FIXED POINT THEOREMS FOR GENERALIZED NONEXPANSIVE MAPPINGS

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(Received 12 July 1972, revised 25 August 1972)

Communicated by E. Strzelecki

1. Introduction

Let S, T be self-mappings on a (non-empty) complete metric space (X, d). Let a_i, $i = 1, 2, \cdots, 5$, be non-negative real numbers such that $\sum_{i=1}^{5} a_i < 1$ and for any x, y in X,

$$(1) \qquad d(S(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, S(x)) + a_4 d(x, S(x)) + a_5 d(y, T(y)).$$

The Banach contraction mapping theorem says that T has a unique fixed point if $S = T$ and $a_2 = a_3 = a_4 = a_5 = 0$. Kannan [15] proved that T has a unique fixed point if $S = T$ and $a_1 = a_2 = a_3 = 0$. Reich [26] proved that T has a unique fixed point if $S = T$ and $a_2 = a_3 = 0$. Hardy and Rogers [13] proved that T has a unique fixed point if $S = T$. Gupta and Srivastava [27] proved that S, T have a unique common fixed point if $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5$. We proved [30] that S, T have a unique common fixed point if $a_2 = a_3$ and $a_4 = a_5$. When $S = T$, because of the symmetry in x, y, one can, without loss of generality, assume that $a_2 = a_3$ and $a_4 = a_5$. So our result generalizes all of the results mentioned above. In general, there is however no such symmetry as $a_2 = a_3$ and $a_4 = a_5$. There are examples [30] of S, T which satisfy the above conditions, but (1) does not hold for any a_i, $i = 1, 2, \cdots, 5$, in $[0, 1]$ with $a_2 = a_3$, $a_4 = a_5$ and $\sum_{i=1}^{5} a_i < 1$. By extending the idea of Rakotch [25], we [30] introduced monotonically non-increasing self-mappings α_i, $i = 1, 2, \cdots, 5$, on $[0, \infty]$ such that $\alpha_2 = \alpha_3$, $\alpha_4 = \alpha_5$ and $\sum_{i=1}^{5} \alpha_i(t) < 1$ for each $t > 0$. It was proved that S, T have a unique common fixed point if (1) is satisfied with a_i replaced by $\alpha_i(d(x, y))$. By extending the idea of Boyd and Wong [5], we [31] introduced self-mappings

This research was supported by the National Research Council of Canada and by a grant from the Canadian Mathematical Congress.

α_i, $i = 1,2,\cdots,5$, on $[0, \infty)$ such that $\alpha_2 = \alpha_3$, $\alpha_4 = \alpha_5$, $\sum_{i=1}^5(\alpha_i(t) < t$ for all $t > 0$ and each α_i is upper semicontinuous. It is assumed that for any distinct x, y in X, (1) is satisfied with a_i replaced by $\alpha_i(d(x,y))/d(x,y)$. We proved that either S or T has a fixed point and if both S and T have fixed points, then each of S, T has a unique fixed point and these two fixed points coincide. Thus when $S = T$, T has a unique fixed point. In fact, in this case, the condition "each α_i is upper semicontinuous" can be weakened to "each α_i is upper semicontinuous from the right". The conclusion of the above result is best possible in the sense that there are examples [30] of S, T which satisfy the above conditions but S has two fixed points and T has none. However, in applying the above results, it may be difficult to find the required α_i's even if they exist. We shall obtain some fixed point theorems by replacing each a_i in (1) with a number $\alpha_i(x,y)$ depending on $\{x,y\}$, i.e. each α_i is a symmetric function of $X \times X$ into $[0, \infty)$. Thus each α_i need not be a composite function of d with any function on the real line and it is possible that

(2) $$\sum_{i=1}^{5} \sup\{\alpha_i(x,y) \colon x, y \in X\} > 1.$$

One such example is given in Section 2. Related results are obtained for mappings in a Banach space.

