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Functions Universal for all Translation Operators in Several Complex Variables

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Abstract. We prove the existence of a (in fact many) holomorphic function f in $\mathbb{C}^d$ such that, for any $a \neq 0$, its translations $f(\cdot + na)$ are dense in $H(\mathbb{C}^d)$.

1 Introduction

The roots of this paper go back to an old paper of Birkhoff [3] in which he proves that, for any $a \neq 0$, there exists an entire function f such that its translates $f(\cdot + na)$ are dense in the space of all entire functions $H(\mathbb{C})$ endowed with the compact-open topology. In modern terms, this means that the operators $\tau_a: H(\mathbb{C}) \to H(\mathbb{C}), f \mapsto f(\cdot + a)$ are hypercyclic, and we shall denote by $HC(\tau_a)$ the set of hypercyclic functions with respect to τ_a, namely the set of functions whose translates by na, $n = 1, 2, \dots$, are dense. Since Birkhoff's theorem, the theory of hypercyclic operators has grown, and we refer the reader to the books [2,5] for more on this subject.

Regarding hypercyclicity of translations, a major breakthrough was made by Costakis and Sambarino in [4]. They were able to show that one can choose the same hypercyclic function for all non-zero translation operators. In other words, $\bigcap_{a\neq 0} HC(\tau_a)$ is non empty. In Tsirivas' subsequent works (see [7–9]) as well as in a paper by the first author [1], the authors were interested in considering common universal functions for sequences of translations $\tau_{\lambda_n a}$. In particular, in [1], one is interested in translation operators acting on $H(\mathbb{C}^d)$ with $d \geq 2$. It is shown that $\bigcap_{a\in\mathbb{R}^d\setminus\{0\}} HC(\tau_a)$ is a residual subset of $H(\mathbb{C}^d)$. There are two main difficulties for going from Costakis and Sambarino's results to this last one:

(a) The method of [4] is one-dimensional and works very well for one-dimensional families of operators. Then an algebraic trick allows one to go from $\mathbb{R}$ to $\mathbb{C}$. It was not clear how to go further, especially on $\mathbb{C}^d$.

(b) Polynomial approximation is more difficult in $H(\mathbb{C}^d)$, $d \geq 2$, than in $H(\mathbb{C})$. In particular, there is no satisfactory Runge or Mergelyan theorem in $H(\mathbb{C}^d)$, and one has to work with the delicate notion of polynomially convex sets. That is why the result of [1] was for translations by real elements even though we are working in $\mathbb{C}^d$. In this paper, we overcome this last difficulty, and we are able to prove the following result.

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Theorem 1.1 *The set* $\bigcap_{a\in\mathbb{C}^d\setminus\{0\}} HC(\tau_a)$ *is a residual subset of* $H(\mathbb{C}^d)$.

Our method of proof uses arithmetical tools from [1], in particular the forthcoming Lemma 2.5. It allows us to obtain a redundant net in any compact subset of $\mathbb{C}^d$, for any dimension d. We then use classical results on polynomially convex sets of $\mathbb{C}^d$ to show that we can do a polynomial approximation of any holomorphic function defined on a union of sufficiently disjoint hypercubes.

2 Tools for the Construction

2.1 Polynomial Convexity

Let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote the complex, real, and natural numbers, respectively, and let $\mathbb{N}_0 = \{0,1,2,\dots\}$. For a compact subset K of $\mathbb{C}^d$, we denote by $\widehat{K}$ the polynomially convex hull of K:

$$\widehat{K} = \Big\{ z \in \mathbb{C}^d;\ \text{for every polynomial } p, |p(z)| \le \max_{w\in K} |p(w)| \Big\}.$$

A compact set $K \subset \mathbb{C}^d$ is said to be *polynomially convex* if it is equal to its polynomially convex hull; that is, if $K = \widehat{K}$. For example, compact convex sets are polynomially convex and a compact subset of $\mathbb{C}$ is polynomially convex if and only if its complement is connected.

Runge's Polynomial Approximation Theorem states that if a compact subset K of $\mathbb{C}$ has connected complement, then every function holomorphic on (a neighborhood of) K can be uniformly approximated by polynomials. The following extension of the Runge Theorem to higher dimensions is known as the Oka–Weil Theorem (see [6]).

