D. Yang
Nagoya Math. J.
Vol. 177 (2005), 1–29

SOME FUNCTION SPACES RELATIVE TO MORREY-CAMPANATO SPACES ON METRIC SPACES

DACHUN YANG

Abstract. In this paper, the author introduces the Morrey-Campanato spaces $L_p^s(X)$ and the spaces $C_p^s(X)$ on spaces of homogeneous type including metric spaces and some fractals, and establishes some embedding theorems between these spaces under some restrictions and the Besov spaces and the Triebel-Lizorkin spaces. In particular, the author proves that $L_p^s(X) = B_{\infty,\infty}^s(X)$ if $0 < s < \infty$ and $\mu(X) < \infty$. The author also introduces some new function spaces $A_p^s(X)$ and $B_p^s(X)$ and proves that these new spaces when $0 < s < 1$ and $1 < p < \infty$ are just the Triebel-Lizorkin space $F_{p,\infty}^s(X)$ if X is a metric space, and the spaces $A_p^1(X)$ and $B_p^1(X)$ when $1 < p \le \infty$ are just the Hajłasz-Sobolev spaces $W_p^1(X)$. Finally, as an application, the author gives a new characterization of the Hajłasz-Sobolev spaces by making use of the sharp maximal function.

§1. Introduction

On metric spaces including fractals, how to reasonably introduce some well-known functions on the Euclidean spaces is the main subject of a lot of recent papers and books; see [26], [21], [29], [30], [16], [17]. The main purpose of this paper is to introduce the Morrey-Campanato spaces $L_p^s(X)$ and the spaces $C_p^s(X)$ on spaces of homogeneous type including metric spaces and some fractals, whose versions on $\mathbb{R}^n$ and its domains are studied by DeVore and Sharpley in [6], Christ in [3] and Miyachi in [24], [25]; see also [28] for more references. Moreover, $L_1^0(\Omega)$ is just the usual space $bmo(\Omega)$ if Ω is a bounded C^∞ domain in $\mathbb{R}^n$; see [28, p. 49]. We will establish some embedding theorems between these spaces under some restrictions and the Besov spaces and the Triebel-Lizorkin spaces in [12], [13], [16], [17]. In particular, we will prove that $L_p^s(X) = B_{\infty,\infty}^s(X)$ if $0 < s < \infty$ and $\mu(X) < \infty$ (see Theorem 2.1 below), which is known if X is a bounded C^∞

Received October 10, 2002.
2000 Mathematics Subject Classification: Primary 42B35; Secondary 46E35, 43-99.
This research was supported by the NNSF (No. 10271015) and the RFDP (No. 20020027004) of China.

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

domain in $\mathbb{R}^n$; see [28, pp. 50, 247–248]. Motivated by [21], we also introduce some new function spaces $A_p^s(X)$ and $B_p^s(X)$, which can be regarded as the fractional versions of the function spaces studied in [21]. However, the metric spaces studied in [21] have the segment property and we do not need this property by assuming some other properties. It is easy to find a metric space satisfying our assumptions (see Definition 2.1 below), which has no segment property. For example, consider $X = [0,1] \cup [2,3]$ with the euclidean distance and the 1-dimensional Lebesgue measure. We will prove that these new spaces $A_p^s(X)$ and $B_p^s(X)$, when $0 < s < 1$, $1 < p < \infty$ and X is a metric space, are just the Triebel-Lizorkin space $F_{p,\infty}^s(X)$ (see Theorem 2.4 below), and that the spaces $A_p^1(X)$ and $B_p^1(X)$, when $1 < p \le \infty$ and X is a metric space, are just the Hajłasz-Sobolev spaces $W_p^1(X)$ in [9] (see Theorem 3.3 below). Finally, as an application of this result, we will establish a new characterization of the Sobolev space $W_p^1(X)$ by means of the sharp maximal function introduced by Triebel in [28, p. 246] (see Theorem 3.4 below), which is also known if X is a bounded C^∞ domain in $\mathbb{R}^n$; see [28, pp. 50, 247–248].

We remark that although some of our results are known if X is a bounded C^∞ domain in $\mathbb{R}^n$, some new ideas and techniques are needed to obtain their counterparts on spaces of homogeneous type. In particular, we need the Calderón reproducing formulae established by Han in [11] and we will also use some ideas from [21].

Section 2 is devoted to the study of the Morrey-Campanato spaces and the $C_p^s(X)$ spaces, and the new characterization of the Hajłasz-Sobolev spaces $W_p^1(X)$ is given in Section 3.

§2. Morrey-Campanato spaces and C_p^s spaces

Let us first recall some definitions and notation on spaces of homogeneous type. A quasi-metric ρ on a set X is a function $\rho\colon X \times X \to [0,\infty)$ satisfying that

(i) $\rho(x,y) = 0$ if and only if $x = y$;

(ii) $\rho(x,y) = \rho(y,x)$ for all x, $y \in X$;

(iii) there exists a constant $A \in [1,\infty)$ such that for all x, y and $z \in X$,

$$\rho(x,y) \le A[\rho(x,z) + \rho(z,y)].$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Any quasi-metric defines a topology, for which the balls

$$B(x,r)=\{y\in X: \rho(y,x)<r\}$$

for all $x\in X$ and all $r>0$ form a basis.

In what follows, we set $\operatorname{diam} X=\sup\{\rho(x,y): x,\, y\in X\}$. We also make the following conventions. We denote by $f\sim g$ that there is a constant $C>0$ independent of the main parameters such that $C^{-1}g<f<Cg$. Throughout the paper, we will denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote $\mathbb{N}\cup\{0\}$ simply by $\mathbb{Z}_+$ and for any $q\in[1,\infty]$, we denote by q' its conjugate index, namely, $1/q+1/q'=1$. If X_1 and X_2 are two quasi-Banach spaces, $B_1\subset B_2$ means that there is a constant $C>0$ such that for all $f\in B_1$,

$$\|f\|_{B_2}\le C\|f\|_{B_1}.$$

Definition 2.1. ([16]) Let $d>0$ and $0<\theta\le 1$. A space of homogeneous type, $(X,\,\rho,\,\mu)_{d,\theta}$, is a set X together with a quasi-metric ρ and a nonnegative Borel regular measure μ on X with $\operatorname{supp}\mu=X$ such that for some constant $C_0>0$ and for all $0<r<\operatorname{diam} X$ and all $x,\, x',\, y\in X$,

$$\mu(B(x,r))\sim r^d \tag{2.1}$$

and

$$|\rho(x,y)-\rho(x',y)|\le C_0\rho(x,x')^\theta[\rho(x,y)+\rho(x',y)]^{1-\theta}. \tag{2.2}$$

Obviously, d can be regarded as the Hausdorff dimension of X (see [23]). Moreover, if ρ is a metric, then θ in (2.2) can be 1; and, if $X=\mathbb{R}^n$, ρ is the usual Euclidean metric and μ is the n-dimensional Lebesgue measure, then $d=n$ and $\theta=1$.

Space of homogeneous type defined above is a variant of space of homogeneous type introduced by Coifman and Weiss in [4]. In [22], Macias and Segovia have proved that one can replace the quasi-metric ρ of space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar\rho$ which yields the same topology on X as ρ such that $(X,\,\bar\rho,\,\mu)$ is the space defined by Definition 2.1 with $d=1$.

Moreover, the spaces of homogeneous type in Definition 2.1 include the Euclidean space, the C^∞-compact Riemannian manifolds, the boundaries of Lipschitz domains and, in particular, the Lipschitz manifolds introduced recently by Triebel in [31] and the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [29] that the isotropic and anisotropic d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set (see also [23]), the generalized Sierpinski carpet, the fern-like fractals, Picasso-Xmas-Tree fractals and Oval-Ferny fractals; see [30], [1], [2] and [8]. We particularly point out that the spaces of homogeneous type in Definition 2.1 also include the post critically finite self-similar fractals studied by Kigami in [20] and by Strichartz in [26], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau in [8].

We now recall the definition of the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. To do so, let us first recall the definition of the spaces of test functions on X in [15]; see also [11].

