-dimensional q-exterior algebra or a double Nakayama algebra, where in both cases the deformation parameter is not a root of unity.*

It follows that existence of ext-finite nonprojective modules is not equivalent with failure of (Fg).

The theorem remains true for an arbitrary field if one takes for A an algebra defined by quiver and relations.

Section 2 contains the relevant background. In Section 3 we prove the main new part of the theorem, and in Section 4 we construct ext-finite nonprojective modules for the algebras for which $\lambda = 2$. We work with finite-dimensional left A-modules and, if M, N are such A-modules, then we write $\mathrm{Hom}(M, N)$ instead of $\mathrm{Hom}_A(M, N)$ and similarly $\mathrm{Ext}^k(M, N)$ means $\mathrm{Ext}^k_A(M, N)$. Relevant background may be found in [1] or [3].

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2. Preliminaries

2.1 We assume throughout that A is a finite-dimensional weakly symmetric algebra over an algebraically closed field K, and we assume that A is indecomposable. This is no restriction since we will focus on indecomposable modules. Suppose that M is a finite-dimensional A-module. Then $\mathrm{rad}(M)$ is the submodule of M such that $M/\mathrm{rad}(M)$ is the largest semisimple factor module of M, sometimes called the 'top' of M. The module $\mathrm{rad}(M)$ is equal to JM, where J is the radical of A. The socle of M, denoted by $\mathrm{soc}(M)$, is the largest semisimple submodule of M.

2.2 A finite-dimensional A-module M has a projective cover, that is, there is a surjective map $\pi_M : P \to M$, where P is projective and $P/\mathrm{rad}(P) \cong M/\mathrm{rad}(M)$. The kernel of π_M is unique up to isomorphism, and is denoted by $\Omega(M)$. Repeatedly taking projective covers gives a minimal projective resolution of M,

$$\cdots \to P_m \xrightarrow{d_m} P_{m-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0,$$

where $d_0 = \pi_M$ and d_m is a projective cover of $\Omega^m(M)$ for $m \geq 1$. If A is selfinjective and M is indecomposable and nonprojective, then also $\Omega(M)$ is indecomposable and nonprojective. In fact, Ω induces an equivalence of the stable module category of A.

2.3 We assume that A is weakly symmetric. This means that A is selfinjective, and any indecomposable projective module has a simple socle, with $\mathrm{soc}(P) \cong P/\mathrm{rad}(P)$. Hence, for any simple module, its projective cover is also its injective hull. This implies also that for any nonprojective indecomposable A-module M we have that $M/\mathrm{rad}(M)$ is isomorphic to $\mathrm{soc}\,\Omega(M)$.

Let $S_1, S_2, \ldots, S_r$ be the simple A-modules, and let P_i be the projective cover of S_i for $1 \leq i \leq r$. We assume throughout that $J^3 = 0$ but $J^2 \neq 0$. If so, then every indecomposable projective module P_i must have radical length three; this is well known (and it is easy to see, recalling that we assume A to be indecomposable). So, we have $P_i/\mathrm{rad}(P_i) \cong S_i \cong \mathrm{soc}(P_i)$ and $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is semisimple and nonzero. So, we can write

$$\mathrm{rad}(P_i)/\mathrm{soc}(P_i) \cong \bigoplus_{j=1}^{r} d_{ij} S_j,$$

where $d_{ij} \geq 0$ and not all d_{ij} are zero. It is also true that for all i, j we have $d_{ij} = d_{ji}$. We will give the proof in 2.6 below.

This is sufficient information to compute dimensions of Ω-translates of M. The crucial property is the following, which is well known. For convenience we give the proof.

Lemma 2.1. *Assume that M is a module with no simple or projective summands. Then* $\mathrm{soc}(M) = \mathrm{rad}(M)$.

Proof. Since M has no projective (hence injective) summand, it has radical length ≤ 2. Therefore, $\mathrm{rad}(M)$ is annihilated by J and hence is contained in $\mathrm{soc}(M)$. The socle of

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M is semisimple and hence $\mathrm{soc}(M) = \mathrm{rad}(M) \oplus C$ for some submodule C of $\mathrm{soc}(M)$. We must show that $C = 0$.

Let $\pi : M \to M/\mathrm{rad}(M)$ be the canonical surjection; then $\pi(C)$ is isomorphic to C: we write $C' = \pi(C)$.

The module $M/\mathrm{rad}(M)$ is semisimple, so we can write $M/\mathrm{rad}(M) = C' \oplus G$ for some semisimple module G. Let $\tilde{G}$ be the submodule of M containing $\mathrm{rad}(M)$ such that $\tilde{G}/\mathrm{rad}(M) = G$.