Theorem 2.1 *Let K be a polynomially convex compact subset of $\mathbb{C}^d$. Then, for every function f holomorphic on K and for every $\epsilon > 0$, there exists a polynomial p such that*

$$|p(z) - f(z)| < \epsilon, \quad \textit{for all} \quad z \in K.$$

An important tool in constructing polynomially convex sets is the following Separation Lemma by Eva Kallin (see [6]).

Lemma 2.2 *Let X and Y be two polynomially convex compact subsets of $\mathbb{C}^d$. If there exists a polynomial p which separates X and Y in the sense that $\widehat{p(X)} \cap \widehat{p(Y)} = \varnothing$, then the union $X \cup Y$ of X and Y is also polynomially convex.*

We identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ by means of either of the two natural complex structures on $\mathbb{R}^{2d}$, and henceforth $|x|$ denotes the ℓ_∞-norm on $\mathbb{R}^{2d}$. For $x = (x^{(1)},\dots,x^{(2d)}) \in \mathbb{R}^{2d}$ we denote by $Q(x,R)$ the closed hypercube

$$Q(x,R) = Q\big((x^{(1)},\dots,x^{(2d)}),R\big) = \{y \in \mathbb{R}^{2d} : |x-y| \le R\},$$

which may also be considered as the closed ball of center x and radius R with respect to the norm $|\cdot|$. If $z \in \mathbb{C}^d$ corresponds to the point $x \in \mathbb{R}^{2d}$, we shall, by abuse of notation, write $Q(z,R)$ to mean the subset of $\mathbb{C}^d$ identified with the hypercube $Q(x,R)$ in $\mathbb{R}^{2d}$. When we say that a subset K of $\mathbb{R}^{2d}$ is polynomially convex, we mean that, as a subset of $\mathbb{C}^d$, it is polynomially convex. Since compact convex sets are polynomially convex, it follows that hypercubes are polynomially convex.

We need to prove that several sets are polynomially convex.

Lemma 2.3 *Let K, L be two compact polynomially convex subsets of $\mathbb{R}^{2d}$. Assume that there exists $a \in \mathbb{R}$ such that $x^{(1)} < a < y^{(1)}$ for all $(x, y) \in K \times L$. Then $K \cup L$ is polynomially convex.*

Proof Let $z^{(1)}$ be a complex coordinate generated by the real coordinate $x^{(1)}$. The polynomial $f(z) = z^{(1)}$ separates K and L, and so by Kallin's Separation Lemma, $K \cup L$ is polynomially convex. ■

Lemma 2.4 *Let $R > 0$. For every $1 \leq \ell \leq 2d$, let $(y_j^{(\ell)})_{0 \leq j \leq \Omega_l}$ be a finite family of points in $\mathbb{R}$ such that, for all $j \neq j'$, $|y_j^{(\ell)} - y_{j'}^{(\ell)}| > 2R$. Then*

$$\bigcup_{j_1, \ldots, j_{2d}} Q\big((y_{j_1}^{(1)}, \ldots, y_{j_{2d}}^{(2d)}), R\big)$$

is polynomially convex.

Proof For simplicity, let us write

$$X = \bigcup_{j_1, \ldots, j_{2d}} Q\big((y_{j_1}^{(1)}, \ldots, y_{j_{2d}}^{(2d)}), R\big).$$

Recalling the identification $\mathbb{R}^{2d} = \mathbb{C}^d$ and denoting by $X^{(n)}$ the projection of X on the complex coordinate $z^{(n)}$, $n = 1, \ldots, d$, we have $X = \prod_{n=1}^d X^{(n)}$, because of the separation hypotheses. Since each $X^{(n)}$ is a disjoint (again by the separation hypothesis) union of closed squares, it is polynomially convex (here, we are just working in $\mathbb{C}$) and since a product of polynomially convex sets is again polynomially convex, it follows that X is polynomially convex. ■

2.2 Construction of Sequences of Integers

We will need the following lemma about the construction of sequences of integers having some redundant properties. The following Lemma is [1, Corollary 2.8] applied to the whole sequence of integers.