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THEOREM 1. *Let S, T be self-mappings on a complete metric space. Suppose that there exist functions α_i, $i = 1,2,\cdots,5$, of $X \times X$ into $[0, \infty)$ such that*

(a) $r \equiv \sup\{\sum_{i=1}^5 \alpha_i(x\ y) \colon x, y \in X\} < 1$;

(b) $\alpha_2 = \alpha_3$, $\alpha_4 = \alpha_5$;

(c) *for any distinct x, y in X,*

$$d(S(x), T(y)) \leqq a_1 d(x,y) + a_2 d(x, T(y)) + a_3 d(y, S(x))$$
$$+ a_4 d(x, S(x)) + a_5 d(y, T(y)),$$

where $a_i = \alpha_i(x,y)$.

Then S or T has a fixed point. If both S and T have fixed points, then each of S, T has a unique fixed point and these two fixed points coincide.

PROOF. Let $x_0 \in X$,

$$x_{2n+1} = S(x_{2n}),\ x_{2n+2} = T(x_{2n+1}),\ n = 0,1,2,\cdots .$$

We shall prove that S or T has a fixed point. For this purpose, we may assume that $x_n \neq x_{n+1}$ for each n. From (c) with $a_i = \alpha_i(x_0, x_1)$,

$$d(x_1, x_2) = d(S(x_0), T(x_1)) \leqq (a_1 + a_4)d(x_0, x_1) + a_2 d(x_0, x_2) + a_5 d(x_1, x_2).$$

Since $d(x_0, x_2) \leqq d(x_0, x_1) + d(x_1, x_2)$,

(3) $$(1 - a_2 - a_5)d(x_1, x_2) \leqq (a_1 + a_2 + a_4)d(x_0, x_1).$$

https://doi.org/10.1017/S1446788700022849 Published online by Cambridge University Press

From (a) and (b), $a_2 + a_4 \leqq r/2 < 1/2$ and

$$(4) \qquad \frac{a_1 + a_2 + a_4}{1 - a_2 - a_5} \leqq \frac{r - a_2 - a_4}{1 - a_2 - a_4} \leqq \max\left\{\frac{r - x}{1 - x} : x \in [0, 1/2]\right\} \leqq r.$$

From (3) and (4),

$$d(x_1, x_2) \leqq rd(x_1, x_0).$$

By induction, we have

$$(5) \qquad d(x_{n+2}, x_{n+1}) \leqq rd(x_{n+1}, x_n), \qquad n = 0, 1, 2, \cdots .$$

By (5) and induction,

$$d(x_{n+1}, x_n) \leqq r^n d(x_1, x_0), \qquad n = 0, 1, 2, \cdots .$$

Since $r < 1$,

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

and therefore $\{x_n\}$ is Cauchy. By completeness of (X, d), $\{x_n\}$ converges to some point x in X. Since $x_{n+1} \neq x_n$ for each n, either $x_{2n+1} \neq x$ for infinitely many n or $x_{2n} \neq x$ for infinitely many n. By symmetry, we may assume that $x_{2n+1} \neq x$ for infinitely many n. Thus there is a subsequence $\{k(n)\}$ of $\{n\}$ such that $x_{2k(n)+1} \neq x$ for each n. Let $n \geqq 1$. Then

$$(6) \qquad \begin{aligned} d(x, T(x)) &\leqq d(x, x_{2k(n)+1}) + d(x_{2k(n)+1}, T(x)) \\ &= d(x, x_{2k(n)+1}) + d(S(x_{2k(n)}), T(x)). \end{aligned}$$