Then we have that $\tilde{G} \cap C = 0$ and $M = \tilde{G} + C$: namely, if $x \in \tilde{G} \cap C$, then $x + \mathrm{rad}(M) \in G \cap C' = 0$ and therefore $x \in \mathrm{rad}(M)$, and then it is in the intersection of $\mathrm{rad}(M)$ with C and is zero. Furthermore, we have $M/\mathrm{rad}(M) = G + C'$, which implies that $M = \tilde{G} + C$. So, if $C \neq 0$, then it is a semisimple summand of M, and by the assumption $C = 0$. □

2.4 Let M be a module such that $\mathrm{soc}(M) = \mathrm{rad}(M)$, so that both the socle of M and $M/\mathrm{rad}(M)$ are semisimple. We write $\underline{\dim}(\mathrm{soc}(M)) = \underline{s} = (s_1, s_2, \ldots, s_r)$, where s_i is the multiplicity of S_i in $\mathrm{soc}(M)$, and similarly we write $\underline{\dim}(M/\mathrm{rad} M) = \underline{t} = (t_1, t_2, \ldots, t_r)$, where t_i is the multiplicity of S_i in $M/\mathrm{rad}\, M$. Then we define the 'dimension vector' for M to be

$$\underline{\dim}(M) := (\underline{t} \mid \underline{s}).$$

The usual dimension vector would be $\underline{t} + \underline{s}$.

The dimension vectors of the Ω-translates of M are usually completely determined in terms of the matrix E.

Lemma 2.2. *Let X be the $2r \times 2r$ matrix which in block form is given by*

$$X = \begin{pmatrix} E & -I_r \\ I_r & 0 \end{pmatrix}.$$

Assume that M has no simple or projective summands, and $\Omega(M)$ is not simple. Then

$$\underline{\dim}\Omega(M)^T = X\underline{\dim}(M)^T.$$

Proof. Consider the projective cover of M,

$$0 \to \Omega(M) \to P_M \to M \to 0.$$

Then $P_M \cong \bigoplus_{i=1}^n t_i P_i$ since $P_M/\mathrm{rad}(P_M)$ must be isomorphic to $M/\mathrm{rad}(M)$. Since M has no projective (hence injective) summands, the socle of $\Omega(M)$ is isomorphic to $\mathrm{soc}(P_M)$ that is $\bigoplus t_i S_i$.

Also, since $\Omega(M)$ has no simple or projective summand, we know that $\mathrm{soc}\,\Omega(M) = \mathrm{rad}\,\Omega(M)$.

Factoring out the socle of $\Omega(M)$, we get a short exact sequence

$$0 \to \Omega(M)/\mathrm{soc}\,\Omega(M) \to P_M/\mathrm{soc}(P_M) \to M \to 0.$$

If we restrict this to the radical of $P_M/\mathrm{soc}(P_M)$, then we get a split exact sequence

$$0 \to \Omega(M)/\mathrm{soc}\,\Omega(M) \to \mathrm{rad}(P_M)/\mathrm{soc}(P_M) \to \mathrm{soc}(M) \to 0.$$

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Hence, the dimension vector of $\Omega(M)/\operatorname{soc}\Omega(M)$ is equal to

$$E\underline{t}^T - \underline{s}^T,$$

as required. □

This is still true if $\Omega(M)$ is simple. Since we want to iterate the calculation, we exclude this.

2.5 If none of the modules $\Omega^m(M)$ for $m = 1, 2, \ldots, k+1$ is simple, it follows that the dimension vector of $\Omega^k(M)$ is equal to $X^k \underline{\dim}(M)^T$. The matrix X^k is of the form

$$X^k = \begin{pmatrix} f_k(E) & -f_{k-1}(E) \\ f_{k-1}(E) & -f_{k-2}(E) \end{pmatrix}.$$

Here $f_k(x)$ is the kth Chebyshev polynomial, given by

$$f_0(x) = 1, \quad f_1(x) = x, \quad f_k(x) = xf_{k-1}(x) - f_{k-2}(x) \quad (k \geq 2).$$

The polynomial $f_k(x)$ is the characteristic polynomial of the $k \times k$ incidence matrix of the Dynkin diagram of type A_k, that is, it has entries $a_{i,i\pm 1} = 1$ and $a_{ij} = 0$ otherwise.

Also, $f_k(x) = U_k(x/2)$, where $U_k(x)$ is a version of a Chebyshev polynomial of the second kind. These polynomials are studied extensively in numerical mathematics; see for example [14].