Definition 2.2. Fix $\gamma > 0$ and $\theta \geq \beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \leq C_1 \dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \leq C_1 \left(\dfrac{\rho(x, y)}{r + \rho(x, x_0)}\right)^\beta \dfrac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$

$$\text{for } \rho(x, y) \leq \frac{1}{2A}[r + \rho(x, x_0)],$$

where $C_1 > 0$ is independent of x, y and r. If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} = \inf\{C_1 : \text{(i) and (ii) hold}\}.$$

Here and in what follows, θ is the same as in (2.2).

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with the equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$.

Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a finite constant $C>0$ such that for all $f\in\mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)|\le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h,f\rangle$ the natural pairing of elements $h\in(\mathcal{G}(\beta,\gamma))'$ and $f\in\mathcal{G}(\beta,\gamma)$. It is easy to see that, for all $h\in(\mathcal{G}(\beta,\gamma))'$, $\langle h,f\rangle$ is well defined for all $f\in\mathcal{G}(x_0,r,\beta,\gamma)$ with $x_0\in X$ and $r>0$. Moreover, in what follows, we will denote by $\mathring{\mathcal{G}}(\beta,\gamma)$, for $0<\beta,\ \gamma<\theta$, the completion of $\mathcal{G}(\theta,\theta)$ in $\mathcal{G}(\beta,\gamma)$.

To state the definition of the inhomogeneous Besov spaces $B^s_{p,q}(X)$ and the inhomogeneous Triebel-Lizorkin spaces $F^s_{p,q}(X)$ studied in [12], we need the following approximations to the identity which were first introduced in [11].

Definition 2.3. A sequence $\{S_k\}_{k=0}^\infty$ of linear operators is said to be an approximation to the identity of order $\epsilon\in(0,\theta]$ if there exist C_2, $C_3>0$ such that for all $k\in\mathbb{Z}_+$ and all x, x', y and $y'\in X$, $S_k(x,y)$, the kernel of S_k is a function from $X\times X$ into $\mathbb{C}$ satisfying

(i) $S_k(x,y)=0$ if $\rho(x,y)\ge C_2 2^{-k}$ and $\|S_k\|_{L^\infty(X\times X)}\le C_3 2^{dk}$;

(ii) $|S_k(x,y)-S_k(x',y)|\le C_3 2^{k(d+\epsilon)}\rho(x,x')^\epsilon$;

(iii) $|S_k(x,y)-S_k(x,y')|\le C_3 2^{k(d+\epsilon)}\rho(y,y')^\epsilon$;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|$
$\qquad\le C_3 2^{k(d+2\epsilon)}\rho(x,x')^\epsilon\rho(y,y')^\epsilon$;

(v) $\displaystyle\int_X S_k(x,y)\,d\mu(y)=1$;

(vi) $\displaystyle\int_X S_k(x,y)\,d\mu(x)=1$.

Here, that $S_k(x,y)$ is the kernel of S_k means that for suitable functions f,

$$S_kf(x)=\int_X S_k(x,y)f(y)\,d\mu(y).$$

We point out that by a similar Coifman's construction to that in [5], one can construct an approximation to the identity with compact supports as in Definition 2.3 for those spaces of homogeneous type in Definition 2.1.

Now, we can introduce the spaces $B^s_{p,q}(X)$ and $F^s_{p,q}(X)$ via the approximations to the identity defined above, which were first studied in [12].

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Definition 2.4. Suppose $s \in (-\theta, \theta)$ and that $\{S_k\}_{k=0}^{\infty}$ is an approximation to the identity of order θ and let $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. The inhomogeneous Besov space $B^s_{p,q}(X)$ for $1 \le p,\, q \le \infty$ is the collection of $f \in \big(\mathring{\mathcal{G}}(\beta, \gamma)\big)'$ for $|s| < \beta < \theta$ and $0 < \gamma < \theta$ such that

$$\|f\|_{B^s_{p,q}(X)} = \left\{\sum_{k=0}^{\infty} \Big[2^{ks}\|D_k(f)\|_{L^p(X)}\Big]^q\right\}^{1/q} < \infty.$$

The inhomogeneous Triebel-Lizorkin space $F^s_{p,q}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$ is the collection of $f \in \big(\mathring{\mathcal{G}}(\beta, \gamma)\big)'$ for $|s| < \beta < \theta$ and $0 < \gamma < \theta$ such that

$$\|f\|_{F^s_{p,q}(X)} = \left\|\left\{\sum_{k=0}^{\infty} \big[2^{ks}|D_k(f)|\big]^q\right\}^{1/q}\right\|_{L^p(X)} < \infty.$$

It was proved in [12] that the above definitions of the spaces $B^s_{p,q}(X)$ and $F^s_{p,q}(X)$ are independent of the choices of approximations to the identity and the pair (β, γ) with $\max(0, -s) < \beta < \theta$ and $0 < \gamma < \theta$. Moreover, in [16], it was also proved that the above definitions are also independent of the equivalent quasi-metrics satisfying (2.2). We say that a quasi-metric ρ is equivalent to another quasi-metric ρ' if there is a constant $C > 0$ such that for all $x,\, y \in X$,

$$C^{-1}\rho'(x, y) \le \rho(x, y) \le C\rho'(x, y).$$

Moreover, it was proved in [32] that the Besov spaces on d-sets in $\mathbb{R}^n$ defined by two different and equivalent methods, namely, traces and quarkonial decompositions in the sense of Triebel in [29], [30] are the same spaces as those introduced in [12], [17] by regarding the d-set as a space of homogeneous type when $0 < s < 1$, $1 < p < \infty$ and $1 \le q \le \infty$.

For $s \in \mathbb{R}$, $C_4 > 0$, $u \in (0, \infty]$ and $x \in X$, we introduce the sharp maximal function

$$f^s_{u,C_4}(x) = \sup_{0<t<C_4} t^{-s}\left(\fint_{B(x,t)}\left|f(y) - \fint_{B(x,t)} f(z)\, d\mu(z)\right|^u d\mu(y)\right)^{1/u},$$

where $\fint_{B(x,t)} f(y)\, d\mu(y)$ means the average on $B(x,t)$ of f, that is,

$$\fint_{B(x,t)} f(y)\, d\mu(y) = \frac{1}{\mu(B(x,t))}\int_{B(x,t)} f(y)\, d\mu(y).$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Using this sharp maximal function, we can now introduce the so-called Morrey-Campanato spaces and the spaces $C_p^s(X)$, whose versions on $\mathbb{R}^n$ and its domains have been introduced by DeVore and Sharpley in [6] and Christ in [3]; see also [24], [25] and [28, pp. 48–49, 246] for some applications of these spaces, the detailed references, further explanations and a short history of the versions on $\mathbb{R}^n$ and its domains of these spaces.

DEFINITION 2.5. Let $C_4 > 0$.

(i) Let $1 \le p < \infty$ and $-d/p \le s < 1$. Then

$$L_p^s(X) = \Big\{ f \in L^p(X) : \|f\|_{L_p^s(X)} = \|f\|_{L^p(X)} + \sup_{x\in X} f_{p,C_4}^s(x) < \infty \Big\}.$$

(ii) Let $0 < s < 1$, $0 < p \le \infty$ and $\bar{p} = \max(1, p)$. Then

$$C_p^s(X) = \Big\{ f \in L^{\bar{p}}(X) : \|f\|_{C_p^s(X)} = \|f\|_{L^p(X)} + \|f_{p,C_4}^s\|_{L^p(X)} < \infty \Big\}.$$

Remark 2.1. It is easy to see that the definition of the spaces $L_p^s(X)$ and $C_p^s(X)$ are independent of the choice of $C_4 > 0$.

The following is one of the main theorems of this section.

THEOREM 2.1. *Let* $1 \le p < \infty$.

(i) $L_p^{-d/p}(X) = L^p(X)$ *with equivalent norms, and for* $\min(-\theta, -d/p) < s < \theta$, $L_p^s(X) \subset B_{\infty,\infty}^s(X)$, *that is, there is a constant* $C > 0$ *such that for all* $f \in L_p^s(X)$,

$$\|f\|_{B_{\infty,\infty}^s(X)} \le C\|f\|_{L_p^s(X)}.$$

(ii) *If* $0 < s < \theta$ *and* $\mu(X) < \infty$, *then* $B_{\infty,\infty}^s(X) \subset L_p^s(X)$, *that is, there is a constant* $C > 0$ *such that for all* $f \in B_{\infty,\infty}^s(X)$,

$$\|f\|_{L_p^s(X)} \le C\|f\|_{B_{\infty,\infty}^s(X)}.$$

Remark 2.2. The space $B_{\infty,\infty}^s(X)$ for $0 < s < \theta$ is usually called the Hölder-Zygmund space; see [28], [27].

