ON METACYCLIC FIBONACCI GROUPS

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1. Introduction

Let F_n be the free group on $\{a_i: i \in \mathbf{Z}_n\}$, where the set of congruence classes mod n is used as an index set for the generators. Let ϕ be the permutation $(1, 2, 3, \ldots, n)$ of $\mathbf{Z}_n$ and denote by θ the automorphism of F_n induced by ϕ, namely

$$a_i\theta = a_{i\phi}.$$

Let r and k be integers such that $r \geqq 2$, $k \geqq 0$ and let N be the normal closure of the set

$$\{(a_1a_2\ldots a_r a_{r+k}^{-1})\theta^m: 1 \leqq m \leqq n\}$$

in F_n. Define the *generalised Fibonacci group* $F(r, n, k)$ by

$$F(r, n, k) = F_n/N.$$

We shall call $(a_1a_2\ldots a_r a_{r+k}^{-1})\theta^{m-1} = 1$ the relation (m) of $F(r, n, k)$. The groups $F(r, n, 1)$ are the *Fibonacci groups* discussed in (**3**), where it is proved that these groups are metacyclic if $r \equiv 1 \bmod n$. In (**3**) two questions are posed relating to the case $r \equiv 1 \bmod n$, namely to find the orders of these groups and also 2-generator 2-relation presentations for them. The first of these questions was solved in (**2**) and in this paper we solve the second problem.

The generalised Fibonacci groups $F(r, n, k)$ are discussed in (**1**) where it is stated that $F(6, 5, 3)$ is isomorphic to $F(6, 5, 1)$. We prove a generalisation of this result, namely that when $r \equiv 1 \bmod n$ $F(r, n, k)$ is isomorphic to $F(r, n, 1)$ for any k coprime to n.

2. A 2-generator presentation for $F(r, n, k)$

Suppose $r \equiv 1 \bmod n$ and let $r = nt+1$. Define $b_0 = 0$, $b_1 = t$ and, inductively, $b_j = rb_{j-1}+t$. Denote by x the element $a_1a_2\ldots a_n$ of $F(r, n, k)$. In (**2**) we showed that $\langle x\rangle$ has index n in $F(r, n, k)$ when k is coprime to n and that x has order b_n. A coset enumeration was carried out using the modified Todd-Coxeter algorithm. Defining the cosets $2, 3, \ldots, n$ by $i.a_i = i+1$, $1 \leqq i \leqq n-1$, the following relations between coset representatives were obtained

$$i\,.\,a_{\alpha k+i} = \begin{cases} x^{b_\alpha}.(i+1) & 1 \leqq i \leqq n-1, \\ x^{b_\alpha+1}.1 & i = n. \end{cases}$$

Let $y = a_{1+k}$. We show that x and y together generate $F(r, n, k)$ and obtain the following theorem.

Theorem 1. *Let* $r \equiv 1 \bmod n$ *and let* k *be coprime to* n. *Then*

$$F(r, n, k) = \langle x, y \mid y^{-1}xy = x^{r^h},\ y^n = x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n} = 1 \rangle,$$

where $hk \equiv 1 \bmod n$ *and* $1 \leqq h \leqq n-1$.

Proof. We use the modified Todd-Coxeter algorithm for the subgroup $H = \langle x, y \rangle$, making use of the results already stated for $\langle x \rangle$ and following through the collapses which occur with the addition of the information $1.a_{1+k} = y.1$. Using relation (1) we obtain $1.a_{1+k} = x^t.2$ and so $2 = x^{-t}y.1$. But $1.a_{\alpha k+1} = x^{b_\alpha}.2 = x^{b_\alpha - t}y.1$. Since k is coprime to n,

$$1.a_i = x^{b_{h(i-1)}-t}y.1; \quad 1 \leqq i \leqq n.$$

Therefore the subgroup H has index one in $F(r, n, k)$ and thus

$$F(r, n, k) = \langle x, y \rangle.$$

Notice that we can always replace x^{b_β} by $x^{b_{\bar\beta}}$ where $\bar\beta \equiv \beta \bmod n$ and $0 \leqq \bar\beta < n$ since $x^{b_n} = 1$.

In addition to the relation $x^{b_n} = 1$ the modified Todd-Coxeter algorithm gives us the following relations for H. From the subgroup generator $x = a_1 a_2 \ldots a_{n-1} a_n$ we obtain

$$(A) \qquad x = \prod_{i=1}^{n} x^{b_{h(i-1)}-t}y,$$

and from the relation (m) of $F(r, n, k)$ we obtain

$$(B_m) \qquad \left(\prod_{i=1}^{n} x^{b_{h(m-2+i)}-t}y\right)^t x^{b_{h(m-1)}-b_{h(m-1)+1}} = 1.$$

Notice that (B_1) is the tth power of relation (A). Now (B_m) and (B_{m+1}) together imply

$$yx^{b_{hm+1}-b_{hm}}y^{-1} = x^{b_{h(m-1)+1}-b_{h(m-1)}}.$$

But $b_{h(m-1)+1} - b_{h(m-1)} = ntb_{h(m-1)} + t$, and thus $y^{-1}x^{ntb_{h(m-1)}+t}y = x^{ntb_{hm}+t}$. We therefore obtain

$$(C_m) \qquad y^{-1}x^{tr^{h(m-1)}}y = x^{tr^{hm}}.$$

However (C_m), $1 \leqq m \leqq n$, together with (A) imply (B_m), $1 \leqq m \leqq n$.