2.6 We recall that if S is a simple module, then, for any $k \geq 1$ and for any module M,

$$\operatorname{Ext}^k(M, S) = \operatorname{Hom}(\Omega^k(M), S).$$

We give the argument. Take the exact sequence

$$0 \to \Omega^k(M) \to P_{k-1} \to \Omega^{k-1}(M) \to 0$$

and apply $\operatorname{Hom}(-, S)$. If $\pi : P_{k-1} \to S$ is any homomorphism, then clearly this restricts to the zero map $\Omega^k(M) \to S$. Hence, the inclusion map from $\operatorname{Hom}(\Omega^{k-1}(M), S)$ to $\operatorname{Hom}(P_{k-1}, S)$ is an isomorphism. Therefore, $\operatorname{Hom}(\Omega^k(M), S) \cong \operatorname{Ext}^1(\Omega^{k-1}(M), S)$, which is isomorphic to $\operatorname{Ext}^k(M, S)$.

We claim that $d_{ij} = d_{ji}$, which shows that the matrix E is symmetric: we may assume that $i \neq j$. Then, since P_j is the projective cover of S_j,

$$d_{ij} = \dim \operatorname{Hom}(P_j, P_i).$$

But P_i is also the injective hull of S_i, so the dimension is also equal to the number of times S_i occurs in P_j, which is equal to d_{ji}.

3. The main result

Assume that A is weakly symmetric and indecomposable with $J^3 = 0 \neq J^2$ and let λ be the largest eigenvalue of the matrix E. In this section we will show that if A has an ext-finite nonprojective module, then $\lambda = 2$. This is Proposition 3.6, and it proves the main part of Theorem 1.1.

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If there is an ext-finite nonprojective module, then we can take such a module M which is indecomposable. We will analyse the dimension vectors of the modules $\Omega^k(M)$ for large k.

We may assume that $\Omega^k(M)$ is not simple for $k \geq 0$: namely at most finitely many of the $\Omega^k(M)$ can be simple, since otherwise it would follow that M is periodic, but then M would not be ext-finite. So, there is some m such that for $k \geq m$ none of the modules $\Omega^k(M)$ is simple. We replace M by $\Omega^m(M)$. With M, also $\Omega^m(M)$ is ext-finite and not projective; recall that Ω induces an equivalence of the stable category. The vanishing of extensions implies that dimension vectors satisfy an orthogonality condition, as follows.

Lemma 3.1. *Assume that* $\mathrm{Ext}^k(M, M) = 0$ *for* $k > n$. *Let* $(\underline{t} \mid \underline{s})$ *be the dimension vector of* M *and* $(\underline{t}^{(k+1)} \mid \underline{s}^{(k+1)})$ *be the dimension vector of* $\Omega^{k+1}(M)$. *Then, for all* $k > n$,

$$(\underline{s} \mid -\underline{t}) \cdot (\underline{t}^{(k+1)} \mid \underline{s}^{(k+1)}) = 0.$$

Proof. By the assumption, and by 2.1, $\mathrm{soc}(M) = JM$, so we have a short exact sequence

$$0 \to M_2 = \mathrm{soc}(M) \to M \to M_1 = M/JM \to 0, \qquad (*)$$

where M_1 and M_2 are semisimple. Write $M_1 = \bigoplus_i t_i S_i$ and $M_2 = \bigoplus_i s_i S_i$. Then $\dim(M) = (\underline{t} \mid \underline{s})$.

We apply the functor $\mathrm{Hom}(M, -)$ to $(*)$, which gives the long exact sequence of homology. Part of this is

$$\cdots \to \mathrm{Ext}^k(M, M) \to \mathrm{Ext}^k(M, M_1) \to \mathrm{Ext}^{k+1}(M, M_2) \to \mathrm{Ext}^{k+1}(M, M) \to .$$

Consider $\mathrm{Ext}^k(M, M_1)$; this is isomorphic to $\bigoplus_i t_i \mathrm{Ext}^k(M, S_i)$, and $\mathrm{Ext}^k(M, S_i)$ is isomorphic to $\mathrm{Hom}(\Omega^k(M), S_i)$ (see Section 2.6). This has dimension

$$\sum_i t_i t_i^{(k)}.$$

Similarly, $\mathrm{Ext}^{k+1}(M, M_2)$ has dimension

$$\sum_i s_i t_i^{(k+1)}.$$

By exactness, we get for $k > n$ that $\mathrm{Ext}^k(M, M_1) \cong \mathrm{Ext}^{k+1}(M, M_2)$. Equating dimensions,

$$\sum_i t_i t_i^{(k)} = \sum_i s_i t_i^{(k+1)}.$$

By 2.3, we know that $\underline{t}^{(k)} = \underline{s}^{(k+1)}$. Using this, and rewriting the last identity, we get the claim. □