To establish Theorem 2.1, we need the following inhomogeneous Calderón reproducing formulae established in [11].

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

LEMMA 2.1. *Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity of order $\epsilon_1 \in (0,\theta]$ as defined in Definition 2.3. Let $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist a family of linear operators $\widetilde{D}_k$ for $k \in \mathbb{Z}_+$ such that for $f \in \mathcal{G}(\beta_1,\gamma_1)$ with $0 < \beta_1,\ \gamma_1 < \epsilon_1$,*

$$f = \sum_{k=0}^{\infty} D_k \widetilde{D}_k(f), \tag{2.3}$$

where the series converge in the norm of $\mathcal{G}(\beta_1',\gamma_1')$ for $0 < \beta_1' < \beta_1$ and $0 < \gamma_1' < \gamma_1$. Moreover, the kernel, $\widetilde{D}_k(x,y)$, of the operator $\widetilde{D}_k$ for $k \in \mathbb{Z}_+$ satisfies the conditions

(i) $\displaystyle |\widetilde{D}_k(x,y)| \le C\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$,

(ii) $\displaystyle |\widetilde{D}_k(x,y) - \widetilde{D}_k(x,y')| \le C\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$

$$\textit{for}\ \ \rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y)),$$

and

(iii) $\displaystyle \int_X \widetilde{D}_k(x,y)\,d\mu(y) = \int_X \widetilde{D}_k(x,y)\,d\mu(x) = \begin{cases} 1, & k=0; \\ 0, & k\in\mathbb{N}, \end{cases}$

where $\epsilon \in (0,\epsilon_1)$.

The following lemma can be found in [12]; see also [15].

LEMMA 2.2. *Let $\{\widetilde{D}_k\}_{k\in\mathbb{Z}_+}$ be as in Lemma 2.1 with $\theta > |s|$.*

(i) *For $1 \le p,\ q \le \infty$ and all $f \in B^s_{p,q}(X)$,*

$$\left\{\sum_{k=0}^{\infty} 2^{ksq}\|\widetilde{D}_k f\|^q_{L^p(X)}\right\}^{1/q} \le C\|f\|_{B^s_{p,q}(X)},$$

where C is independent of f.

(ii) *For $1 < p < \infty$, $1 < q \le \infty$ and all $f \in F^s_{p,q}(X)$,*

$$\left\|\left\{\sum_{k=0}^{\infty} 2^{ksq}|\widetilde{D}_k f|^q\right\}^{1/q}\right\|_{L^p(X)} \le C\|f\|_{F^s_{p,q}(X)},$$

where C is independent of f.

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Proof of Theorem 2.1. We first show (i). Let $f \in L_p^{-d/p}(X)$. From Definition 2.5, we know that $f \in L^p(X)$ and

$$\|f\|_{L^p(X)} \le \|f\|_{L_p^{-d/p}(X)}. \tag{2.4}$$

We now suppose that $f \in L^p(X)$ and the Hölder inequality and $\mu(B(x,t)) \sim t^d$ tell us that for all $x \in X$,

$$\begin{aligned} f_{p,C_4}^{-d/p}(x) &= \sup_{0<t<C_4} t^{d/p}\left(\fint_{B(x,t)}\left|f(y) - \fint_{B(x,t)} f(z)\,d\mu(z)\right|^p d\mu(y)\right)^{1/p} \\ &\le C\|f\|_{L^p(X)}. \end{aligned}$$

Thus, $f \in L_p^{-d/p}(X)$ and

$$\|f\|_{L_p^{-d/p}(X)} = \|f\|_{L^p(X)} + \sup_{x\in X} f_{p,C_4}^{-d/p}(x) \le C\|f\|_{L^p(X)}. \tag{2.5}$$

The estimates (2.4) and (2.5) show that $L_p^{-d/p}(X) = L^p(X)$ with equivalent norms.

Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. Let $\min(-\theta, -d/p) < s < \theta$ and $f \in L_p^s(X)$. We denote $B_1 = B(x, 2C_1 2^{-k})$. The Hölder inequality yields that

$$\begin{aligned} |D_0 f(x)| &= \left|\int_X D_0(x,y) f(y)\,d\mu(y)\right| \\ &\le \|f\|_{L^p(X)}\left\{\int_X |D_0(x,y)|^{p'}\,d\mu(y)\right\}^{1/p'} \\ &\le C\|f\|_{L^p(X)}, \end{aligned} \tag{2.6}$$

and, by the fact that

$$\int_X D_k(x,y) f(y)\,d\mu(y) = 0 \tag{2.7}$$

and $\operatorname{supp} D_k(x,\,\cdot\,) \subset B_1$ for $k \in \mathbb{N}$, we have

$$\begin{aligned} |D_k f(x)| &= \left|\int_X D_k(x,y) f(y)\,d\mu(y)\right| \\ &= \left|\int_X D_k(x,y)\left[f(y) - \fint_{B_1} f(z)\,d\mu(z)\right] d\mu(y)\right| \end{aligned} \tag{2.8}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

$$
\begin{aligned}
&\le C \fint_{B_1} \left| f(y) - \fint_{B_1} f(z)\, d\mu(z) \right| d\mu(y) \\
&\le C \left\{ \fint_{B_1} \left| f(y) - \fint_{B_1} f(z)\, d\mu(z) \right|^p d\mu(y) \right\}^{1/p}.
\end{aligned}
$$

Thus, the estimates (2.6) and (2.8) tell us that

$$
\begin{aligned}
\|f\|_{B^s_{\infty,\infty}(X)} &= \sup_{k \in \mathbb{Z}_+} \sup_{x \in X} 2^{ks} |D_k f(x)| \\
&\le C\|f\|_{L^p(X)} + C \sup_{x \in X} f^s_{p,2C_1}(x) \\
&\le C\|f\|_{L^s_p(X)}.
\end{aligned}
$$

This proves (i).

We now turn to the proof of (ii). Let $f \in B^s_{\infty,\infty}(X)$. By the Hölder inequality, we have that

$$
\begin{aligned}
\|f\|_{L^p(X)} &= \left\{ \int_X \left| \sum_{k=0}^{\infty} D_k f(x) \right|^p d\mu(x) \right\}^{1/p} \\
&\le \left\{ \int_X \left[\sum_{k=0}^{\infty} 2^{-ks} \sup_{k \in \mathbb{Z}_+,\, x \in X} 2^{ks} |D_k f(x)| \right]^p d\mu(x) \right\}^{1/p} \\
&\le C\mu(X) \|f\|_{B^s_{\infty,\infty}(X)}.
\end{aligned} \tag{2.9}
$$

On the other hand, let $C_4 = 1$ and $2^{-k_0-1} \le t < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}_+$. Let $y,\, z \in B(x, 2^{-k_0})$. By Lemma 2.1, we decompose $f(y) - f(z)$ into

$$
\begin{aligned}
f(y) - f(z) &= \sum_{k=0}^{\infty} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\
&= \sum_{k=0}^{k_0} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\
&\quad + \sum_{k=k_0+1}^{\infty} \left[D_k \widetilde{D}_k f(y) - D_k \widetilde{D}_k f(z) \right] \\
&= I_1 + I_2.
\end{aligned} \tag{2.10}
$$

For I_1, noting that $\rho(y,z) \le C2^{-k_0}$ if $\rho(y,x) \le C2^{-k_0}$ and $\rho(z,x) \le C2^{-k_0}$, by Lemma 2.2, we have

