Dear Editor,

I should like to draw your attention to the fact that the *Journal of Applied Probability* **18** (1981), 245–252, published the paper 'Weak convergence of the simple birth-and-death process' by C. Ivan. In this, the author considered a particular case of our result:

> Ежов, И. И и Королюк, В. С. Предельные теоремы для одного класса условных марковских процессов. Preprint 78.3 (1978), Inst. Mat., Akad. Nauk Ukrain, SSR, Kiev. (English translation: Ežov, I.I. and Koroljuk, V.S. Limit theorems for a class of conditional Markov processes. In *Selected Translations on Mathematical Statistics and Probability* **15** (1981), 181–211.)

We therefore claim priority for our work.

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Yours sincerely,
V. S. KOROLJUK

Editor's note. We have not been able to inform Dr Ivan of the above prior to publication because we do not know his present address. If he reads Professor Koroljuk's letter, we appeal to him to respond to it.

Dear Editor,

The main purpose of this letter is to point out some serious differences of opinion about the paper by Choo and Conolly [2], in which they analysed a queueing system where arrivals are Poisson and service times are general independent. If the arriving customer finds the server idle, he immediately enters service; if, on the other hand, he finds the server busy, he leaves immediately and tries his luck again after an exponential amount of time. This system has been considered in detail also by Aleksandrov [1] and Falin [4].

In the case of exponential service times, Choo and Conolly analysed the system time W of a customer. The system time is measured from the instant a customer enters the system until the instant he completes service. In this paragraph we recapitulate Choo and Conolly's main argument. Let the service

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times and retrial times have means $1/\mu$ and $1/\nu$, respectively, let λ be the arrival rate and define $\rho = \lambda/\mu$ and $c = \lambda/\nu$. Let S and T be exponential with parameters λ and ν, respectively. It is known that the server is idle with probability $1-\rho$. Then

$$W = \begin{cases} S & \text{with probability } 1-\rho \\ T + W_1 & \text{with probability } \rho. \end{cases} \tag{1}$$

Next

$$W_1 = \begin{cases} S & \text{with probability } 1-P_B \\ T + W_1 & \text{with probability } P_B, \end{cases} \tag{2}$$

where P_B is the probability that a customer reapplying for service finds the server busy. (The meaning of W_1 has to be inferred from Equation (2).) Choo and Conolly give the following formula for P_B:

$$P_B = (\rho + \nu/\mu)/(1 + \nu/\mu). \tag{3}$$

Using (1)–(3), they derive an expression for the Laplace transform (LT) of W and W_1. They also mention that in the case $c = 0$ the LT of W reduces to that of the waiting time in a simple $M/M/1$ queue.

The above analysis does not seem right for the following reasons. Equation (1) stands to reason. However, the use of Equation (2) is wrong, because W_1 on the right-hand side of (2) is not independent of W_1 on the left-hand side of (2). (They do have identical marginal distributions.) This is why it gives rise to incompatible results. The case $c = \infty$ is *not* the simple $M/M/1$ system. It is the $M/M/1$ system with random service. It is well known (see Cooper [3]) that the *distribution* of waiting time in a random service system is *not* the same as in an FCFS system. (Only the expected values of the waiting times are equal.)

In the rest of this letter we present the correct analysis of the waiting time (excluding service), which will now be denoted by W. Let Q_1 be the number of customers in service ($Q_1 = 0, 1$) and Q_2 be the number of retrial customers ($Q_2 = 0, 1, 2, \cdots$) in steady state. It is known that (see Equation (2.3a) in [2])

$$q_j = P\{Q_1 = 1;\, Q_2 = j\} = \binom{j+c}{j}(1-\rho)^{c+1}\rho^{j+1} \qquad (j \geqq 0)$$

where

$$\binom{j+c}{j} = (c+1)(c+2)\cdots(c+j)/j!$$

Let W_j be the waiting time (excluding service) of a customer who finds $Q_1 = 1$ and $Q_2 = j$ when he tries (or re-tries) for service ($j \geqq 0$). Define

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$$w_j(s) = E[e^{-sW_j}]$$
$$w(s) = E[e^{-sW}; W > 0].$$