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We analyse $(\underline{s} \mid -\underline{t}) \cdot (\underline{t}^{(k+1)} \mid \underline{s}^{(k+1)})$, which is equal to

$$(\underline{s} \mid -\underline{t})X^{k+1}(\underline{t} \mid \underline{s})^T \tag{1k}$$

for $k > n$. We substitute X^{k+1} and expand; then (1k) becomes

$$\underline{s}f_{k+1}(E)\underline{t}^T - \underline{t}f_k(E)\underline{t}^T - \underline{s}f_k(E)\underline{s}^T + \underline{t}f_{k-1}(E)\underline{s}^T. \tag{2k}$$

The matrix $f_{k-1}(E)$ is symmetric, so we can interchange $\underline{t}$ and $\underline{s}$ in the last term. Then, using the recurrence relation for the Chebyshev polynomials,

$$f_{k+1}(x) = xf_k(x) - f_{k-1}(x),$$

the expression (2k) becomes

$$\underline{s}Ef_k(E)\underline{t}^T - \underline{t}f_k(E)\underline{t}^T - \underline{s}f_k(E)\underline{s}^T. \tag{3k}$$

Since E is real symmetric, there is an orthogonal matrix R such that $R^TER = D$, a diagonal matrix. We substitute $E = RDR^T$, and we set $\underline{\alpha} := \underline{s}R$ and $\underline{\beta} := \underline{t}R$. With this, noting also that $Rf_k(E)R^T = f_k(RER^T)$, expression (3k) becomes

$$\underline{\alpha}Df_k(D)\underline{\beta}^T - \underline{\beta}f_k(D)\underline{\beta}^T - \underline{\alpha}f_k(D)\underline{\alpha}^T. \tag{4k}$$

The matrices involved are diagonal; let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of D. Then (4k) is equal to

$$\sum_{i=1}^{r}(\alpha_i\beta_i\lambda_i - \beta_i^2 - \alpha_i^2)f_k(\lambda_i).$$

If we denote the distinct eigenvalues of D by $\mu_1, \ldots, \mu_m$, then this becomes

$$\sum_{j=1}^{m}\Big(\sum_{\lambda_i=\mu_j}\alpha_i\beta_i\lambda_i - \beta_i^2 - \alpha_i^2\Big)f_k(\mu_j). \tag{5k}$$

Then Lemma 3.1 shows that (5k) is zero for all $k > n$. The coefficients $c_j := (\sum_{\lambda_i=\mu_j}\alpha_i\beta_i\lambda_i - \beta_i^2 - \alpha_i^2)$ do not depend on k. We take any m of these equations for $k > n$ and write them in matrix form. That is, consider a matrix

$$C := \begin{pmatrix} f_N(\mu_1) & f_N(\mu_2) & \cdots & f_N(\mu_m) \\ f_{N+i_1}(\mu_1) & f_{N+i_1}(\mu_2) & \cdots & f_{N+i_1}(\mu_m) \\ \cdots & & & \\ f_{N+i_{m-1}}(\mu_1) & f_{N+i_{m-1}}(\mu_2) & \cdots & f_{N+i_{m-1}}(\mu_m) \end{pmatrix}$$

with $N > n$ and $0 < i_1 < i_2 < \cdots < i_{m-1}$. Then, for any such C,

$$C\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = 0.$$

Since N can be arbitrarily large, one may expect that for some choice of parameters, the matrix C is nonsingular, and hence show that the c_i must be zero.

Lemma 3.2. *There are $N > n$ and $0 < i_1 < i_2 < \cdots < i_m$ such that C is nonsingular.*

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Proof. We use induction on m. Assume first that $m = 1$. We have $f_0(\mu_1) = 1 \neq 0$. Whenever $f_{u-1}(\mu_1) \neq 0$ and $f_u(\mu_1) = 0$, then $f_{u+1}(\mu_1) = -f_{u-1}(\mu_1) \neq 0$. So, at worst, every second one of the values can be zero.

Now we fix some $N > n$ such that $f_N(\mu_1) \neq 0$. Consider the matrix with rows

$$R_{N+j} := (f_{N+j}(\mu_1), f_{N+j}(\mu_2), \ldots, f_{N+j}(\mu_m))$$

for $j = 0, 1, 2, \ldots, k$ and k large, $k > m + 2$. We replace R_{N+k} by $R_{N+k} + R_{N+k-2} - \mu_1 R_{N+k-1}$ and obtain as the new last row

$$[0, (\mu_2 - \mu_1) f_{N+k-1}(\mu_2), \ldots, (\mu_m - \mu_1) f_{N+k-1}(\mu_m)].$$

Similarly we replace R_{N+k-1} and so on. This process ends when row R_{N+2} has become

$$[0, (\mu_2 - \mu_1) f_{N+1}(\mu_2), \ldots, (\mu_m - \mu_1) f_{N+1}(\mu_m)].$$

By construction, $f_N(\mu_1) \neq 0$, and we take the row of $f_N(\mu_i)$ as the first row of our required submatrix.