(2.11)

$$
\begin{aligned}
|I_1| &\le \sum_{k=0}^{k_0} \int_X |D_k(y,u) - D_k(z,u)| \big|\widetilde{D}_k f(u)\big| \, d\mu(u) \\
&\le C \sum_{k=0}^{k_0} 2^{k(d+\theta)} \int_{\{u\in X: \rho(y,u)\le C2^{-k} \text{ or } \rho(z,u)\le C2^{-k}\}} \rho(y,z)^\theta \big|\widetilde{D}_k f(u)\big| \, d\mu(u) \\
&\le C \sum_{k=0}^{k_0} 2^{-(k_0-k)\theta} \sup_{u\in X} \big|\widetilde{D}_k f(u)\big| \\
&= C 2^{-k_0 s} \left[\sum_{k=0}^{k_0} 2^{-(k_0-k)(\theta-s)} \right] \sup_{k\in\mathbb{Z}_+,\, u\in X} 2^{ks} \big|\widetilde{D}_k f(u)\big| \\
&\le C 2^{-k_0 s} \|f\|_{B^s_{\infty,\infty}(X)},
\end{aligned}
$$

and for I_2, we have

$$
\begin{aligned}
|I_2| &\le \sum_{k=k_0+1}^{\infty} \Big[\big|D_k \widetilde{D}_k f(y)\big| + \big|D_k \widetilde{D}_k f(z)\big| \Big] \qquad (2.12) \\
&\le \sum_{k=k_0+1}^{\infty} 2^{-ks} \left[\int_X |D_k(y,u)| d\mu(u) + \int_X |D_k(z,u)| d\mu(u) \right] \\
&\quad \times \sup_{k\in\mathbb{Z}_+,\, u\in X} 2^{ks} \big|\widetilde{D}_k f(u)\big| \\
&\le C 2^{-k_0 s} \|f\|_{B^s_{\infty,\infty}(X)}.
\end{aligned}
$$

The definition of $f^s_{p,1}$ and the estimates (2.11) and (2.12) imply that

(2.13)

$$
\begin{aligned}
f^s_{p,1}(x) &= \sup_{0<t<1} t^{-s} \left(\fint_{B(x,t)} \left| f(y) - \fint_{B(x,t)} f(z)\, d\mu(z) \right|^p d\mu(y) \right)^{1/p} \\
&\le \sup_{0<t<1} t^{-s} \left(\fint_{B(x,t)} \left[\fint_{B(x,t)} |f(y) - f(z)|\, d\mu(z) \right]^p d\mu(y) \right)^{1/p} \\
&\le C \sup_{k_0\in\mathbb{Z}_+} 2^{k_0 s} \left(\fint_{B(x,2^{-k_0})} \left[\fint_{B(x,2^{-k_0})} |f(y) - f(z)|\, d\mu(z) \right]^p d\mu(y) \right)^{1/p} \\
&\le C \|f\|_{B^s_{\infty,\infty}(X)}.
\end{aligned}
$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Finally, (2.9) and (2.13) yield that

$$\|f\|_{L_p^s(X)} \le C\|f\|_{B^s_{\infty,\infty}(X)}.$$

This finishes the proof of Theorem 2.1. $\square$

On the relation between $C_p^s(X)$ and the Triebel-Lizorkin space $F^s_{p,q}(X)$, we have the following result.

THEOREM 2.2. *Let $0 < s < \theta$.*

(i) *If $1 < p < \infty$, then $C_p^s(X) \subset F^s_{p,\infty}(X)$, namely, there is a constant $C > 0$ such that for all $f \in C_p^s(X)$,*

$$\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{C_p^s(X)}.$$

(ii) *If $1 < p_2 < p_1 < \infty$ and $\mu(X) < \infty$, then $F^s_{p_1,\infty}(X) \subset C^s_{p_2}(X)$, namely, there is a constant $C > 0$ such that for all $f \in F^s_{p_1,\infty}(X)$,*

$$\|f\|_{C^s_{p_2}(X)} \le C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}.$$

Proof. We first show (i). Let $f \in C_p^s(X)$ and $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. We first have

$$|D_0 f(x)| = \left|\int_X S_0(x,y)f(y)\,d\mu(y)\right| \le CMf(x), \tag{2.14}$$

where M is the Hardy-Littlewood maximal function. Thus, for $1 < p < \infty$, by the $L^p(X)$-boundedness of M (see [4], [18]), we obtain

$$\|D_0 f\|_{L^p(X)} \le C\|Mf\|_{L^p(X)} \le C\|f\|_{L^p(X)}. \tag{2.15}$$

Let $B_1 = B(x, 2C_1 2^{-k})$. The estimate (2.8) tells us that for $k \in \mathbb{N}$ and all $x \in X$,

$$2^{ks}|D_k f(x)| \le Cf^s_{p,C_1}(x), \tag{2.16}$$

where C is independent of x. The estimates (2.15) and (2.16) yield that

$$C_p^s(X) \subset F^s_{p,\infty}(X)$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

and

$$\|f\|_{F^s_{p,\infty}(X)} = \Big\| \sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f| \Big\|_{L^p(X)} \leq C\|f\|_{L^p(X)} + C\big\|f^s_{p,C_1}\big\|_{L^p(X)} \leq C\|f\|_{C^s_p(X)}. \tag{2.17}$$

This proves (i).

We now turn to prove (ii). By the properties of $F^s_{p,q}(X)$ in [12], [14] (see also [16]), we have that

$$F^s_{p_1,\infty}(X) \subset F^0_{p_1,2}(X) = L^{p_1}(X) \subset L^{p_2}(X),$$

since $s>0$, $p_2<p_1$ and $\mu(X)<\infty$. Thus,

$$\|f\|_{L^{p_2}(X)} \leq \mu(X)^{1/p_2-1/p_1}\|f\|_{L^{p_1}(X)} \leq C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}. \tag{2.18}$$

Moreover, without loss of generality, we may assume that $C_4=1$ in the definition of $C^s_{p_2}(X)$ by Remark 2.1. Let $2^{-k_0-1}\leq t<2^{-k_0}$ for some $k_0\in\mathbb{Z}_+$ and we decompose $f(y)-f(z)$ as in (2.10) of the proof of Theorem 2.1 by means of Lemma 2.1. Then, for $y,\ z\in B(x,2^{-k_0})$,

$$\begin{aligned}
|I_1| &\leq \sum_{k=0}^{k_0}\int_X |D_k(y,u)-D_k(z,u)|\big|\widetilde{D}_k f(u)\big|\,d\mu(u)\\
&\leq C\sum_{k=0}^{k_0} 2^{k(d+\theta)}\int_{\{u\in X:\rho(y,u)\leq C2^{-k}\ \text{or}\ \rho(z,u)\leq C2^{-k}\}} \rho(y,z)^\theta\big|\widetilde{D}_k f(u)\big|\,d\mu(u)\\
&\leq C\sum_{k=0}^{k_0} 2^{(k-k_0)\theta}2^{kd}\int_{\{u\in X:\rho(y,u)\leq C2^{-k}\ \text{or}\ \rho(z,u)\leq C2^{-k}\}} \big|\widetilde{D}_k f(u)\big|\,d\mu(u)\\
&\leq C2^{-k_0 s}\sum_{k=0}^{k_0} 2^{(k-k_0)(\theta-s)}\\
&\quad\times\Bigg\{2^{kd}\int_{\{u\in X:\rho(y,u)\leq C2^{-k}\}}\Big(\sup_{k\in\mathbb{Z}_+}2^{ks}\big|\widetilde{D}_k f(u)\big|\Big)d\mu(u)\\
&\qquad + 2^{kd}\int_{\{u\in X:\rho(z,u)\leq C2^{-k}\}}\Big(\sup_{k\in\mathbb{Z}_+}2^{ks}\big|\widetilde{D}_k f(u)\big|\Big)d\mu(u)\Bigg\}
\end{aligned} \tag{2.19}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

$$\le C2^{-k_0 s}\left[\sum_{k=0}^{k_0} 2^{(k-k_0)(\theta-s)}\right]$$
$$\times \left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(y) + M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(z)\right\}$$
$$\le C2^{-k_0 s}\left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(y) + M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(z)\right\}$$

and

$$|I_2| \le \sum_{k=k_0+1}^{\infty} \int_X \left[|D_k(y,u)| + |D_k(z,u)|\right]\left|\widetilde{D}_k f(u)\right| d\mu(u) \tag{2.20}$$
$$\le C\left[\sum_{k=k_0+1}^{\infty} 2^{-ks}\right]$$
$$\times \left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(y) + M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(z)\right\}$$
$$\le C2^{-k_0 s}\left\{M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(y) + M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)(z)\right\}.$$

Then similarly to (2.13), the estimates (2.19) and (2.20) and the Minkowski inequality yield that