Obviously,

(4) $$w(s) = \sum_{j=0}^{\infty} q_j w_j(s).$$

A first step analysis yields

(5)
$$\begin{aligned} w_j(s) = {} & \mu\nu/(\lambda + \mu + s)(\lambda + (j+1)\nu + s) \\ & + (\lambda/(\lambda + \mu + s))w_{j+1}(s) \\ & + (\mu\lambda/(\lambda + \mu + s)(\lambda + (j+1)\nu + s))w_j(s) \\ & + (j\mu\nu/(\lambda + \mu + s)(\lambda + (j+1)\nu + s))w_{j-1}(s) \end{aligned}$$

for $j \geqq 0$. ($w_{-1}(s)$ can be arbitrarily defined to be 0.)

It does not seem possible to obtain closed-form expressions for $w_j(s)$ and $w(s)$. However, it is possible to represent them as convergent infinite series (whenever $\rho < 1$) of recursively defined terms as follows.

Define

$$D_{-2} = 0; \qquad D_{-1} = 1; \qquad t_{-1} = 0; \qquad T_0 = q_0;$$
$$\begin{aligned} D_j &= (1 - \lambda\mu/(\lambda + \mu + s)(\lambda + (j+1)\nu + s))D_{j-1} \\ &\quad - (j\lambda\mu\nu/(\lambda + \mu + s)^2(\lambda + (j+1)\nu + s))D_{j-2}; && (j \geqq 0) \\ t_j &= (\mu\nu/(\lambda + \mu + s)(\lambda + (j+1)\nu + s))D_{j-1} \\ &\quad + (j\mu\nu/(\lambda + \mu + s)(\lambda + (j+1)\nu + s))t_{j-1}; && (j \geqq 0) \\ T_j &= (\lambda/(\lambda + \mu + s))T_{j-1} + q_j D_{j-1} && (j \geqq 1). \end{aligned}$$

With the above definitions we can write

(6) $$w_k(s) = \sum_{j=k}^{\infty} (\lambda/(\lambda + \mu + s))^{j-k}(t_j/D_{j-1}D_j)$$

(7) $$w(s) = \sum_{k=0}^{\infty} T_j t_j / D_{j-1} D_j.$$

Equations (6) and (7) behave extremely well numerically. The rate of convergence decreases as ρ increases to 1. Choo and Conolly's results imply that

(8) $$w(s) = \rho(1 - P_B)\nu/(s + \nu(1 - P_B)).$$

Figure 1 shows the graphs obtained by using Equations (7) and (8) for the case $\rho = 0.4$, $c = 1$. As can be seen the two graphs match only at $s = 0$.

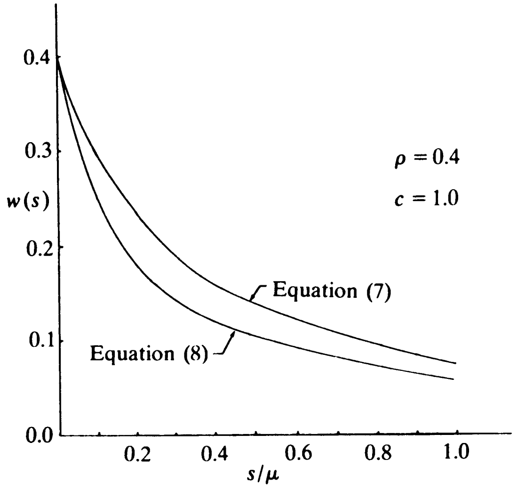

Figure 1

As a special case ($c = 0$) Equation (7) gives us a computationally efficient method for computing the LT of the waiting time in an $M/M/1$ queue with random service.

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Chapel Hill

Yours truly,
VIDYADHAR G. KULKARNI

References

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Dear Editor,

Falin (private communication) and Kulkarni [2] have independently pointed to an error in the waiting-time analysis given in [1]. They explain the common observation that the chance that an 'orbiting' customer, i.e., one who is making retrials for service, finds the service empty, depends on the number of customers in orbit when the retrial is made. This parallels the waiting time under random service in conventional queueing systems with a single stream of applications.

A corrected version of the analysis, and an interesting procedure for computing the distribution, is given in [2]. Falin has indicated that he too was working

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