Lemma 2.5 *For all $d \geq 1$ and all $A > 0$, there exist $\rho > 1$ and an increasing sequence of integers (μ_n) such that $\mu_{n+1} \geq \rho \mu_n$ for any $n \geq 1$ and, for all $P > 0$, we can find $s_1 \in \mathbb{N}$, finite subsets E_r of $\mathbb{N}^{r-1}$ for $r = 2, \ldots, 2d+1$, maps $s_r \colon E_r \to \mathbb{N}$ for $r = 2, \ldots, 2d$ and a one-to-one map $\phi \colon E_{2d+1} \to \mathbb{N}$ such that the following hold.*

- *For any $r = 2, \ldots, 2d+1$,*

  $$E_r = \big\{(k_1, \ldots, k_{r-1}) \in \mathbb{N}_0^{r-1} : \ k_1 < s_1, \ k_2 < s_2(k_1), \ldots, k_{r-1} \leq s_{r-1}(k_1, \ldots, k_{r-2})\big\}.$$

- *For every $r = 1, \ldots, 2d$, for every $(k_1, \ldots, k_{r-1}) \in E_r$, where $E_1 = \varnothing$,*

  $$\sum_{j=1}^{s_r(k_1, \ldots, k_{r-1})} \frac{1}{\mu_{\phi(k_1, \ldots, k_{r-1}, j, 0, \ldots, 0)}} \geq \frac{A}{\mu_{\phi(k_1, \ldots, k_{r-1}, 0, \ldots, 0)}}.$$

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- $\phi(0,\dots,0) \geq P$.
- *If* $(k_1,\dots,k_{2d}) > (k'_1,\dots,k'_{2d})$ *in the lexicographical order, then*

$$\phi(k_1,\dots,k_{2d}) > \phi(k'_1,\dots,k'_{2d}).$$

When $r = 1$, the second point of the lemma simply means that

$$\sum_{j=1}^{s_1} \frac{1}{\mu_{\phi(j,0,\dots,0)}} \geq \frac{A}{\mu_{\phi(0,\dots,0)}}.$$

3 The Construction

Lemma 3.1 *Let K be a compact subset of $(0,+\infty)^{2d}$. Assume that for all $\varepsilon > 0$ and all $R > 0$, we can find $N \geq 1$, a finite increasing sequence of integers $(\lambda_n)_{n=1,\dots,N}$, and a finite number $(x_{n,k})_{1\leq n\leq N,\ 1\leq k\leq p_n}$ of elements of K satisfying the following:*

(i) *The hypercubes $Q(\lambda_n x_{n,k}, R)$, $1 \leq n \leq N$, $1 \leq k \leq p_n$, are pairwise disjoint and are disjoint from $Q(0,R)$.*
(ii) *The compact set $Q(0,R) \cup \bigcup_{1\leq n\leq N,\ 1\leq k\leq p_n} Q(\lambda_n x_{n,k}, R)$ is polynomially convex.*
(iii) *For every $x \in K$, there exist $n, m \in \{1,\dots,N\}$ and $k \in \{1,\dots,p_n\}$ such that $|\lambda_m x - \lambda_n x_{n,k}| < \varepsilon$.*

Then $\bigcap_{a\in K} HC(\tau_a)$ is a residual subset of $H(\mathbb{C}^d)$.

Proof Let U, V be nonempty open subsets of $H(\mathbb{C}^d)$. It is sufficient to show that

$$U \cap \left\{ f \in H(\mathbb{C}^d);\ \forall x \in K,\ \exists m \in \mathbb{N},\ \tau_{mx} f \in V \right\}$$

is nonempty (see for instance [2, Proposition 7.4]). Let $\delta, \rho > 0$ and $g, h \in H(\mathbb{C}^d)$ be such that

$$U \supset \left\{ f \in H(\mathbb{C}^d);\ \|f - g\|_{\mathcal{C}(Q(0,\rho))} < 2\delta \right\}$$
$$V \supset \left\{ f \in H(\mathbb{C}^d);\ \|f - h\|_{\mathcal{C}(Q(0,\rho))} < 2\delta \right\},$$