We apply the inductive hypothesis to the matrix with rows consisting of $R_{N+2}, \ldots, R_{N+k}$ omitting the first column. Note that from each column we can take a nonzero scalar factor $\mu_i - \mu_1$. The remaining matrix has the same shape again with $m - 1$ columns. So, by the inductive hypothesis, it has $m - 1$ rows which form a nonsingular submatrix. □

Example 1. The roots of $f_r(x)$ are precisely the eigenvalues of the $r \times r$ matrix E with $e_{i,i\pm1} = 1$ and $e_{ij} = 0$ otherwise (see Section 2.5). By the Cayley–Hamilton theorem, we know that $f_r(E) = 0$. In [11], it is proved that the sequence of matrices $(f_m(E))$ is periodic. In fact, one can see from the proof there that there are r successive rows which are linearly independent, but there are rows of zeros.

For example, if $r = 2$, then the eigenvalues are ± 1 and the rows are

$$\begin{matrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \\ \cdots & \end{matrix}$$

Corollary 3.3. *If (1k) is zero for all $k > n$, then, for all j with $1 \leq j \leq m$,*

$$\sum_{\lambda_i = \mu_j} (\alpha_i \beta_i \lambda_i - \beta_i^2 - \alpha_i^2) = 0.$$

This follows from the previous lemma.

Let $\lambda_1 = \lambda$, the largest eigenvalue of E. We assume that A is indecomposable and therefore E is an irreducible matrix. Therefore, λ has multiplicity one as an eigenvalue

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of E, and there is a real eigenvector $\underline{v}$ with $v_i > 0$ for all i. We may take it as a unit vector, and then $\underline{v}^T$ is the first column of R, where $R^T ER = D$. Recall that $\underline{\alpha} = \underline{s}R$ and $\underline{\beta} = \underline{t}R$. These have first components

$$\alpha_1 = \sum_i s_i v_i, \quad \beta_1 = \sum_i t_i v_i.$$

Since $\underline{s}$ and $\underline{t}$ are nonzero in $\mathbb{Z}^r_{\geq 0}$, it follows that α_1 and β_1 are positive. Because λ has multiplicity one, the sum in Corollary 3.4 for λ has only one term, and we deduce the following result.

Corollary 3.4. *The numbers α_1/β_1 and β_1/α_1 are roots of the equation*

$$X^2 - \lambda X + 1 = 0.$$

The aim is to show that $\alpha_1 = \beta_1$; this needs some more information. We may do the same calculation with $\Omega^m(M)$ instead of M for any $m \geq 0$; denote the corresponding numbers by $\beta_1^{(m)}$ and $\alpha_1^{(m)}$. For any such m, the two quotients must therefore be roots of the above quadratic equation, that is,

$$\frac{\beta_1^{(m)}}{\alpha_1^{(m)}} + \frac{\alpha_1^{(m)}}{\beta_1^{(m)}} = \lambda. \tag{$**$}$$

We can say more.

Lemma 3.5. *We have $\beta_1^{(m+1)} = \lambda\beta_1^{(m)} - \alpha_1^{(m)}$.*

Proof. To prove this, it suffices to take $m = 0$. We have $\underline{t}^{(1)} = E\underline{t}^T - \underline{s}$ and therefore

$$\underline{t}_i^{(1)} = (E\underline{t}^T)_i - s_i.$$

Now $(E\underline{t}^T)_i = \sum_{k=1}^r e_{ik}t_k = \sum_{k=1}^r e_{ki}t_k$ (recall that E is symmetric). Then

$$\beta_1^{(1)} + \alpha_1^{(0)} = \sum_{i=1}^r (E\underline{t}^T)_i v_i.$$

We substitute and change the order of summation and get that this is equal to

$$\sum_{k=1}^r \Big(\sum_{i=1}^r e_{ki}v_i\Big) t_k.$$

The coefficient of t_k is the kth entry of $E\underline{v}^T = \lambda\underline{v}$, which is λv_k. So,

$$\beta_1^{(1)} + \alpha_1^{(0)} = \lambda \sum_k v_k t_k = \lambda\beta_1^{(0)},$$

as stated. □

Proposition 3.6. *If M is ext-finite, then $\alpha_1 = \beta_1$. In particular, $\lambda = 2$.*

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PROOF. (1) First we claim that $\alpha_1^{(m)}/\beta_1^{(m)} = \alpha_1^{(m+1)}/\beta_1^{(m+1)}$.