From (c) with $a_i = \alpha_i(x_{2k(n)}, x)$,

$$(7) \qquad \begin{aligned} d(S(x_{2k(n)}), T(x)) &\leqq a_1 d(x_{2k(n)}, x) + a_2 d(x_{2k(n)}, T(x)) \\ &\quad + a_3 d(x, x_{2k(n)+1}) \\ &\quad + a_4 d(x_{2k(n)}, x_{2k(n)+1}) + a_5 d(x, T(x)) \\ &\leqq d(x_{2k(n)}, x) + \frac{r}{2} d(x_{2k(n)}, T(x)) \\ &\quad + d(x, x_{2k(n)+1}) \\ &\quad + d(x_{2k(n)}, x_{2k(n)+1}) + \frac{r}{2} d(x, T(x)). \end{aligned}$$

From (6) and (7), we have by letting $n \to \infty$,

$$d(x, T(x)) \leqq rd(x, T(x)).$$

Since $r < 1$, $T(x) = x$. Let x be a fixed point of S and let y be a fixed point of T. We need to prove that $x = y$. Suppose not. From (c) with $a_i = \alpha_i(x, y)$,

$$d(x, y) = d(S(x), T(y)) \leqq (a_1 + a_2 + a_3) d(x, y) < d(x, y),$$

a contradiction.

THEOREM 2. *Let T be a self-mapping on a complete metric space (X, d). Suppose that there exist symmetric functions α_i, $i = 1, 2, \cdots, 5$, of $X \times X$ into $[0, 1)$ such that*

(a) $r \equiv \sup\{\sum_{i=1}^{5} \alpha_i(x, y): x, y \in X\} < 1$;

(b) *for any x, y in X,*

$$d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, T(x)) + a_4 d(x, T(x)) + a_5 d(y, T(y)),$$

where $a_i = \alpha_i(x, y)$.
Then T has a unique fixed point.

PROOF. Let $x, y \in X$. Calculating

$$(d(T(x), T(y)) + d(T(y), T(x)))/2$$

by (b), we obtain

$$d(T(x), T(y)) \leqq b_1 d(x, y) + b_2 d(x, T(y)) + b_3 d(y, T(x)) + b_4 d(x, T(x)) + b_5 d(y, T(y)),$$

where

$$b_1 = (\alpha_1(x, y) + \alpha_1(y, x))/2,$$

$$b_2 = (\alpha_2(x, y) + \alpha_3(y, x))/2, \quad b_3 = (\alpha_2(y, x) + \alpha_3(x, y))/2,$$

$$b_4 = (\alpha_4(x, y) + \alpha_5(y, x))/2, \quad b_5 = (\alpha_4(y, x) + \alpha_5(x, y))/2.$$

Since each α_i is symmetric, $b_2 = b_3$, $b_4 = b_5$ and

$$\sum_{i=1}^{5} b_i = \sum_{i=1}^{5} \alpha_i(x, y) \leqq r.$$

So we may assume that $\alpha_2 = \alpha_3$ and $\alpha_4 = \alpha_5$. By Theorem 1, T has a unique fixed point.

EXAMPLE. Let X be the unit interval with the usual distance. Let T be the self-mapping on X defined by

$$T(x) = \frac{109}{60} x - \frac{3}{2} \quad \text{if} \quad x \in [10/11, 1] \tag{8}$$

$$T(x) = \frac{1}{6} x \quad \text{if} \quad x \in [0, 10/11). \tag{9}$$

Then T is a continuous increasing self-mapping on X. For x, y in $[0, 10/11)$, we define

$$\alpha_1(x, y) = 1/6, \alpha_2(x, y) = \alpha_3(x, y) = \alpha_4(x, y) = \alpha_5(x, y) = 0.$$

https://doi.org/10.1017/S1446788700022849 Published online by Cambridge University Press

For x, y in X with x or y in $[10/11, 1]$, we define

$$\alpha_1(x,y) = \alpha_2(x,y) = \alpha_3(x,y) = 0, \alpha_4(x,y) = \alpha_5(x,y) = 19/41.$$