where $\|\cdot\|_{\mathcal{C}(Q(0,\rho))}$ denotes the sup-norm for $\mathcal{C}(Q(0,\rho))$. We set $R = 2\rho$. By uniform continuity of h on $Q(0,2\rho)$, there exists $\eta \in (0,\rho)$ such that

$$\|h(\cdot - z_0) - h\|_{\mathcal{C}(Q(0,\rho))} < \delta$$

provided $|z_0| < \eta$. We set $\varepsilon = \min(\delta,\eta)$, and the assumptions of the lemma give us sequences (λ_n) and $(x_{n,k})$. By (i) and (ii), there exists an entire function $f \in H(\mathbb{C}^d)$ such that $\|f - g\|_{\mathcal{C}(Q(0,\rho))} < \varepsilon < 2\delta$ and

$$\|f(\cdot + \lambda_n x_{n,k}) - h\|_{\mathcal{C}(Q(0,R))} < \delta$$

for any n, k. Now let $x \in K$ and let n, m and k be such that (iii) holds. Then for any $z \in Q(0,\rho)$, observing that $z + \lambda_m x - \lambda_n x_{n,k}$ belongs to $Q(0,R)$, we get

$$\begin{aligned} |\tau_{\lambda_m x} f(z) - h(z)| &\leq \left| f(z + \lambda_m x - \lambda_n x_{n,k} + \lambda_n x_{n,k}) - h(z + \lambda_m x - \lambda_n x_{n,k}) \right| \\ &\quad + \left| h(z + \lambda_m x - \lambda_n x_{n,k}) - h(z) \right| < 2\delta, \end{aligned}$$

which concludes the proof. ■

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We will use a version of the previous lemma for special K and restrict the covering property to compact subsets of K.

Lemma 3.2 *Let K be a compact subset of $(0,+\infty)^{2d}$ of the form $K = \prod_{\ell=1}^{2d}[a_\ell, a'_\ell]$. Assume that, for all $\varepsilon > 0$, for all $R > 0$, there exists $\gamma > 0$ such that for every compact hypercube $L \subset K$ with diameter less than γ, for every $M \in \mathbb{N}$, we can find $N \geq M$, a finite increasing sequence of integers $(\lambda_n)_{n=M,\dots,N}$ with $\lambda_M \geq M$, and a finite number $(x_{n,k})_{M\leq n\leq N,\ 1\leq k\leq p_n}$ of elements of L satisfying the following:*

(i) *The hypercubes $Q(\lambda_n x_{n,k}, R)$, $M \leq n \leq N$, $1 \leq k \leq p_n$, are pairwise disjoint.*
(ii) *The compact set $\bigcup_{M\leq n\leq N,\ 1\leq k\leq p_n} Q(\lambda_n x_{n,k}, R)$ is polynomially convex.*
(iii) *For every $x \in L$, there exist $n, m \in \{M,\dots,N\}$ and $k \in \{1,\dots,p_n\}$ such that*

$$|\lambda_m x - \lambda_n x_{n,k}| < \varepsilon.$$

Then $\bigcap_{a\in K} HC(\tau_a)$ is a residual subset of $H(\mathbb{C}^d)$.

Proof We show that the assumptions of Lemma 3.1 are automatically satisfied. Put $a = \min a_\ell > 0$ and $a' = \max a'_\ell$. A positive real number $\gamma > 0$ being fixed, K may be decomposed as $K = L_1 \cup \cdots \cup L_J$, where each L_j is a compact hypercube with diameter less than γ. We set $N_0 = 0$, $\lambda_0 = 1$, $p_0 = 0$, and we construct inductively sequences (λ_n) and $(x_{n,k})$ as in Lemma 3.1. Assume that the construction has been done until step $j-1$ $(1 \leq j \leq J)$ and let us do it for step j. Let M_j be sufficiently large such that $M_j > N_{j-1}$, $M_j a - \lambda_{N_{j-1}} a' - 2R > 0$. We then apply the assumptions of Lemma 3.2 to $L = L_j$ and $M = M_j$ to get $N_j \geq M_j$ and sequences (λ_n), $M_j \leq n \leq N_j$ and elements $(x_{n,k})$ of L_J, $M_j \leq n \leq N_j, 1 \leq k \leq p_n$.