By Lemma 3.5, and since $\alpha_1^{(m+1)} = \beta_1^{(m)}$ (recall that $\underline{s}^{(m+1)} = \underline{t}^{(m)}$),

$$\frac{\beta_1^{(m+1)}}{\alpha_1^{(m+1)}} = \lambda - \frac{\alpha_1^{(m)}}{\beta_1^{(m)}}.$$

Using also (∗∗), we deduce that

$$\frac{\beta_1^{(m+1)}}{\alpha_1^{(m+1)}} + \frac{\alpha_1^{(m+1)}}{\beta_1^{(m+1)}} = \lambda = \lambda + \frac{\alpha_1^{(m+1)}}{\beta_1^{(m+1)}} - \frac{\alpha_1^{(m)}}{\beta_1^{(m)}}$$

and hence the claim follows.

The set of positive numbers $\{\alpha_1^{(m)}, m \geq 0\}$ is bounded below, and it is a discrete subset of $\mathbb{R}$; therefore, it has a minimum. That is, we may choose M in its Ω-orbit so that the number $\alpha_1^{(1)} \leq \alpha_1^{(m)}$ for all $m \geq 0$.

Then $\beta_1^{(1)} = \alpha_1^{(2)} \geq \alpha_1^{(1)} = \beta_1^{(0)}$ and $\alpha_1^{(1)} \leq \alpha_1^{(0)}$. It follows that

$$\frac{\alpha_1^{(0)}}{\beta_1^{(0)}} = \frac{\alpha_1^{(0)}}{\alpha_1^{(1)}} \geq 1,$$
$$\frac{\alpha_1^{(1)}}{\beta_1^{(1)}} = \frac{\alpha_1^{(1)}}{\alpha_1^{(2)}} \leq 1$$

and hence the fractions must be equal to 1.

So, the quadratic equation of Corollary 3.4 has one root equal to 1. The product of the roots is 1, so both roots are equal to 1 and then $\lambda = 2$. □

We have proved that for $\lambda \neq 2$, the algebra has no ext-finite modules.

REMARK. Assume that $\lambda = 2$. For the algebras without (Fg) (which are of type $\tilde{A}$ or local), the vector $\underline{v}$ is a multiple of $(1, 1, \ldots, 1)$ and, if $\alpha_1^{(m)} = \beta_1^{(m)}$ for all m, then the socle and the top of any Ω-translate of M have the same dimension. So, M has even dimension, and it follows that M cannot be an Ω-translate of a simple module. Namely, the Ω-translates of simple modules have odd dimensions for these algebras.

4. Algebras where $\lambda = 2$

Assume that A is an algebra as in Theorem 1.1, such that the largest eigenvalue λ of E is equal to 2. We will now show that if A does not satisfy (Fg), then A has ext-finite nonprojective modules. This will prove the other direction of Theorem 1.1.

By [12], when $\lambda = 2$ and condition (Fg) fails, the algebra is Morita equivalent to either the q-exterior algebra or to an algebra of type $\tilde{A}$, which we call a double Nakayama algebra. In both cases, there is a deformation parameter which is not a root of unity (and nonzero). In both cases we will construct explicitly ext-finite nonprojective modules.

4.1. The q-exterior algebra. Let $\Lambda = \Lambda(q) = K\langle x, y\rangle/(x^2, y^2, xy + qyx)$ and $0 \neq q \in K$. We assume that q is not a root of unity. It was discovered by R. Schulz, some years ago, that this algebra has ext-finite nonprojective modules; see [15, Section 4]. The modules below are essentially the same as studied in [15], but for completeness we give the details.

For $0 \neq \lambda \in K$, we define a Λ-module $M = C(\lambda)$ as follows. It is two dimensional and x, y act by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}.$$

This module is clearly indecomposable and not projective, and it is easy to check that $C(\lambda) \cong C(\mu)$ only if $\lambda = \mu$. We construct $C(\lambda)$ as the submodule of Λ generated by $\zeta = -\lambda q x + y \in \Lambda$, and take a basis $\zeta, x\zeta$.

Lemma 4.1. *We have* $\Omega^m(C(\lambda)) \cong C(q^{-m}\lambda)$ *for* $m \geq 0$.

Proof. We find that $\Omega(M) = \{z \in \Lambda : z\zeta = 0\} = \Lambda\zeta_1$, where $\zeta_1 = y - \lambda x$; and then $y\zeta_1 = \lambda q^{-1} x\zeta_1$. That is, $\Omega(M) \cong C(\lambda q^{-1})$, and the statement follows. □

For convenience we give a proof showing that the module $C(\lambda)$ is ext-finite.