Then each α_i is a symmetric function of $X \times X$ into $[0, \infty)$ and $\sum_{i=1}^{5}\alpha_i \leqq 38/41 < 1$. Suppose that $x, y \in [10/11, 1]$.
Then

$$d(T(x), T(y)) = \frac{109}{60}\, d(x,y) \leqq \frac{109}{660}$$

and

$$\alpha_4(x,y)d(x,T(x)) + \alpha_5(x,y)d(y,T(y)) = \frac{19}{41}\left(3 - \frac{49}{60}x - \frac{49}{60}y\right) \geqq \frac{418}{660} .$$

So T is not nonexpansive on X and

(10) $$d(T(x), T(y)) \leqq \alpha_4(x,y)d(x,T(x)) + \alpha_5(x,y)d(y,T(y)).$$

Now suppose that $x \in [10/11, 1]$, $y \in [0, 10/11)$. Then

$$d(T(x), T(y)) = T(x) - T(y) = \frac{109}{60}x - \frac{3}{2} - \frac{1}{6}\,y;$$

$$\alpha_4(x,y)d(x,T(x)) + \alpha_5(x,y)d(y,T(y)) = \frac{19}{41}\left(\frac{3}{2} - \frac{49}{60}x + \frac{5}{6}y\right).$$

Since

$$\frac{109}{60}x - \frac{3}{2} - \frac{1}{6}y - \frac{19}{41}\left(\frac{3}{2} - \frac{49}{60}x + \frac{5}{6}y\right) = \frac{90}{41}(x-1) - \frac{68}{123}y \leqq 0,$$

(10) is satisfied. Finally, suppose that $x, y \in [0, 10/11)$.
Then

$$d(T(x), T(y)) = \frac{1}{6}\, d(x,y) = \alpha_5(x,y)d(x,y).$$

So T satisfies the conditions of Theorem 2. To see the advantage of Theorem 2, we note that

$$\sum_{i=1}^{5} \sup\{\alpha_i(x,y) : x, y \in X\} = \frac{269}{246} > 1$$

and α_4 is not a composite function of d with any function on the real line ($\alpha_4(0, 1/11) \neq \alpha_4(10/11, 1)$ but $d(0, 1/11) = d(10/11, 1)$). Also, if we replace $1/6$ in (9) by a number in $[0, 1/6)$, we will obtain an example of T which is not continuous on X and satisfies the conditions of Theorem 2.

3

Let X be a complete metric space. For any subset A of X, cl A will denote the closure of A in X and $\delta(A)$ will denote the diameter of A, i.e. $\delta(A) = \sup\{d(x,y):$

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$x, y \in A\}$. Let T be a self-mapping on X. A subset A of X is *T-invariant* if $T(A) \subset A$. Let $x \in X$. $O(x)$ will denote the set $\{T^n(x): n \geqq 0\}$ $(T^0(x) = x)$ of all iterates $T^n(x)$ of x and will be called the *orbit* of x. T has *orbital diminishing diameters* if for any x in X, either $\delta(O(x)) = 0$ or

$$\lim_{n \to \infty} \delta(O(T^n(x)) < \delta(O(x)).$$

This definition was introduced by Belluce and Kirk [3], [4], [18]. Kirk [18] proved that if X is compact and if T is a continuous self-mapping on X which has diminishing orbital diameters, then for any x in X, some subsequence of $\{T^n(x)\}$ converges to a fixed point of T. We shall obtain a related result with a different approach.

THEOREM 3. *Let T be a continuous self-mapping on a compact metric space. Suppose that*

(a) *there exist symmetric functions $\alpha_1, \alpha_2, \alpha_3$, of $X \times X$ into $[0,1]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \leqq 1$ and for any x, y in X,*

$$d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, T(x)),$$

where $a_i = \alpha_i(x, y)$;

(b) *for any T-invariant closed subset A of X with $\delta(A) > 0$, there exist y, z in A such that*

$$\sup\{d(y, T^n(z)): n \geqq 0\} < \delta(A).$$

Then for any x in X, $\{T^n(x)\}$ has a subsequence which converges to a fixed point of T.