(2.21)

$$f^s_{p_2,1}(x)$$
$$\le C \sup_{k_0\in\mathbb{Z}_+} 2^{k_0 s}\left(\fint_{B(x,2^{-k_0})}\left[\fint_{B(x,2^{-k_0})} |f(y)-f(z)|\,d\mu(z)\right]^{p_2} d\mu(y)\right)^{1/p_2}$$
$$\le CM^2\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f(u)\right|\right)(x) + C\left\{M\left[M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f\right|\right)\right]^{p_2}\right\}^{1/p_2},$$

where M is the Hardy-Littlewood maximal operator and M^2 means $M\circ M$, the composition of M. The estimate (2.21), the Hölder inequality, Lemma 2.2 and the $L^p(X)$-boundedness of M for $p\in(1,\infty]$ then imply that

(2.22)

$$\|f^s_{p_2,1}\|_{L^{p_2}(X)} \le C\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\left|\widetilde{D}_k f(u)\right|\right\|_{L^{p_2}(X)}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

$$
\begin{aligned}
&+ C\mu(X)^{1/p_2-1/p_1}\left\|M\left[M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)\right]^{p_2}\right\|^{1/p_2}_{L^{p_1/p_2}(X)}\\
&\le C\mu(X)^{1/p_2-1/p_1}\left\{\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right\|_{L^{p_1}(X)}\right.\\
&\qquad \left. + \left\|M\left(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\right)\right\|_{L^{p_1}(X)}\right\}\\
&\le C\mu(X)^{1/p_2-1/p_1}\left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f(u)\big|\right\|_{L^{p_1}(X)}\\
&\le C\mu(X)^{1/p_2-1/p_1}\|f\|_{F^s_{p_1,\infty}(X)}.
\end{aligned}
$$

The estimates (2.18) and (2.22) imply (ii) and we finish the proof of Theorem 2.2. □

Motivated by [21], we now introduce some new function spaces on spaces of homogeneous type. These spaces under some restrictions will be proved to be the special cases of the Triebel-Lizorkin spaces.

Definition 2.6. Let $1 \le p \le \infty$ and $s > 0$. The space $B^s_p(X)$ is the set of functions $f \in L^p(X)$ satisfying that there exists a function $g \in L^p(X)$ such that

$$
\left| f(x) - \fint_B f(z)\, d\mu(z)\right| \le r(B)^s g(x) \tag{2.23}
$$

for μ-a. e. $x \in B$ and any ball $B \subset X$, where $r(B)$ is the radius of the ball B. Moreover, if $f \in B^s_p(X)$, we define its norm by

$$
\|f\|_{B^s_p(X)} = \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},
$$

where the infimum is taken over all functions g satisfying (2.23).

We remark that in some sense, the function g in Definition 2.6 behaves like the fractional derivative of f, which is also the main subject of [7].

The following theorem will indicate the relation between the space $B^s_p(X)$ and the Triebel-Lizorkin space.

Theorem 2.3. *Let $1 < p < \infty$ and $0 < s < \theta$. Then $B^s_p(X) = F^s_{p,\infty}(X)$ with equivalent norms.*

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Proof. Let $f \in F^s_{p,\infty}(X)$. If $r(B) \geq 1$, then, for $y \in B$, we have

$$\begin{aligned} (2.24)\qquad &\left|f(y) - \frac{1}{\mu(B)}\int_B f(z)\,d\mu(z)\right| \\ &= \left|\frac{1}{\mu(B)}\int_B [f(y) - f(z)]\,d\mu(z)\right| \\ &= \left|\frac{1}{\mu(B)}\int_B \sum_{k=0}^{\infty}\Big[D_k\widetilde{D}_k f(y) - D_k\widetilde{D}_k f(z)\Big]\,d\mu(z)\right| \\ &\leq C\sum_{k=0}^{\infty}\frac{1}{\mu(B)}\int_B \Big[M\big(\widetilde{D}_k f\big)(y) + M\big(\widetilde{D}_k f\big)(z)\Big]\,d\mu(z) \\ &\leq C\sum_{k=0}^{\infty} M^2\big(\widetilde{D}_k f\big)(y) \\ &\leq CM^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y)\sum_{k=0}^{\infty} 2^{-ks} \\ &\leq CM^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y) \\ &\leq Cr(B)^s M^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y), \end{aligned}$$

since $s > 0$, where we used the facts that

$$(2.25)\qquad |f(x)| \leq Mf(x)$$

and

$$(2.26)\qquad |D_k f(x)| \leq Mf(x).$$

If $2^{-(k_0+1)} \leq r(B) < 2^{-k_0}$ for some $k_0 \in \mathbb{Z}_+$, then for $y, z \in B$, we write $f(y) - f(z)$ as in (2.10) and the estimates (2.19) and (2.20) tell us that

$$|f(y) - f(z)| \leq Cr(B)^s\left\{M\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y) + M\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(z)\right\}.$$

Therefore, this estimate and (2.25) yield

$$(2.27)\qquad \left|f(y) - \frac{1}{\mu(B)}\int_B f(z)\,d\mu(z)\right| \leq Cr(B)^s M^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y)$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

for all $y \in B$.

Let

$$g(y) = CM^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)(y).$$

By Lemma 2.2 and the $L^p(X)$-boundedness of M for $p \in (1,\infty)$, we obtain

$$\begin{aligned}
\|g\|_{L^p(X)} &\le C\Big\|M^2\Big(\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big)\Big\|_{L^p(X)} \\
&\le C\Big\|\sup_{k\in\mathbb{Z}_+} 2^{ks}\big|\widetilde{D}_k f\big|\Big\|_{L^p(X)} \\
&= C\|f\|_{F^s_{p,\infty}(X)}.
\end{aligned} \tag{2.28}$$

The estimates (2.24), (2.27) and (2.28) show that $f \in B^s_p(X)$ and

$$\|f\|_{B^s_p(X)} \le C\|f\|_{F^s_{p,\infty}(X)}.$$

Now suppose $f \in B^s_p(X)$. By Definition 2.6, for any given $\epsilon > 0$, there exists a function $g \in L^p(X)$ such that (2.23) holds for any ball $B \subset X$ and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^s_p(X)} + \epsilon. \tag{2.29}$$

Let B_1 be the same as in the proof of Theorem 2.1. For $k \in \mathbb{N}$, by (2.7) and Definition 2.6, we have

$$\begin{aligned}
|D_k f(x)| &= \Big|\int_X D_k(x,y) f(y)\,d\mu(y)\Big| \\
&= \Big|\int_X D_k(x,y)\Big[f(y) - \fint_{B_1} f(z)\,d\mu(z)\Big]d\mu(y)\Big| \\
&\le C2^{-ks}\int_X |D_k(x,y)|\,|g(y)|\,d\mu(y) \\
&\le C2^{-ks} Mg(x).
\end{aligned} \tag{2.30}$$

The estimates (2.14) and (2.30) tell us that

$$\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f(x)| \le C[Mf(x) + Mg(x)]. \tag{2.31}$$

Finally the estimates (2.31) and (2.29) and the $L^p(X)$-boundedness of M yield that

$$\begin{aligned}\|f\|_{F^s_{p,\infty}(X)} &= \Big\|\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f|\Big\|_{L^p(X)}\\ &\le C\big\{\|Mf\|_{L^p(X)} + \|Mg\|_{L^p(X)}\big\}\\ &\le C\big\{\|f\|_{L^p(X)} + \|g\|_{L^p(X)}\big\}\\ &\le C\big[\|f\|_{B^s_p(X)} + \epsilon\big].\end{aligned}$$

Letting $\epsilon \to 0$, we obtain that $B^s_p(X) \subset F^s_{p,\infty}(X)$ and

$$\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{B^s_p(X)}.$$

This finishes the proof of Theorem 2.3. ∎

We now introduce some other space of functions and we will prove that these spaces under some restrictions are just the Triebel-Lizorkin spaces.

Definition 2.7. Let $1 < p \le \infty$ and $s > 0$. The space $A^s_p(X)$ is the set of functions $f \in L^p(X)$ satisfying that there exist some $q \in [1, p)$ and a non-negative function $g \in L^p(X)$ such that

$$\fint_B \left| f(x) - \fint_B f(y)\,d\mu(y)\right| d\mu(x) \le r(B)^s \left(\fint_B g^q(x)\,d\mu(x)\right)^{1/q} \tag{2.32}$$

for every ball $B \subset X$. Moreover, if $f \in A^s_p(X)$, we define its norm by

$$\|f\|_{A^s_p(X)} = \|f\|_{L^p(X)} + \inf_{\{g\}} \|g\|_{L^p(X)},$$

where the infimum is taken over all functions g satisfying (2.32).