Lemma 4.2. *If* $\mu \in K$ *and* $\mu \neq \lambda$ *or* $q\lambda$, *then* $\mathrm{Ext}^1((C(\mu), C(\lambda)) = 0$.

Proof. A projective cover of $C(\mu)$ is of the form

$$0 \to C(\mu q^{-1}) \to \Lambda \to C(\mu) \to 0.$$

Applying $\mathrm{Hom}(-, C(\lambda))$ gives a four-term exact sequence

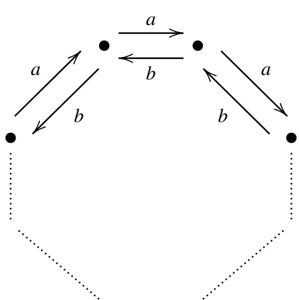
$$0 \to \mathrm{Hom}(C(\mu), C(\lambda)) \to \mathrm{Hom}(\Lambda, C(\lambda)) \to \mathrm{Hom}(C(\mu q^{-1}), C(\lambda)) \\ \to \mathrm{Ext}^1(C(\mu), C(\lambda)) \to 0.$$

With the assumptions, the first and third terms are one dimensional. Also, $\mathrm{Hom}(\Lambda, C(\lambda)$ is two dimensional and hence the ext-space is zero. □

Corollary 4.3. *Let* $M = C(\lambda)$; *then* $\mathrm{Ext}^k(M, M) = 0$ *for* $k \geq 2$. *Hence, M is ext-finite and not projective.*

Proof. We have that $\mathrm{Ext}^k(M, M) \cong \mathrm{Ext}^1(\Omega^{k-1}(M), M) = \mathrm{Ext}^1(C(q^{-k+1}\lambda), C(\lambda)) = 0$. □

4.2. Double Nakayama algebras. We consider algebras of the form $A = A(t) = KQ/I$, where KQ is the path algebra of a quiver of the form



We label the vertices by $\mathbb{Z}_r$, and the arrows are $a_i : i \mapsto i+1$ and $b_i : i+1 \mapsto i$. The ideal I is generated by $a_{i+1}a_i, b_ib_{i+1}$ and

$$b_ia_i + a_{i-1}b_{i-1} \quad (i \neq 0), \qquad b_0a_0 + ta_{r-1}b_{r-1},$$

where $0 \neq t \in K$. We call this algebra, and any Morita equivalent version, a double Nakayama algebra. We want to show that if t is not a root of unity, then A has nonprojective ext-finite modules.

The idea is to show that A has a suitably embedded subalgebra isomorphic to a quantum exterior algebra, and then show that the ext-finite modules for this subalgebra, constructed before, induce to ext-finite A-modules.

Note that for an arrow $a_i : i \to i+1$ we have in the algebra that $a_i = e_{i+1}a_ie_i$, where e_i is the idempotent corresponding to vertex i.

Lemma 4.4. *The algebra A has a subalgebra Λ isomorphic to $\Lambda(q)$, where $q^r - t^{-1} = 0$, and A is projective as a left and right Λ-module.*

Proof. Let $x := q^ra_0 + q^{r-1}a_1 + q^{r-2}a_2 + \cdots + qa_{r-1}$ and $y := b_0 + b_1 + b_2 + \cdots + b_{r-1}$. One checks that $xy + qyx = 0$ but $xy \neq 0$; and clearly $x^2 = 0$ and $y^2 = 0$. Take Λ to be the subalgebra with generators x, y.

Consider A as a left Λ-module; one checks that $A = \bigoplus_{i=0}^{r-1} \Lambda e_i$ and that $A = \bigoplus_{i=0}^{r-1} e_i\Lambda$. □

Remark. (1) By the previous lemma, the functor $A \otimes_\Lambda (-)$ is exact and takes projective modules to projective modules. In the following we write $A \otimes (-)$ for $A \otimes_\Lambda (-)$. Also, for any Λ-module N, the module $A \otimes N$ has dimension $r \cdot \dim N$.

(2) We have $xe_i = qa_i = e_{i+1}x$ and $ye_i = b_{i-1} = e_{i-1}y$. Hence,

$$e_i(yx) = (yx)e_i = qb_ia_i.$$

(3) If the Λ-module N has no nonzero projective summands, then $A \otimes N$ has no nonzero projective summands: more generally, a module of a selfinjective algebra has no nonzero projective summands if and only if it is annihilated by the socle of the algebra.

Here, the socle of A is spanned by the elements b_ia_i and, for $w \in N$,

$$q(b_ia_i) \otimes w = e_i(yx) \otimes w = e_i \otimes yxw = 0$$

since yx is in the socle of Λ.