PROOF. Let $x \in X$. Consider cl $O(x)$. Then cl $O(x)$ is compact and T-invariant. By Zorn's lemma, cl $O(x)$ includes a minimal non-empty closed T-invariant subset Y of X. Suppose to the contrary that $\delta(Y) > 0$. Then by (b), there exist y, z in Y such that

$$r \equiv \sup\{d(y, T^n(z)): n \geqq 0\} < \delta(Y).$$

By continuity of T, $d(y, u) \leqq r$ for each $u \in \text{cl}\, O(z)$. Since cl $O(z)$ is T-invariant, by minimality of Y, cl $O(z) = Y$. So the set

$$W = \{u \in Y: d(u, v) \leqq r \text{ for each } v \text{ in } Y\} \tag{11}$$

contains y. By continuity of d, W is closed. Let $u \in W$. By compactness of Y, there exists $v_0 \in Y$ such that

$$d(T(u), v_0) = \sup\{d(T(u), v): v \in Y\}. \tag{12}$$

Since T is continuous, $T(Y)$ is compact. Also, $T(Y)$ is T-invariant. So by minimality of Y, $T(Y) = Y$. Therefore there exists v_1 in Y such that $T(v_1) = v_0$. By (11), (12) and (a) with $a_i = \alpha_i(u, v_1)$,

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$$\begin{aligned} d(T(u), v_0) &= d(T(u), T(v_1)) \\ &\leqq a_1 d(u, v_1) + a_2 d(u, T(v_1)) + a_3 d(v_1, T(u)) \\ &\leqq a_1 r + a_2 r + a_3 d(T(u, v_0) \leqq (1 - a_3) r + a_3 d(T(u), v_0). \end{aligned}$$

So

(13) $$(1 - a_3) d(T(u), v_0) \leqq (1 - a_3) r.$$

Since each α_i is symmetric, we may assume that $a_2 = a_3$. Thus $a_3 \leqq 1/2$. From (13), we have $d(T(u), v_0) \leqq r$. So W is T-invariant. By minimality of Y, $W = Y$. Therefore by (11),

$$\delta(Y) = \delta(W) \leqq r < \delta(Y),$$

a contradiction. Hence $\delta(Y) = 0$ and the point x_0 in Y is a fixed point of T. Since x_0 is a fixed point of T and $x_0 \in \mathrm{cl}\, O(x)$, some subsequence of $\{T^n(x)\}$ converges to x_0.

We remark that in Theorem 3, (b) is satisfied if T has diminishing orbital diameters.

4

Let B be a Banach space. d will denote the metric for B induced by the norm $\| \ \|$ of B. For any subset A of B, co A will denote the convex hull of A. Let X be a bounded closed convex subset of B. Let T be a self-mapping on X. X is *regular* with respect to T if for any non-empty closed convex T-invariant subset A of X, either $\delta(A) = 0$ or there exist y, z in A such that

$$\sup\{d(y, T^n(z)) : n \geqq 0\} < \delta(A).$$

X is *normal* with respect to T if for any non-empty closed convex T-invariant subset A of X, either $\delta(A) = 0$ or there exists z in A such that

$$\sup\{d(z, y) : y \in A\} < \delta(A).$$

X has *normal structure* [6] if X is normal with respect to the identity function on X. It is clear that X is regular with respect to T if it is normal with respect to T; X is normal with respect to T if it has normal structure. Our notions of regularity and normality link T with the convex structure of X.

THEOREM 4. *Let X be a weakly compact convex subset of a Banach space B. Let T be a self-mapping on X. Suppose that*

(a) *X is normal with respect to T;*

(b) *there exist symmetric functions $\alpha_1, \alpha_2, \alpha_3$ of $X \times X$ into $[0, \infty)$ such that $\alpha_1 + \alpha_2 + \alpha_3 \leqq 1$ and for any x, y in X,*

$$d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T)y)) + a_3 d(y, T(x)),$$

where $a_i = \alpha_i(x, y)$.