The following theorem will indicate the relations among these spaces, the Triebel-Lizorkin spaces and the spaces $B^s_p(X)$ under some restrictions.

Theorem 2.4. *Let $1 < p < \infty$ and $0 < s < \theta$. Then $A^s_p(X) = F^s_{p,\infty}(X) = B^s_p(X)$ with equivalent norms.*

Proof. Let $f \in A^s_p(X)$. For any given $\epsilon > 0$, there is a non-negative function $g \in L^p(X)$ and some $q \in (0, p)$ satisfying (2.32) and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{A^s_p(X)} + \epsilon. \tag{2.33}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 2.4. For $k \in \mathbb{N}$, let $B_1 = B(x, 2C_1 2^{-k})$. By (2.7), we then have

$$
\begin{aligned}
|D_k f(x)| &= \left|\int_X D_k(x,y) f(y)\, d\mu(y)\right| \\
&= \left|\int_X D_k(x,y)\left[f(y) - \fint_{B_1} f(z)\, d\mu(z)\right] d\mu(y)\right| \\
&\le C \fint_{B_1} \left|f(y) - \fint_{B_1} f(z)\, d\mu(z)\right| d\mu(y) \\
&\le C2^{-ks}\left(\fint_{B_1} g^q(y)\, d\mu(y)\right)^{1/q} \\
&\le C2^{-ks}\{M(g^q)(x)\}^{1/q}.
\end{aligned} \tag{2.34}
$$

Thus, the estimates (2.14), (2.34) and (2.33), the fact $q < p$ and the $L^{p/q}(X)$-boundedness of M tell us that

$$
\begin{aligned}
\|f\|_{F^s_{p,\infty}(X)} &= \left\|\sup_{k\in\mathbb{Z}_+} 2^{ks}|D_k f|\right\|_{L^p(X)} \\
&\le C\|Mf\|_{L^p(X)} + C\left\|\{M(g^q)\}^{1/q}\right\|_{L^p(X)} \\
&\le C\|f\|_{L^p(X)} + C\|g\|_{L^p(X)} \\
&\le C\big\{\|f\|_{A^s_p(X)} + \epsilon\big\}.
\end{aligned}
$$

Letting $\epsilon \to 0$, we obtain that $A^s_p(X) \subset F^s_{p,\infty}(X)$ and

$$
\|f\|_{F^s_{p,\infty}(X)} \le C\|f\|_{A^s_p(X)}. \tag{2.35}
$$

On another hand, let $f \in B^s_p(X)$. Then, by Definition 2.6, there is a function $g \in L^p(X)$ such that (2.23) holds and

$$
\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B^s_p(X)} + \epsilon. \tag{2.36}
$$

Then the estimate (2.23) and the Hölder inequality tell us that for any ball $B \subset X$ and any $q \in [1, p)$,

$$
\begin{aligned}
\fint_B \left|f(x) - \fint_B f(y)\, d\mu(y)\right| d\mu(x) &\le r(B)^s \fint_B g(x)\, d\mu(x) \\
&\le r(B)^s \left\{\fint_B g^q(x)\, d\mu(x)\right\}^{1/q}.
\end{aligned}
$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Thus, $f \in A_p^s(X)$ and, by (2.36),

$$\|f\|_{A_p^s(X)} \le \|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B_p^s(X)} + \epsilon.$$

Letting $\epsilon \to 0$, we finally obtain that $B_p^s(X) \subset A_p^s(X)$ and

$$\|f\|_{A_p^s(X)} \le \|f\|_{B_p^s(X)}. \tag{2.37}$$

Then (2.35) and (2.37) together with Theorem 2.3 imply Theorem 2.4 and we finish this proof. □

§3. Sobolev spaces

The main purpose of this section is to give several new characterizations of the Hajłasz-Sobolev spaces of order 1 on spaces of homogeneous type. We will first show that the space $A_p^1(X)$ is the same space as the space $B_p^1(X)$ by using some ideas from [21]. In fact, Theorem 3.1 below is similar to Theorem 1 in [21]. However, we do not suppose that the space of homogeneous type, X, has the segment property as in [21].

THEOREM 3.1. *Let $1 \le q < \infty$, f be a locally integrable function on X for which there is a non-negative function $g \in L^q(X)$ such that the Poincaré inequality*

$$\fint_B \left| f(x) - \fint_B f(z)\,d\mu(z) \right| d\mu(x) \le Cr(B) \left(\fint_B g^q(x)\,d\mu(x) \right)^{1/q} \tag{3.1}$$

holds for every ball $B \subset X$. Then for μ-a. e. $x \in B$,

$$\left| f(x) - \fint_B f(z)\,d\mu(z) \right| \le Cr(B) M(g^q)(x)^{1/q},$$

where C is independent of x and B.

Proof. Let $x \in B$ be a Lebesgue point of $\left| f(y) - \fint_B f(z)\,d\mu(z) \right|$ and $g(y)$. Let $B_0 = B$ and $B_j = B(x, 2^{-j} r(B))$ for $j \in \mathbb{N}$. Then

$$\begin{aligned} \left| f(x) - \fint_B f(z)\,d\mu(z) \right| &= \lim_{j\to\infty} \fint_{B_j} \left| f(y) - \fint_B f(z)\,d\mu(z) \right| d\mu(y) \\ &\le \limsup_{j\to\infty} \fint_{B_j} \left| f(y) - \fint_{B_j} f(z)\,d\mu(z) \right| d\mu(y) \\ &\quad + \limsup_{j\to\infty} \left| \fint_{B_j} f(z)\,d\mu(z) - \fint_B f(z)\,d\mu(z) \right| \\ &= I_1 + I_2. \end{aligned}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

For I_1, by (3.1), we have

$$I_1 \le C \limsup_{j\to\infty} r(B_j)\left(\fint_{B_j} g^q(y)\,d\mu(y)\right)^{1/q} = 0\cdot g(x) = 0.$$

We further control I_2 by

$$\begin{aligned}
I_2 &\le \limsup_{j\to\infty} \sum_{l=0}^{j-1}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right| \\
&\le \sum_{l=1}^{\infty}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right| \\
&\quad + \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{B} f(z)\,d\mu(z)\right| \\
&= I_2^1 + I_2^2.
\end{aligned}$$

For I_2^1, the facts that $B_{l+1}\subset B_l$ and $\mu(B_{l+1})\sim\mu(B_l)$ for $l\in\mathbb{N}$ and (3.1) imply that

$$\begin{aligned}
I_2^1 &\le C\sum_{l=1}^{\infty}\fint_{B_l}\left|f(y)-\fint_{B_l} f(z)\,d\mu(z)\right|d\mu(y) \\
&\le C\sum_{l=1}^{\infty} 2^{-l} r(B)\left[\fint_{B_l} g^q(y)\,d\mu(y)\right]^{1/q} \\
&\le Cr(B)M(g^q)(x)^{1/q}.
\end{aligned}$$

We estimate I_2^2. Let $\bar{B} = B(x, 2Ar(B))$. Then $B\cup B_1\subset\bar{B}$ and

$$\mu(\bar{B})\sim\mu(B)\sim\mu(B_1).$$

Therefore,

$$\begin{aligned}
I_2^2 &\le \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{\bar{B}} f(z)\,d\mu(z)\right| + \left|\fint_{\bar{B}} f(z)\,d\mu(z) - \fint_{B} f(z)\,d\mu(z)\right| \\
&\le \fint_{B_1}\left|f(y)-\fint_{\bar{B}} f(z)\,d\mu(z)\right|d\mu(y) + \fint_{B}\left|f(y)-\fint_{\bar{B}} f(z)\,d\mu(z)\right|d\mu(y) \\
&\le C\fint_{\bar{B}}\left|f(y)-\fint_{\bar{B}} f(z)\,d\mu(z)\right|d\mu(y) \\
&\le Cr(B)M(g^q)(x)^{1/q}.
\end{aligned}$$

This proves Theorem 3.1. ∎

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

Using Theorem 3.1, we can establish the relation between the space $A^1_p(X)$ and $B^1_p(X)$.