The relation (C_1) is $y^{-1}x^ty = x^{tr^h}$, and raising this relation to the power $r^{h(m-1)}$ gives the relation (C_m). Hence a presentation for H, and therefore for $F(r, n, k)$, is given by the generators x and y subject to the relations (A), (C_1) and $x^{b_n} = 1$. Since $b_{hi} - t$, $1 \leqq i \leqq n$, is divisible by t, relation (A) simplifies using (C_1) to give $y^n = x^\alpha$ where

$$\alpha = 1 - \sum_{i=1}^{n} (b_{hi} - t)r^{h(n-i)}.$$

But $b_{hi}-t=(r^{hi}-r)/n$ and so $\alpha=r(r^n-1)/(n(r-1))$. Relation (A) becomes $y^n=x^{(r^n-1)/(n(r-1))}$ since $r=1+(r-1)$ and $x^{(r-1)(r^n-1)/(n(r-1))}=1$. Relation (C_1) now simplifies using the modified relation (A). For $\alpha=vt+1$ for some $v\in\mathbf{Z}$, and so $y^{-1}x^{-vt}y=x^{-vtr^h}$ giving

$$y^{-1}xy=x^{-vtr^h+\alpha}=x^{r^h}x^{\alpha(1-r^h)}.$$

Notice we have used the fact that $x^{\alpha}\in Z(H)$, the centre of H. However, $x^{\alpha(1-r^h)}=1$ since

$$\alpha(1-r^h)=\frac{r^n-1}{n}\cdot\frac{(1-r^h)}{r-1}=u\cdot\frac{r^n-1}{n}$$

for some $u\in\mathbf{Z}$. Thus $y^{-1}xy=x^{r^h}$.

Corollary 1. $F(r,n,1)=\langle x,y\mid y^{-1}xy=x^r,\ y^n=x^{(r^n-1)/(n(r-1))}\rangle$, *where* $r\equiv$ mod n.

Proof. It suffices to show that the relations $y^{-1}xy=x^r$ and

$$y^n=x^{(r^n-1)/(n(r-1))}$$

together imply $x^{(r^n-1)/n}=1$. Raising $y^{-1}xy=x^r$ to the power $(r^n-1)/(n(r-1))$ gives

$$y^{-1}x^{(r^n-1)/(n(r-1))}y=x^{r(r^n-1)/(n(r-1))}.$$

Since $x^{(r^n-1)/(n(r-1))}\in Z(H)$, $x^{(r^n-1)/n}=1$.

Corollary 2. $F(r,n,1)\cong F(r,n,k)$ *when* $r\equiv 1$ mod n *and* k *is coprime to* n.

Proof. Let Π be the set of prime factors of h.c.f. $(h,(r-1)(r^n-1))$ and λ the maximal Π'-number dividing $(r-1)(r^n-1)$, then $h+\lambda n$ is coprime to $(r-1)(r^n-1)$ and hence coprime to the order of x and the order of y. The group $F(r,n,1)$ has a presentation

$$\langle x,y\mid y^{-1}xy=x^r,\ y^n=x^{(r^n-1)/(n(r-1))},\ x^{(r^n-1)/n}=1\rangle.$$

With this choice of λ, $x^{h+\lambda n}$ and $y^{h+\lambda n}$ together generate $F(r,n,1)$. With $a=x^{h+\lambda n}$, $b=y^{h+\lambda n}$ it is straightforward to check that

$$b^{-1}ab=a^{r^h},\quad b^n=a^{(r^n-1)/(n(r-1))},\quad a^{(r^n-1)/n}=1.$$

Hence $F(r,n,1)$ is a homomorphic image of $F(r,n,k)$ and, using the fact proved in (2) that $|F(r,n,1)|=|F(r,n,k)|$, the result follows.

An immediate consequence of Corollary 1 and Corollary 2 is the following theorem.

Theorem 2. *Let* $r\equiv 1$ mod n *and let* k *be coprime to* n. *Then* $F(r,n,k)$ *has a* 2-*generator* 2-*relation presentation*

$$F(r,n,k)=\langle x,y\mid y^{-1}xy=x^r,\ y^n=x^{(r^n-1)/(n(r-1))}\rangle.$$

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