Then T has a fixed point.

PROOF. By Zorn's lemma, there exists a minimal non-empty closed convex T-invariant subset Y of X. Suppose to the contrary that $\delta(Y) > 0$. Then by (a), there exists z in Y such that

$$r \equiv \sup\{d(z, y): y \in Y\} < \delta(Y).$$

So

$$W = \{w \in Y: d(w, y) \leqq r \text{ for each } y \text{ in } Y\}$$

contains z. Obviously W is convex and closed. We shall prove that W is T-invariant. Let $w \in W$,

$$r_1 = \sup\{d(T(w), x): x \in Y\}.$$

Since $T(Y) \subset Y$ and Y is closed and convex, one has cl co $T(Y) \subset Y$; hence

$$T(\text{cl co } T(Y)) \subset T(Y) \subset \text{cl co } T(Y).$$

By minimality of Y, cl co $T(Y) = Y$. So by continuity and convexity of d,

$$r_1 = \sup\{d(T(w), T(y)) : y \in Y\}.$$

(For nonexpansive mappings, the above argument or its variants occurred in [1], [2], [3], [14], [16], [17], [19], [20], [21], [22], [23], [28], [29].) Let $w \in W$, $y \in Y$. By (b) with $a_i = \alpha_i(x, y)$,

$$d(T(w), T(y)) \leqq a_1 d(w, y) + a_2 d(w, T(y)) + a_3 d(y, T(w)) \leqq a_1 r + a_2 r + a_3 r_1.$$

Let $\varepsilon > 0$ and select $\bar{y} \in Y$ so that

$$\sup_{y \in Y}(a_1 r + a_2 r + a_3 r_1) \leqq \bar{a}_1 r + \bar{a}_2 r + \bar{a}_3 r_1 + \varepsilon \text{ where } \bar{a}_i = \alpha_i(w, \bar{y}).$$

Then

$$r_1 = \sup_{y \in Y} d(T(w), T(y)) \leqq \sup_{y \in Y}(a_1 r + a_2 r + a_3 r_1) \leqq \bar{a}_1 r + \bar{a}_2 r + \bar{a}_3 r_1 + \varepsilon$$
$$= (1 - \bar{a}_3) r + \bar{a}_3 r_1 + \varepsilon.$$

Hence, since $\bar{a}_3 \leqq \frac{1}{2}$,

$$r_1 \leqq r + \frac{\varepsilon}{1 - \bar{a}_3} \leqq r + 2\varepsilon.$$

Since ε is arbitrary, $r_1 \leqq r$. A contradiction can be obtained as in the proof of Theorem 3. Hence Y is a singleton and the point in Y is a fixed point of T.

By refining the above argument, one can obtain the following result for nonexpansive mappings.

THEOREM 5. *Let X be a weakly compact convex subset of a Banach space. Let T be a nonexpansive self-mapping on X. Suppose that X is regular with respect to T. Then T has a fixed point.*

Let X be a weakly compact convex subset of a Banach space B. Let T be a nonexpansive self-mapping on X. Browder [7] and Göhde [12] proved that T has a fixed point if B is uniformly convex. Belluce and Kirk [1] proved that T has a fixed point if X has normal structure. Then they [3] obtained the following more general results: (i) T has a fixed point if for any x in X, cl co $O(x)$ has normal structure, (ii) T has a fixed point if T has diminishing orbital diameters. Theorem 5 combines all these results into a more general one.