THEOREM 3.2. *Let* $1 < p \le \infty$*. Then* $A^1_p(X) = B^1_p(X)$ *with equivalent norms.*

Proof. Let $f \in A^1_p(X)$. For any given $\epsilon > 0$, Definition 2.7 then tells us that there exist some $q \in [1, p)$ and a non-negative function $g \in L^p(X)$ such that (2.32) holds for any ball $B \subset X$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{A^1_p(X)} + \epsilon.$$

Therefore, Theorem 3.1 yields that

$$\left| f(x) - \fint_B f(z)\, d\mu(z) \right| \le C_5 r(B) M(g^q)(x)^{1/q}$$

for a. e. $x \in B$ and all $B \subset X$. Let $h = C_5 M(g^q)(x)^{1/q}$. Then (2.23) holds with g replaced by h, and

$$\begin{aligned} \|f\|_{L^p(X)} + \|h\|_{L^p(X)} &\le \|f\|_{L^p(X)} + C_5 \big\| M(g^q)^{1/q} \big\|_{L^p(X)} \\ &\le C\big\{ \|f\|_{L^p(X)} + \|g\|_{L^p(X)} \big\} \\ &\le C\big\{ \|f\|_{A^1_p(X)} + \epsilon \big\}, \end{aligned}$$

where we used the $L^{p/q}(X)$-boundedness of M. Let $\epsilon \to 0$. We therefore obtain that $A^1_p(X) \subset B^1_p(X)$ and

$$\|f\|_{B^1_p(X)} \le C \|f\|_{A^1_p(X)}.$$

Conversely, by an argument similar to the proof of Theorem 2.4, it is easy to show that $B^1_p(X) \subset A^1_p(X)$ and

$$\|f\|_{A^1_p(X)} \le C \|f\|_{B^1_p(X)},$$

which finishes the proof of Theorem 3.2. ∎

We now recall the definition of the Sobolev spaces of Hajłasz in [9]; see also [10], [18].

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

DEFINITION 3.1. Let $1 < p \le \infty$. The Sobolev space $W_p^1(X)$ is defined by

$$\begin{aligned} W_p^1(X) = \{u \in L^p(X) : &\text{there is a set } E \subset X,\ \mu(E) = 0, \\ &\text{and a function } g \ge 0,\ g \in L^p(X) \text{ such that} \\ &|u(x) - u(y)| \le \rho(x,y)[g(x) + g(y)] \text{ for all } x,\ y \in X \setminus E\}, \end{aligned}$$

where g is called a generalized gradient of u. Moreover, we define

$$\|u\|_{W_p^1(X)} = \|u\|_{L^p(X)} + \inf_{\{g\}} \|g\|_{L^p(X)},$$

where the infimum is taken over all generalized gradients of the function u in the definition of $W_p^1(X)$.

The following theorem indicates the relations among the Hajłasz-Sobolev space, the space $A_p^1(X)$ and the space $B_p^1(X)$.

THEOREM 3.3. *Let $1 < p \le \infty$. Then $W_p^1(X) = A_p^1(X) = B_p^1(X)$ with equivalent norms.*

Proof. Let $1 < p \le \infty$ and $f \in W_p^1(X)$. For any given $\epsilon > 0$, then there is a non-negative function $g \in L^p(X)$ such that

$$|f(x) - f(y)| \le \rho(x,y)[g(x) + g(y)]$$

for a. e. x, $y \in X$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{W_p^1(X)} + \epsilon.$$

Thus, for any ball $B \subset X$ and a. e. $x \in B$,

$$\begin{aligned} \left| f(x) - \fint_B f(y)\, d\mu(y) \right| &\le \fint_B |f(x) - f(y)|\, d\mu(y) \\ &\le r(B) \fint_B [g(x) + g(y)]\, d\mu(y) \\ &\le r(B)[g(x) + Mg(x)] \\ &\le 2r(B)Mg(x), \end{aligned}$$

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

where we used the estimate (2.25). Thus, (2.23) holds with g replaced by $2Mg$, and

$$\begin{aligned}\|f\|_{B_p^1(X)} &\le \|f\|_{L^p(X)} + 2\|Mg\|_{L^p(X)}\\ &\le \|f\|_{L^p(X)} + 2C\|g\|_{L^p(X)}\\ &\le C\big[\|f\|_{W_p^1(X)} + \epsilon\big],\end{aligned}$$

where we used the $L^p(X)$-boundedness of M. Letting $\epsilon \to 0$, we then obtain that $W_p^1(X) \subset B_p^1(X)$ and

$$\|f\|_{B_p^1(X)} \le C\|f\|_{W_p^1(X)}.$$

Let $f \in B_p^1(X)$. Then, for any given $\epsilon > 0$, there is a non-negative function $g \in L^p(X)$ such that for any ball $B \subset X$ and a. e. $x \in B$,

$$\left|f(x) - \fint_B f(y)\,d\mu(y)\right| \le r(B)g(x)$$

and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B_p^1(X)} + \epsilon.$$

From this, it follows that for a. e. x, $y \in B$,

$$|f(x) - f(y)| \le r(B)[g(x) + g(y)].$$

By suitably choosing $r(B)$, we finally obtain that for a. e. x, $y \in X$,

$$|f(x) - f(y)| \le \rho(x, y)[g(x) + g(y)].$$

Thus, $f \in W_p^1(X)$ and

$$\|f\|_{W_p^1(X)} \le \|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B_p^1(X)} + \epsilon.$$

Letting $\epsilon \to 0$, we obtain that $B_p^1(X) \subset W_p^1(X)$ and

$$\|f\|_{W_p^1(X)} \le \|f\|_{B_p^1(X)}.$$

Thus, $W_p^1(X) = B_p^1(X)$ with equivalent norms. Furthermore, this fact and Theorem 3.2 imply Theorem 3.3. This finishes the proof of Theorem 3.3. ∎

Finally, the following theorem gives another new characterization of the Sobolev space $W_p^1(X)$.

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

THEOREM 3.4. *Let* $1 \le u < p \le \infty$. *Then* $f \in W_p^1(X)$ *if and only if* $f \in L^p(X)$ *and* $f^1_{u,C_4} \in L^p(X)$. *Moreover,*

$$\|f\|_{W_p^1(X)} \sim \|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)}.$$

Proof. Let $f \in W_p^1(X)$. Then Theorem 3.3 tells us that $f \in B_p^1(X)$. By Definition 2.6, for any given $\epsilon > 0$, there is a function $g \in L^p(X)$ such that (2.23) holds for all $B \subset X$ and a. e. $x \in B$, and

$$\|f\|_{L^p(X)} + \|g\|_{L^p(X)} < \|f\|_{B_p^1(X)} + \epsilon.$$

Thus,

$$\begin{aligned} f^1_{u,C_4}(x) &= \sup_{0<t<C_4} t^{-1}\left[\fint_{B(x,t)}\left|f(y) - \fint_{B(x,t)} f(z)\,d\mu(z)\right|^u d\mu(y)\right]^{1/u} \\ &\le \sup_{0<t<C_4}\left[\fint_{B(x,t)} g^u(y)\,d\mu(y)\right]^{1/u} \\ &\le M(g^u)(x)^{1/u}. \end{aligned}$$

Therefore, the $L^{p/u}(X)$-boundedness of M implies that

$$\begin{aligned} \|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)} &\le \|f\|_{L^p(X)} + \left\|M(g^u)^{1/u}\right\|_{L^p(X)} \\ &\le \|f\|_{L^p(X)} + C\|g\|_{L^p(X)} \\ &\le C\big(\|f\|_{B_p^1(X)} + \epsilon\big). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\|f\|_{L^p(X)} + \left\|f^1_{u,C_4}\right\|_{L^p(X)} \le C\|f\|_{B_p^1(X)} \le C\|f\|_{W_p^1(X)}.$$

