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ON SPECTRAL SYNTHESIS AND ERGODICITY IN SPACES OF VECTOR-VALUED FUNCTIONS(1)

BY
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ABSTRACT. Spectral synthesis in $L_\infty(\mathbb{R}, \mathbb{C}^N)$, $N>1$, is considered. It is is proved that sets of spectral synthesis are necessarily sets of spectral resolution.

These results are applied to investigate ergodic and mixing properties of some positive contractions on $L_1(G, \mathbb{C}^N)$.

1. **Introduction and preliminaries.** Let $L_1(\mathbb{R}, H)$ denote the Banach space of H-valued Lebesgue integrable functions on $\mathbb{R}$, where H is a separable, complex Hilbert space. $L_1(\mathbb{R}, H)$ is a module over $L_1(\mathbb{R})$ with convolution as multiplication.

The characterization of closed submodules of $L_1(\mathbb{R}, H)$ is equivalent, by duality, to the problem of spectral synthesis for the translation-invariant, w^*-closed subspaces of $L_\infty(\mathbb{R}, H)$. The minimal ones are those generated by the vector-valued exponentials $ve^{i\lambda x}$, $v \in H$, $\lambda \in \mathbb{R}$.

For an invariant, w^*-closed subspace W, we define the spectrum of W by

$$S_p^v(W) = \{\lambda \in \hat{\mathbb{R}} : he^{i\lambda x} \in W \text{ for some } h \in H, h \neq 0\}. \tag{1.1}$$

A closed set $B \subset \hat{\mathbb{R}}$ is said to be a set of spectral synthesis for $L_\infty(\mathbb{R}, H)$ if each W whose spectrum is B is spanned by the vector-valued exponentials it contains.

In [1] it was proved that spectral synthesis holds for the space of $\mathbb{C}^2$-valued continuous (not necessarily bounded) functions on $\mathbb{R}$.

However, since the well known result of Malliavin on the failure of spectral synthesis for $L_\infty(\mathbb{R})$, it is obvious that spectral synthesis does not hold for $L_\infty(\mathbb{R}, \mathbb{C}^N)$.

In [7] Malliavin has introduced the notion of a set of spectral resolution as a closed subset of $\hat{\mathbb{R}}$ all of whose closed subsets are sets of spectral synthesis for $L_\infty(\mathbb{R})$.

In Section 2 we prove that sets of spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$, $N>1$, are necessarily sets of spectral resolution which are very thin sets of $\hat{\mathbb{R}}$. However,

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every countable closed subset of $\hat{\mathbb{R}}$ admits spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$. Some of these results are applied in Section 3 to obtain vector-valued results concerning ergodic and mixing properties of matrix-valued measures on locally compact Abelian groups. Thus the vector-valued generalizations of Choquet–Deny Theorem [2] and Foguel's result in [5] are obtained.

For $f \in L_p(\mathbb{R}, \mathbb{C}^N)$ we denote by $(f)_k$ the kth coordinate of f. For $f \in L_\infty(\mathbb{R})$ let $S_p(f)$ be the spectrum of f. Finally, for an almost-periodic function f we denote by $M(f)$ the mean value of f as defined in [6, p. 160].

2. **Spectral sets for** $L_\infty(\mathbb{R}, \mathbb{C}^N)$, $N > 1$. The simplest sets of spectral synthesis for $L_\infty(\mathbb{R})$ remain sets of spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$.

THEOREM 1. *Every countable closed subset of* $\hat{\mathbb{R}}$ *is a set of spectral synthesis for* $L_\infty(\mathbb{R}, \mathbb{C}^N)$.

Proof. Let $f \in L_\infty(\mathbb{R}, \mathbb{C}^N)$ whose spectrum A is a closed countable subset of $\hat{\mathbb{R}}$. Let V_f denote the invariant w^*-closed subspace generated by f and let $\tilde{V}_f$ be the w^*-closed subspace spanned by the vector-valued exponentials contained in V_f.