We remark here that by modifying the definitions, conditions and proofs in an obvious way, Theorem 3 can be proved for a compact Hausdorff topological space X associated with a definite family of pseudo-metrics; Theorems 4 and 5 can be proved for a locally convex Hausdorff topological vector space X associated with a family of pseudo-norms. One such example can be found in [14] Also, every uniformly convex Banach space is reflexive [24]. So in this case, to assume that X is weakly compact convex is the same as assuming that X is bounded, closed and convex.

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Let X be a bounded convex subset of a Banach space B. Let $y \in X$,

$$\begin{aligned} r_y &= \sup\{d(x,y) : x \in X\}, \\ r &= \inf\{r_y : y \in X\}, \\ C &= \{x \in X : r_x = r\}. \end{aligned}$$

r is called the *radius* of X and C is called the *Chebyshev centre* of X. It was shown in [17] that if B is reflexive and X is weakly compact convex, then C is a non-empty closed convex subset of X. If C is a singleton $\{x\}$, then x is called the *generalized center* of X. For example, every bounded closed convex subset of a uniformly convex Banach space has a generalized centre [11].

THEOREM 6. *Let X be a bounded convex subset of a Banach space B. Suppose that X has a generalized centre x_0. Let T be a self-mapping on X (not necessarily continuous). Suppose further that*

(a) cl co $T(X) \supset X$;

(b) *there exist symmetric functions $\alpha_1, \alpha_2, \alpha_3$ of $X \times X$ into $[0, \infty)$ such that for all x, y in X,*

$$d(T(x), T(y)) \leqq a_1 d(x,y) + a_2 d(x, T(y)) + a_3 d(y, T(x)),$$

where $a_i = \alpha_i(x,y)$.
Then x_0 is a fixed point of T.

PROOF. Form the definition of generalized centres,

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$$(14) \qquad \{x_0\} = \bigcap_{y\in Y} \{x \in Y: d(x,y) \leqq r\},$$

where r is the radius of Y. Let $x \in Y$. By (b) with $a_i = \alpha_i(x_0, x)$,

$$(15) \qquad d(T(x_0), T(x)) \leqq a_1 d(x_0, x) + a_2 d(x_0, T(x)) + a_3 d(x, T(x_0))$$
$$\leqq a_1 r + a_2 r + a_3 r_1,$$

where $r_1 = \sup\{d(T(x_0), y): y \in Y\}$. By (a),

$$r_1 = \sup\{d(T(x_0), T(y)): y \in Y\}.$$

Arguing as the the proof of Theorem 4, $r_1 \leqq r$. So

$$T(x_0) \in \bigcap_{y\in Y} \{x \in Y: d(x,y) \leqq r\}.$$

So by (14), $T(x_0) = x_0$.

Let X be a weakly compact convex subset of a Banach space B. Let T be a self-mapping on X. T is called a generalized nonexpansive mapping if there exist symmetric functions α_i, $i = 1, 2, \cdots, 5$, of $X \times X$ into X such that $\sum_{i=1}^{5} \alpha_i \leqq 1$ and for all x, y in X,

$$d(T(x), T(y)) \leqq a_1 d(x,y) + a_2 d(x, T(y)) + a_3 d(y, T(x))$$
$$a_4 d(x, T(x)) + a_5 d(y, T(y)),$$

where $a_i = \alpha_i(x, y)$. Let T be a generalized nonexpansive self-mapping on X. We post the following open questions:

(a) $\inf\{d(x, T(x)): x \in X\} = 0$?

(b) Does T have a fixed point?

By using the asymptotic center recently introduced by Edelstein [10], we can prove that if X is uniformly convex and if T is continuous, then the above questions are equivalent. It is also interesting to note that if T has a fixed point, then T must be quasi-nonexpansive [9].

We would like to mention here that the use of five monotonically non-increasing functions α_i on $(0, \infty)$ (with $\sum_{i=1}^{5} \alpha_i < 1$) to contract self-mappings on a complete metric space is invented by Hardy and Rogers [13].

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https://doi.org/10.1017/S1446788700022849 Published online by Cambridge University Press