Now let $M = C(\lambda)$, the Λ-module as in Subsection 4.1.

(4) By (1) and (3),

$$\Omega(A \otimes M) \cong A \otimes \Omega(M) = A \otimes C(q^{-1}\lambda).$$

Lemma 4.5. *If $r > 0$, then the space* $\mathrm{Hom}_A(A \otimes C(q^{-r}\lambda), A \otimes M)$ *has dimension r.*

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PROOF. By adjointness,

$$\mathrm{Hom}_A(A \otimes C(q^{-r}\lambda), A \otimes M) \cong \mathrm{Hom}_\Lambda(C(q^{-r}\lambda), A \otimes M), \qquad (*)$$

where $A \otimes M$ is restricted to Λ. We work with the Λ-homomorphisms. One checks that the Λ-socle of $A \otimes M$ is equal to $A \otimes \operatorname{soc} M = \operatorname{rad}_\Lambda(A \otimes M)$ and hence this has dimension r.

The space ($*$) contains all maps with image in the Λ-socle of $A \otimes M$ and this has dimension r. So, we must show that for $r \neq 0$ there are no other homomorphisms, that is, we have no monomorphism from $C(q^{-r}\lambda)$ to $A \otimes M$ for $r \neq 0$.

Assume that there is a monomorphism; then its image is a cyclic Λ-submodule of $A \otimes M$ of dimension two. So, let ξ be a generator for a cyclic two-dimensional submodule of $A \otimes M$. We may assume that ξ is of the form

$$\xi = \sum_{i \in \mathbb{Z}_r} c_i(e_i \otimes \zeta)$$

(if $w \in \operatorname{soc}(A)$, then $w \otimes \xi = 0$. Furthermore, if $w \in \operatorname{rad}(A)$ and $w \otimes \xi$ is in the socle of $A \otimes M$, then it may be omitted from a cyclic generator).

We require that $x\xi$ and $y\xi$ are linearly dependent. By the identities in Remark 4.5,

$$x\xi = \sum_{j \in \mathbb{Z}_r} c_{j-1}(e_j \otimes x\zeta), \quad y\xi = \sum_{j \in \mathbb{Z}_r} c_{j+1}(e_j \otimes y\zeta) = \sum_{j \in \mathbb{Z}_r} \lambda c_{j+1}(e_j \otimes x\zeta).$$

Assume that $y\xi = \mu x\xi$ for some scalar $\mu \neq 0$. The set $\{e_j \otimes x\zeta : j \in \mathbb{Z}_r\}$ is linearly independent, so we must have

$$c_{j-1}\mu = \lambda c_{j-1} \quad (j \in \mathbb{Z}_r).$$

So, we get if r is even, $c_j = (\mu^{-1}\lambda)^{r/2} c_j$ for all j and, if r is odd, $c_j = (\mu^{-1}\lambda)^{r-1} c_j$ for all j.

Hence, if there is such element ξ, then $\mu = \lambda \cdot \omega$ for some root of unity ω.

If $\mu = q^{-r}\lambda$, then $\mu = \lambda \cdot \omega$ only if $r = 0$, and our claim is proved. □

PROPOSITION 4.6. *Let $M = C(\lambda)$. Then $A \otimes M$ is ext-finite and not projective.*

PROOF. We have by the remark that $A \otimes M$ is not projective, and that $\Omega^r(A \otimes M) \cong A \otimes C(q^{-r}\lambda)$. Take the exact sequence

$$0 \to A \otimes C(q^{-r-1}\lambda) \to A \otimes \Lambda \to A \otimes C(q^{-r}\lambda) \to 0$$

and apply the functor $\mathrm{Hom}_A(-, A \otimes M)$. Then, by using adjointness, we get the four-term exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Lambda(C(q^{-r}\lambda), A \otimes M) \to \mathrm{Hom}_\Lambda(\Lambda, A \otimes M) \to \mathrm{Hom}_\Lambda(C(q^{-r-1}\lambda), A \otimes M) \\ &\to \mathrm{Ext}^1_\Lambda(C(q^{-r}), A \otimes M) \cong \mathrm{Ext}^r_A(\Omega^r(A \otimes M), A \otimes M) \to 0. \end{aligned}$$

If $r > 0$, then the first and third terms of the sequence have dimension r. Also, the second term has dimension $2r$ and hence the fourth term is zero. Hence, for all $r > 0$, we have $\mathrm{Ext}^1_A(\Omega^r(A \otimes M), A \otimes M) = 0$. This is isomorphic to $\mathrm{Ext}^{r+1}_A(A \otimes M, A \otimes M)$ and hence $A \otimes M$ is ext-finite. □

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