Suppose that $\phi \in L_1(\mathbb{R}, \mathbb{C}^N)$ annihilates the subspace $\tilde{V}_f$. Let $F = \sum_{i=1}^N (f)_i * (\phi)_i$. To prove the theorem we must show that $F = 0$. Obviously, $S_p(F) \subset A$.

Let $\lambda_0 \in A$ and let $\psi \in L_1(\mathbb{R})$ such that $\hat{\psi}(\lambda_0) \neq 0$ and $\hat{\psi}$ has compact support.

Let

$$\chi = F * \psi = \sum_{i=1}^{N} \psi_i * (\phi)_i$$

where

$$\psi_i = (f)_i * \psi, \qquad i = 1, 2, \ldots, N.$$

It follows that $S_p(\psi_i)$ $(i = 1, 2, \ldots, N)$ and $S_p(\chi)$ are compact and countable implying by [6, p. 168] that ψ_i $(i = 1, 2, \ldots, N)$ and χ are almost periodic functions.

Let $H_T \in L_1(\mathbb{R})$, $T > 0$, be defined by

$$H_T(x) = \begin{cases} \dfrac{1}{2T} e^{i\lambda_0 x} & |x| < T \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f * \psi * H_T \in V_f$ for each $T > 0$.
By [6, p. 161] we have

$$\psi_j * H_T \xrightarrow[T \to \infty]{} M(\psi_j e^{-i\lambda_0 x}) e^{i\lambda_0 x} \qquad (j = 1, 2, \ldots, N)$$

uniformly in $L_\infty(\mathbb{R})$. Hence $\tilde{V}_f$ contains the vector-valued exponential $ve^{i\lambda_0 x}$ where

$$(v)_j = M(\psi_j e^{-i\lambda_0 x}), \qquad j = 1, 2, \ldots, N.$$

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It follows, as ϕ annihilates $\tilde{V}_f$ that

$$\sum_{j=1}^{N} (\hat{\phi})_j(\lambda_0)M(\psi_j e^{-i\lambda_0 x}) = 0. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} M(\chi e^{-i\lambda_0 x}) &= \sum_{j=1}^{N} M\big((\psi_j * (\phi)_j)e^{-i\lambda_0 x}\big) \\ &= \sum_{j=1}^{N} (\hat{\phi})_j(\lambda_0)M(\psi_j e^{-i\lambda_0 x}). \end{aligned}$$

It follows, therefore, by (1) and [6, p. 161] that λ_0 is not a member of the norm spectrum of χ. Hence, the norm spectrum of χ must be empty, implying that $\chi = 0$. Consequently, $\lambda_0 \notin S_p(F)$ for each $\lambda_0 \in A$, implying that $S_p(F) = \emptyset$. Finally, by Wiener's Theorem, it follows that $F = 0$, as required.

REMARK 2. The result in Theorem 1 is to some extent the analogue of the fact that spectral synthesis holds in $\mathbb{C}^2$-valued continuous functions on $\mathbb{R}$ [1]. In the latter case mean-periodic functions play the role of almost-periodic functions.

Sets of spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$, $N > 1$, are necessarily very thin sets of $\hat{\mathbb{R}}$, as described in

THEOREM 3. *A closed subset of $\hat{\mathbb{R}}$ is a set of spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$, $N > 1$, only if it is a set of spectral resolution for $L_\infty(\mathbb{R})$.*

Proof. Assume that a closed set C of $\hat{\mathbb{R}}$ is not a set of spectral resolution. If C fails to be a set of spectral synthesis for $L_\infty(\mathbb{R})$ then, obviously, C is not a set of spectral synthesis for $L_\infty(\mathbb{R}, \mathbb{C}^N)$.

We may assume, therefore, that C contains properly a closed set A which is not of spectral synthesis for $L_\infty(\mathbb{R})$. Let $g, h \in L_\infty(\mathbb{R})$ be such that $S_p(g) = A$ and $S_p(h) = C$.

Let $f \in L_\infty(\mathbb{R}, \mathbb{C}^N)$ where $(f)_1 = h - g$, $(f)_2 = -h$ and $(f)_i = 0$ for $2 < i \le N$. Let V_f denote the invariant w^*-closed subspace of $L_\infty(\mathbb{R}, \mathbb{C}^N)$ generated by f.

