On Appell's Function $P(\theta, \phi)$

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§ 1. Appell's functions, $P(\theta, \phi)$, $Q(\theta, \phi)$ and $R(\theta, \phi)$ are defined by the expansion[1]

$$e^{j\theta + j^2\phi} = P(\theta, \phi) + jQ(\theta, \phi) + j^2 R(\theta, \phi)$$

where $j^3 = 1$, affording, both for the third order and the field of two variables, a very direct generalization of the circular functions, as

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

They can be written as follows:

$$P(\theta, \phi) = \tfrac{1}{3}\left(e^{\theta+\phi} + e^{j\theta+j^2\phi} + e^{j^2\theta+j\phi}\right),$$
$$Q(\theta, \phi) = \tfrac{1}{3}\left(e^{\theta+\phi} + j^2 e^{j\theta+j^2\phi} + j e^{j^2\theta+j\phi}\right),$$
$$R(\theta, \phi) = \tfrac{1}{3}\left(e^{\theta+\phi} + j e^{j\theta+j^2\phi} + j^2 e^{j^2\theta+j\phi}\right),$$

and they satisfy the fundamental relation

$$P^3 + Q^3 + R^3 - 3PQR = 1.$$

I showed recently that they are of great help to solve numerous problems connected with the equation

$$\Delta_3 v = \frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 v}{\partial z^3} - 3\frac{\partial^3 v}{\partial x\,\partial y\,\partial z} = 0$$

and allied equations.[2]

The object of this short note is to state some elementary remarks on the function $P(n\theta, n\phi)$ where n is an integer, and to make more conspicuous the analogy between it and $\cos n\theta$.

§ 2. Let us consider the expression

$$E = \log\left(1 - ae^{\theta+\phi}\right)\left(1 - ae^{j\theta+j^2\phi}\right)\left(1 - ae^{j^2\theta+j\phi}\right),$$

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[1] P. Appell, *C. R.*, **84** (1877), 540.

[2] *Atti della Pont. Accad. delle Scienze*, *Anno* 83 (1929-30), 128. *Cf.* J. Devisme, *C. R.*, **193** (1931), 981.

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where a is an arbitrary constant, and try to expand it in ascending powers of a. We have

$$E = \log(1 - ae^{\theta+\phi}) + \log(1 - ae^{j\theta+j^2\phi}) + \log(1 - ae^{j^2\theta+j\phi})$$

$$= -\sum_n \frac{a^n\, e^{n(\theta+\phi)}}{n} - \sum_n a^n \frac{e^{n(j\theta+j^2\phi)}}{n} - \sum_n a^n \frac{e^{n(j^2\theta+j\phi)}}{n}$$

$$= -\sum_n 3a^n P(n\theta,\, n\phi)/n.$$

Now we can write, as $1 + j + j^2 = 0$,

$$E = \log[1 - ae^{\theta+\phi} - ae^{j\theta+j^2\phi} - ae^{j^2\theta+j\phi} + a^2e^{-j\theta-j^2\phi} + a^2e^{-j^2\theta-j\phi} + a^2e^{-\theta-\phi} - a^3]$$
$$= \log[1 - 3aP(\theta,\, \phi) + 3a^2P(-\theta,\, -\phi) - a^3].$$

So we obtain the function $P(n\theta,\, n\phi)$ through the generating function

$$-\log[1 - 3aP(\theta,\, \phi) + 3a^2P(-\theta,\, -\phi) - a^3],$$

the coefficient of a^n being $\{3P(n\theta,\, n\phi)\}/n$.

The noteworthy analogy with the circular functions arises from the fact that the coefficient of a^n in the expansion of

$$-\log[1 - 2a\cos\theta + a^2]$$

is $(2\cos n\theta)/n$.

§ 3. The expansion just obtained,

$$-\log[1 - 3aP(\theta,\, \phi) + 3a^2P(-\theta,\, -\phi) - a^3] = \sum_n 3a^n \frac{P(n\theta,\, n\phi)}{n},$$

shows that $P(n\theta,\, n\phi)$ can be expressed as a *polynomial* with respect to $P(\theta,\, \phi)$ and $P(-\theta,\, -\phi)$. We observe that

$$P(-\theta,\, -\phi) = P^2(\theta,\, \phi) - Q(\theta,\, \phi)\, R(\theta,\, \phi),$$

so that $P(n\theta,\, n\phi)$ is a polynomial with respect to P and QR. For instance

$$P(2\theta,\, 2\phi) = P^2 + 2QR = 3P^2(\theta,\, \phi) - 2P(-\theta,\, -\phi)$$
$$P(3\theta,\, 3\phi) = 1 + 9PQR = 9P^3(\theta,\, \phi) - 9P(\theta,\, \phi)\, P(-\theta,\, -\phi) + 1.$$

Our expansion leads readily to the following general result:

$$\frac{P(n\theta,\, n\phi)}{n} = \sum_p \sum_q \frac{(-1)^q\, 3^{p+q}}{n+2p+q}\; {}_pC_{(n+2p+q)/3}\; {}_qC_{(n+q-p)/3}\; P^p(\theta,\, \phi)\, P^q(-\theta,\, -\phi)$$

with $p \leqslant n$, $q \leqslant \frac{1}{2}(n-p)$. The symbol ${}_sC_r$ stands for the number of combinations of r objects s at a time. Of course $\frac{1}{3}(n + 2p + q)$ and $\frac{1}{3}(n + q - p)$ must be positive integers.

§4. Similar formulae can, of course, be written for $Q(n\theta, n\phi)$ and $R(n\theta, n\phi)$. We may use the relations

$$Q(-\theta, -\phi) = Q^2 - RP$$
$$R(-\theta, -\phi) = R^2 - PQ.$$

If we take $\phi = 0$, the function P reduces to one of the *sines of the third order*,

$$f_1(\theta) = \tfrac{1}{3}(e^{\theta} + e^{j\theta} + e^{j^2\theta}),$$

and we obtain the expansion

$$-\log[1 - 3a f_1(\theta) + 3a^2 f_1(-\theta) - a^3] = \sum_n 3a^n f_1(n\theta)/n,$$

showing that $f_1(n\theta)$ can be expressed as a polynomial with respect to $f_1(\theta)$ and $f_1(-\theta)$.

We may, perhaps, suggest the following researches:

(*a*) To express $P(n\theta, n\phi)$ as an hypergeometric function of two variables and of the third order (one of the functions introduced by J. Kampé de Fériet).

(*b*) To extend the result to sines of the 4^{th}, etc., order, with one or two variables.

(*c*) To find a generating function for $P(h\theta, k\phi)$.

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