We claim that the union of the sequences (λ_n), $M_j \leq n \leq N_J$ and $(x_{n,k})$, $M_j \leq n \leq N_J, 1 \leq k \leq p_n$, for $j = 1,\dots,J$, satisfies the hypotheses and hence the conclusion of Lemma 3.1. Notice that the sequence (λ_n) is increasing, since $N_{j-1} < M_j$. The covering property (iii) of Lemma 3.1 clearly follows from Lemma 3.2(iii).

We then show that all the hypercubes $Q(\lambda_n x_{n,k}, R)$ are pairwise disjoint, even if they are constructed at different steps.

First of all, for fixed j, and $n \in \{M_j,\dots,N_j\}$, the finite sequence $x_{n,k}$ was chosen according to the hypothesis of Lemma 3.2, so we have that the hypercubes $Q(\lambda_n x_{n,k}, R)$ are indeed pairwise disjoint.

For n and m coming from different $j's$, the crucial point is to observe that, for any $x \in L_{j-1}$ and any $y \in L_j$, for any $n \in \{M_{j-1},\dots,N_{j-1}\}$, for any $m \in \{M_j,\dots,N_j\}$,

$$\lambda_n x^{(1)} + R \leq \lambda_{N_{j-1}} a' + R < \lambda_{M_j} a - R \leq \lambda_m y^{(1)} - R. \tag{3.1}$$

The way we choose to initialize the construction (with $M_1 a > 2R$) guarantees that $Q(0,R)$ is also disjoint from all these hypercubes, and so our construction satisfies Lemma 3.1(i).

For each $j = 1,\dots,J$, the set

$$X_j = \bigcup_{n=M_j}^{N_j} \bigcup_{k=1}^{p_n} Q(\lambda_n x_{n,k}, R)$$

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is polynomially convex, and an easy induction based on Lemma 2.3 and (3.1) ensures that

$$Q(0,R)\cup\bigcup_{j=1}^{J}\bigcup_{n=M_j}^{N_j}\bigcup_{k=1}^{p_n}Q(\lambda_n x_{n,k},R)=Q(0,R)\cup\bigcup_{j=1}^{J}X_j$$

is polynomially convex. We have verified (i), (ii), and (iii) of Lemma 3.1. This concludes the proof. ∎

Proposition 3.3 *Let K be a compact subset of $(0,+\infty)^{2d}$. Then $\bigcap_{a\in K}HC(\tau_a)$ is a residual subset of $H(\mathbb{C}^d)$.*

Proof Without loss of generality, we can assume that $K=\prod_{\ell=1}^{2d}[a_\ell,a'_\ell]$. We intend to apply Lemma 3.2. Thus, let $R,\varepsilon>0$. We first apply Lemma 2.5 to $A=4R/\varepsilon$ to get some $\rho>1$ and some sequence of integers (μ_n) with $\mu_{n+1}\geq\rho\mu_n$. We then define $\gamma>0$ as any positive real number such that, given any $x\in K$, $\rho x^{(\ell)}-x^{(\ell)}-\gamma>0$ for all $\ell=1,\dots,2d$. Now let L be a compact hypercube in K with diameter less than γ and let $M\in\mathbb{N}$. Without loss of generality, we can assume that $L=\prod_{\ell=1}^{2d}[b_\ell,b_\ell+\gamma]$. We then apply Lemma 2.5 with $P\geq M$ such that

$$\mu_P\inf_{\ell=1,\dots,2d}(\rho b_\ell-b_\ell-\gamma)>2R.$$

We get maps $s_1,\dots,s_{2d}$ and ϕ. We can now define our covering of L. Bearing in mind that the domain of ϕ is finite, we set

$$n_0=\min_{(k_1,\dots,k_{2d})}\phi(k_1,\dots,k_{2d})\geq M,\ N=\max_{(k_1,\dots,k_{2d})}\phi(k_1,\dots,k_{2d})$$