By [9] we deduce that $S_p^V(f) = C$. We will characterize the vector-valued exponentials $ae^{i\lambda x}$, $\lambda \in C - A$, contained in V_f. Let $\lambda_0 \in C - A$ and let $\{\phi_n\} \in L_1(\mathbb{R})$ be such that $\{h * \phi_n\}$ converges in w^* to $e^{i\lambda_0 x}$. Let $\psi \in L_1(\mathbb{R})$ where $\psi * g = 0$ and $\hat{\psi}(\lambda_0) \ne 0$. Then

$$f * \phi_n * \psi \xrightarrow[n\to\infty]{w^*} ae^{i\lambda_0 x}$$

where

$$(a)_1 = -(a)_2 = \hat{\psi}(\lambda_0) \quad \text{and} \quad (a)_i = 0 \quad \text{for} \quad 2 < i \le N.$$

Suppose now that $be^{i\lambda_0 x} \in V_f$ for some $b \in \mathbb{C}^N$, $b \ne 0$.

Let $\tilde{\psi} \in L_1(\mathbb{R}, \mathbb{C}^N)$ where $(\tilde{\psi})_1 = (\tilde{\psi})_2 = \psi$ and $(\tilde{\psi})_i = 0$ for $2 < i \leq N$. Obviously, $\tilde{\psi}$ annihilates V_f and in particular is orthogonal to $be^{i\lambda_0 x}$, implying that

$$b_1\hat{\psi}(\lambda_0) + b_2\hat{\psi}(\lambda_0) = 0.$$

Consequently, $b = \lambda a$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Let $\chi \in L_1(\mathbb{R})$ such that $\hat{\chi}$ vanishes on A, $\int \chi(t)g(t)\,dt \neq 0$.

Let $\tilde{\chi} \in L_1(\mathbb{R}, \mathbb{C}^N)$ be defined by $(\tilde{\chi})_1 = (\tilde{\chi})_2 = \chi$ and $(\tilde{\chi})_i = 0$ for $2 < i \leq N$. Then $\tilde{\chi}$ annihilates all the vector-valued exponentials in V_f while $\sum_{i=1}^N ((\chi)_i * (f)_i)(0) = -(\chi * g)(0) \neq 0$, which completes the proof.

Remark 4. For infinite dimensional H we obtain by [9] the following:

Spectral synthesis fails completely for $L_\infty(\mathbb{R}, H)$ where H is infinite dimensional. That is, no subset of $\hat{\mathbb{R}}$ admits spectral synthesis.

Remark 5. Theorem 1 and Theorem 3 may be extended to general LCA groups.

3. **Ergodic and mixing properties of Matrix-valued measures.** Let $P = (\sigma_{i,j})$ be an $N \times N$ matrix whose entries are probability measures on a LCA group G. P defines a positive contraction (denoted again by P) acting on the Banach space $L_1(G, \mathbb{C}^N)$ by

$$(Pf)_k = \frac{1}{N} \sum_{j=1}^{N} \sigma_{k,j} * f_j, \qquad k = 1, 2, \ldots, N, \quad f \in L_1(G, \mathbb{C}^N).$$

Let $L_1^0(G, \mathbb{C}^N)$ denote the closed submodule of all $f \in L_1(G, \mathbb{C}^N)$ with $\hat{f}_k(e) = 0$, $k = 1, 2, \ldots, N$, where e denotes the unit element of $\hat{G}$. Following [8] we say that P is ergodic by convolutions if for all

$$f \in L_1^0(G, \mathbb{C}^N) \qquad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_1 = 0$$

and P is mixing by convolutions if for all $f \in L_1^0(G, C^N)$ $\lim_{n\to\infty} \|P_f^n\|_1 = 0$. {Here $\|f\|_1 = \sum_{k=1}^N \|(f)_k\|_1$, $f \in L_1(G, \mathbb{C}^N)$.} We say that a matrix-valued measure P on G is adapted if the group generated by the support of P is dense in G.