and let $n\in\{n_0,\dots,N\}$. Then either n is not a $\phi(k_1,\dots,k_{2d})$, in which case we set $p_n=0$, that is, we do nothing; or n is equal to $\phi(k_1,\dots,k_{2d})$ for a (necessarily) unique $(k_1,\dots,k_{2d})$. We then define the set $\{x_{n,k}\}_{1\leq k\leq p_n}$ as

$$\begin{aligned}L\cap\Big\{\Big(&b_1+\frac{4R\alpha_1}{\mu_{\phi(0,\dots,0)}}+\frac{\varepsilon}{\mu_{\phi(1,0,\dots,0)}}+\cdots+\frac{\varepsilon}{\mu_{\phi(k_1,0,\dots,0)}},\\&b_2+\frac{4R\alpha_2}{\mu_{\phi(k_1,0,\dots,0)}}+\frac{\varepsilon}{\mu_{\phi(k_1,1,0,\dots,0)}}+\cdots+\frac{\varepsilon}{\mu_{\phi(k_1,k_2,\dots,0)}},\\&\quad\vdots\\&b_{2d}+\frac{4R\alpha_{2d}}{\mu_{\phi(k_1,\dots,k_{2d-1},0)}}+\frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{2d-1},1)}}+\cdots+\frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{2d})}}\Big)\\&\alpha_1,\dots,\alpha_{2d}\in\mathbb{N}_0\Big\}.\end{aligned}$$

We also set $\lambda_n=\mu_{\phi(k_1,\dots,k_{2d})}$ and we show that the assumptions of Lemma 3.2 are satisfied. First of all, the hypercubes $Q(\lambda_n x_{n,k},R)$ are pairwise disjoint. Indeed, let $(n,k)\neq(m,j)$. Then we have two cases:

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- $n \neq m$: for instance, $n < m$. In this case, looking at the first coordinate of $\lambda_n x_{n,k}$ and $\lambda_m x_{m,j}$, we get, using the fact that $\phi(k_1, \dots, k_{2d}) \geq P$:

$$|\lambda_m x_{m,j} - \lambda_n x_{n,k}| \geq \lambda_m b_1 - \lambda_n(b_1 + \gamma) \geq \rho\lambda_n b_1 - \lambda_n(b_1 + \gamma) \geq \mu_P(\rho b_1 - b_1 - \gamma) > 2R. \tag{3.2}$$

- $n = m$: Then $x_{n,k}$ and $x_{n,j}$ may be written as above, with two different sequences $(\alpha_1, \dots, \alpha_{2d})$ and $(\beta_1, \dots, \beta_{2d})$. Let $\ell \in \{1, \dots, 2d\}$ be such that $\beta_\ell \neq \alpha_\ell$. Looking now at this coordinate, we get

$$|\lambda_n x_{n,k} - \lambda_n x_{n,j}| \geq \frac{4R\lambda_n}{\mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots)}} > 2R, \tag{3.3}$$

since $\lambda_n = \mu_{\phi(k_1,\dots,k_{2d})} \geq \mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots)}$.

The covering property is also easy to verify using the construction of $(x_n)_{n,k}$. Let $x \in L$. There exists $\alpha_1 \in \mathbb{N}_0$ such that

$$b_1 + \frac{4R\alpha_1}{\mu_{\phi(0,\dots,0)}} \leq x^{(1)} \leq b_1 + \frac{4R(\alpha_1 + 1)}{\mu_{\phi(0,\dots,0)}}.$$

Now, by construction of ϕ, using Lemma 2.5 (recall that $A = 4R/\varepsilon$), there exists $k_1 < s_1$ such that

$$b_1 + \frac{4R\alpha_1}{\mu_{\phi(0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1,0,\dots,0)}} \leq x^{(1)} \leq b_1 + \frac{4R\alpha_1}{\mu_{\phi(0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1+1,0,\dots,0)}}.$$

This k_1 being fixed, there exists $\alpha_2 \geq 0$ such that

$$b_2 + \frac{4R\alpha_2}{\mu_{\phi(k_1,0,\dots,0)}} \leq x^{(2)} \leq b_2 + \frac{4R(\alpha_2 + 1)}{\mu_{\phi(k_1,0,\dots,0)}}.$$