For a measure μ let $\check{\mu}$ be defined by $\check{\mu}(A) = \mu(-A)$ and let $\hat{P}(\chi_0)$ denote the numerical matrix $\{\hat{\sigma}_{i,j}(\chi_0)\}$, $\chi_0 \in \hat{G}$.

The following is a vector-valued generalization of Choquet–Deny Theorem [2]:

Theorem 6. *Let $P = (\sigma_{i,j})$ where σ_{ij}, $1 \leq i, j \leq N$ are probability measures on a LCA group G. Then P is ergodic by convolutions if and only if P is adapted.*

Proof. Let W denote the w^*-closed, translation-invariant subspace of $L_\infty(G, \mathbb{C}^N)$ of functions f satisfying

$$P^* f = f$$

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where
$$P^* = (\sigma^{\vee}_{j,i}).$$

It follows, by [3, 4], that P is ergodic by convolutions, if and only if, W consists of constant functions.

Let $a\chi_0 \in W$ where $a \in \mathbb{C}^N$, $a \neq 0$ and $\chi_0 \in \hat{G}$. Then

$$\hat{P}^*(\chi_0)a = Na$$

implying that $\hat{\sigma}_{i,j}(\chi_0) = 1$ for all $1 \leq i, j \leq N$. Hence χ_0 is the constant 1 on Supp $\sigma_{i,j}$, $1 \leq i, j \leq N$ and on Supp P. If P is adapted then $\chi_0 = e$, implying that $S_p^V(W) = \{e\}$ (See 1.1). It follows, by Theorem 1 and Remark 5, that W consists of constant functions which completes the proof.

For a matrix A let $\rho(A)$ denote the spectral radius of A.

The following is a vector-valued generalization of a result of Foguel [5] concerning the mixing properties of a measure:

THEOREM 7. *Let $P = (\sigma_{i,j})$ where $\sigma_{i,j}$, $1 \leq i, j \leq N$, are probability measures on a LCA group G. Then P is mixing by convolutions if and only if $\rho(\hat{P}(\chi)) < N$ for $\chi \in \hat{G}$, $\chi \neq e$. In particular, if σ_{i_0,j_0} is mixing by convolutions (on $L_1(G)$) for some (i_0, j_0) then P is mixing by convolutions.*

Proof. Let $P^* = (\sigma_{\check{j},i})$ and $\check{P}^* = (\sigma_{j,i})$. Let W be the w^*-closed, translation-invariant subspace of $L_\infty(G, \mathbb{C}^N)$ of functions satisfying

$$\check{P}^* P^* f = f.$$

It follows, by [3, 4], that P is mixing by convolutions, if and only if, W consists of constant functions. Suppose that $\rho(\hat{P}(\chi)) < N$, $\chi \in \hat{G}$, $\chi \neq e$.

Let $a\chi_0 \in W$, $a \in \mathbb{C}^N$, $a \neq 0$ and $\chi_0 \in \hat{G}$. It follows that

$$B_1 B_2 a = N^2 a$$

where

$$B_1 = (\hat{\sigma}_{j,i}(\chi_0)), \qquad B_2 = (\overline{\hat{\sigma}_{j,k}}(\chi_0)).$$

However, we have $\rho(B_i) < N$, $i = 1, 2$, for $\chi \neq e$, implying that $\chi_0 = e$. Hence $S_p^V(W) = \{e\}$ which, by Theorem 1 and Remark 5, implies that W consists of constant functions, as required. If σ_{i_0,j_0} is mixing by convolutions then $|\hat{\sigma}_{i_0,j_0}(\chi)| < 1$, $\chi \in \hat{G}$, $\chi \neq e$, implying that $\rho(\hat{P}(\chi)) < N$ and the result follows.

REMARK 7. Let

$$P = \begin{pmatrix} \delta_1 & \delta_\alpha \\ \delta_0 & \delta_0 \end{pmatrix}$$

where α is irrational and δ_x denotes the Dirac measure concentrated at $x \in \mathbb{R}$. It follows, by Theorem 6, that P is mixing by convolution on $L_1(\mathbb{R}, \mathbb{C}^2)$ although none of its entries is even adapted.

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