Iterating this construction, we find $\alpha_1, \dots, \alpha_{2d} \geq 0$ and $k_1, \dots, k_{2d}$ such that, for all $\ell = 1, \dots, 2d$,

$$b_\ell + \frac{4R\alpha_\ell}{\mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},k_\ell,0,\dots,0)}} \leq x^{(\ell)} \leq b_\ell + \frac{4R\alpha_\ell}{\mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},k_\ell+1,0,\dots,0)}}.$$

Let $n = \phi(k_1, \dots, k_{2d})$ and let $x_{n,k}$ correspond to these values of $\alpha_1, \dots, \alpha_{2d}$. Then,

$$|\lambda_n x - \lambda_n x_{n,k}| \leq \mu_{\phi(k_1,\dots,k_{2d})} \times \sup_{\ell=1,\dots,2d} \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_\ell+1,0,\dots,0)}} \leq \varepsilon.$$

It remains to be shown that $\bigcup_{M \leq n \leq N,\ 1 \leq k \leq p_n} Q(\lambda_n x_{n,k}, R)$ is polynomially convex, bearing in mind that we are only taking $n \geq n_0$. For such $M \leq n \leq N$, $n = \phi(k_1, \dots, k_{2d})$, we set $H_n = \bigcup_{1 \leq k \leq p_n} Q(\lambda_n x_{n,k}, R)$, and we first show that H_n is polynomially convex. For $\ell = 1, \dots, 2d$, let $\Omega_\ell \geq 0$ be the greatest integer such that

$$b_\ell + \frac{4R\Omega_\ell}{\mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},k_\ell+1,0,\dots,0)}} \leq b_\ell + \gamma.$$

For $0 \le j \le \Omega_\ell$, we also set

$$y_j^{(\ell)} = b_\ell + \frac{4Rj}{\mu_{\phi(k_1,\dots,k_{\ell-1},0,\dots,0)}} + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},1,0,\dots,0)}} + \cdots + \frac{\varepsilon}{\mu_{\phi(k_1,\dots,k_{\ell-1},k_\ell+1,0,\dots,0)}}$$

so that

$$\{x_{n,k};\ 1 \le k \le p_n\} = \Big\{ \big(y_{j_1}^{(1)}, \dots, y_{j_{2d}}^{(2d)} \big);\ 0 \le j_\ell \le \Omega_\ell,\ \ell = 1,\dots,2d \Big\}.$$

Since, as observed above (see (3.3)), $|y_{j_l}^{(l)} - y_{j_l'}^{(l)}| > 2R$ if $j_\ell \ne j_\ell'$, it follows from Lemma 2.4 that H_n is polynomially convex. Bearing in mind that $H_n = \varnothing$, for $n < n_0$, we then conclude that $H_M \cup \cdots \cup H_N$ is polynomially convex by an easy induction using either Lemma 2.3 or Lemma 2.4. Indeed, for $n = n_0, \dots, m_0 - 1$, for any $1 \le k \le p_n$ and any $1 \le j \le p_m$,

$$\lambda_n x_{n,k}^{(1)} + R \le \lambda_n (b_1 + \gamma) + R < \lambda_{n+1} b_1 - R \le \lambda_{n+1} x_{n+1,j}^{(1)} - R. \tag{3.4}$$ ∎

Proof of Theorem 1.1 So far, we have shown that if K is a compact subset of $(0,+\infty)^{2d}$, then $\bigcap_{a\in K} HC(\tau_a)$ is a residual subset of $H(\mathbb{C}^d)$. This property remains true if $K = K_1 \times \cdots \times K_{2d}$ where each K_i is either a subset of $(0,+\infty)$; or a subset of $(-\infty,0)$; or $K_i = \{0\}$ and at least one K_i, say K_{i_0}, is different from $\{0\}$. The construction is exactly similar except that, on each coordinate such that $K_i = \{0\}$, we do nothing (we fix $x_{n,k}^{(i)} = 0$) and, wherever we need a separation property (see for instance (3.1), (3.2), (3.4)), we look at the i_0-th coordinate. Moreover, in this case, the hypercubes K and L will have lower dimension. We finally conclude by writing $\mathbb{R}^{2d}\backslash\{0\}$ as a countable union of such compact sets. ∎

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