Let now $f \in L^p(X)$ and $f^1_{u,C_4} \in L^p(X)$. Let x be a Lebesgue point of f. For any fixed $0 < t < C_4$, consider the Lebesgue point of the function

$$f(v) - \fint_{B(x,t)} f(z)\,d\mu(z).$$

Moreover, let $B_0 = B(x,t)$ and $B_j = B(y, 2^{-j}t/A)$ for $j \in \mathbb{N}$. Then, for

https://doi.org/10.1017/S002776300000903X Published online by Cambridge University Press

a. e. $y \in B(x, t/2A)$, we have

$$\begin{aligned}\left|f(y) - \fint_{B(x,t)} f(z)\,d\mu(z)\right| &= \lim_{j\to\infty} \fint_{B_j} \left|f(v) - \fint_{B(x,t)} f(z)\,d\mu(z)\right| d\mu(v)\\ &\le \limsup_{j\to\infty} \fint_{B_j} \left|f(v) - \fint_{B_j} f(z)\,d\mu(z)\right| d\mu(v)\\ &\quad + \limsup_{j\to\infty}\left|\fint_{B_j} f(z)\,d\mu(z) - \fint_{B_0} f(z)\,d\mu(z)\right|\\ &= J_1 + J_2.\end{aligned}$$

The Hölder inequality tells us that

$$\begin{aligned}J_1 &\le \limsup_{j\to\infty}\left\{\fint_{B_j}\left|f(v) - \fint_{B_j} f(z)\,d\mu(z)\right|^u d\mu(v)\right\}^{1/u}\\ &\le \limsup_{j\to\infty} 2^{-j} t f^1_{u,C_4}(y)\\ &= 0.\end{aligned}$$

We now estimate J_2. The Hölder inequality and the fact that $\mu(B_{l+1}) \sim \mu(B_l)$ for $l \in \mathbb{N}$ yield that

$$\begin{aligned}J_2 &\le \limsup_{j\to\infty} \sum_{l=0}^{j-1}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right|\\ &\le \sum_{l=1}^{\infty}\left|\fint_{B_{l+1}} f(z)\,d\mu(z) - \fint_{B_l} f(z)\,d\mu(z)\right|\\ &\quad + \left|\fint_{B_1} f(z)\,d\mu(z) - \fint_{B_0} f(z)\,d\mu(z)\right|\\ &\le \sum_{l=1}^{\infty}\left\{\fint_{B_{l+1}}\left|f(v) - \fint_{B_l} f(z)\,d\mu(z)\right|^u d\mu(v)\right\}^{1/u}\\ &\quad + C\left\{\fint_{B_0}\left|f(v) - \fint_{B_0} f(z)\,d\mu(z)\right| d\mu(v)\right\}^{1/u}\\ &\le C\sum_{l=1}^{\infty} 2^{-l} t f^1_{u,1}(y) + Ct f^1_{u,1}(x)\\ &\le Ct\left[f^1_{u,1}(y) + f^1_{u,1}(x)\right].\end{aligned}$$

Thus, for a. e. $y \in B(x, t/2A)$,

$$|f(y) - f(x)| \leq Ct\big[f^1_{u,1}(y) + f^1_{u,1}(x)\big].$$

By suitably choosing t, we obtain that for a. e. x, $y \in X$,

$$|f(y) - f(x)| \leq C_6\rho(x,y)\big[f^1_{u,1}(y) + f^1_{u,1}(x)\big].$$

Therefore, letting $g(x) = C_6 f^1_{u,1}(x)$, we then verify that $f \in W^1_p(X)$ and

$$\begin{aligned}\|f\|_{W^1_p(X)} &\leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}\\ &\leq \|f\|_{L^p(X)} + C_6\|f^1_{u,1}\|_{L^p(X)}\\ &\leq C\big[\|f\|_{L^p(X)} + \|f^1_{u,C_4}\|_{L^p(X)}\big].\end{aligned}$$

This proves Theorem 3.4. □

Acknowledgements. The author would like to express his sincere thanks to Professor Hans Triebel and Professor Guozhen Lu for some helpful discussions on this topic.

The author also thanks the referee for his/her very careful reading and many valuable remarks which made this article more readable.

References

[1] M. T. Barlow, *Diffusions on fractals*, Lectures on probability theory and statistics (Saint-Flour, 1995) (P. Bernard, ed.), Lecture Notes in Math. **1690**, Springer, Berlin (1998), pp. 1–121.

[2] M. T. Barlow and R. F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets*, Canad. J. Math., **51** (1999), 673–744.

[3] M. Christ, *The extension problem for certain function spaces involving fractional orders of differentiability*, Ark. Mat., **22** (1984), 63–81.

[4] R. Coifman and G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogènes, Lecture Notes in Math. **242**, Springer-Verlag, Berlin, 1971.

[5] G. David, J. L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana, **1** (1985), 1–56.

[6] R. A. DeVore and R. C. Sharpley, *Maximal functions measuring smoothness*, Memoirs Amer. Math. Soc., **47** (1984), no. 293, 1–115.

[7] A. E. Gatto, *Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition*, J. Funct. Anal., **188** (2002), 27–37.

[8] A. Grigor'yan, J. Hu and K. Lau, *Heat kernels on metric-measure spaces and an application to semi-linear elliptic equations*, Trans. Amer. Math. Soc., **355** (2003), 2065–2095.

[9] P. Hajłasz, *Sobolev spaces on an arbitrary metric spaces*, Potential Anal., **5** (1996), 403–415.

[10] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Memoirs Amer. Math. Soc., **145** (2000), no. 688, 1–101.

[11] Y. Han, *Inhomogeneous Calderón reproducing formula on spaces of homogeneous type*, J. Geometric Anal., **7** (1997), 259–284.

[12] Y. Han, S. Lu and D. Yang, *Inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type*, Approx. Th. and its Appl., **15(3)** (1999), 37–65.

[13] Y. Han, S. Lu and D. Yang, *Inhomogeneous Triebel-Lizorkin spaces on spaces of homogeneous type*, Math. Sci. Res. Hot-Line, **3(9)** (1999), 1–29.

[14] Y. Han, S. Lu and D. Yang, *Inhomogeneous discrete Calderón reproducing formulas for spaces of homogeneous type*, J. Fourier Anal. Appl., **7** (2001), 571–600.

[15] Y. Han and E. T. Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and classical function spaces*, Memoirs Amer. Math. Soc., **110** (1994), no. 530, 1–126.

[16] Y. Han and D. Yang, *New characterizations and applications of inhomogeneous Besov and Triebel-Lizorkin spaces on homogeneous type spaces and fractals*, Dissertationes Math. (Rozprawy Mat.), **403** (2002), 1–102.

[17] Y. Han and D. Yang, *Some new spaces of Besov and Triebel-Lizorkin type on homogeneous spaces*, Studia Math., **156** (2003), 67–97.

[18] J. Heinonen, Lectures on Analysis on Metric Spaces, Springer-Verlag, Berlin, 2001.

[19] A. Jonsson and H. Wallin, Function Spaces on Subsets of $\mathbb{R}^n$, Math. Reports, Vol. **2**, Harwood Academic Publ., London, 1984.

[20] J. Kigami, Analysis on fractals, Cambridge Tracts in Math. **143**, Cambridge University Press, Cambridge, 2001.

[21] Y. Liu, G. Lu and R. L. Wheeden, *Some equivalent definitions of high order Sobolev spaces on stratified groups and generalizations to metric spaces*, Math. Ann., **323** (2002), 157–174.

[22] R. A. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math., **33** (1979), 257–270.

[23] P. Mattila, Geometry of sets and measures in Euclidean spaces, Cambridge University Press, Cambridge, 1995.

[24] A. Miyachi, *Multiplication and factorization of functions in Sobolev spaces and in C_p^α spaces on general domains*, Math. Nachr., **176** (1995), 209–242.

[25] A. Miyachi, *Atomic decomposition for Sobolev spaces and for the C_p^α spaces on general domains*, Tsukuba J. Math., **21** (1997), 59–96.

[26] R. S. Strichartz, *Function spaces on fractals*, J. Funct. Anal., **198** (2003), 43–83.

[27] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.

[28] H. Triebel, Theory of Function Spaces II, Birkhäuser Verlag, Basel, 1992.

[29] H. Triebel, Fractals and Spectra, Birkhäuser Verlag, Basel, 1997.

[30] H. Triebel, The Structure of Functions, Birkhäuser Verlag, Basel, 2001.

[31] H. Triebel, *Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Revista Mat. Complutense, **15** (2002), 1–50.

[32] D. Yang, *Besov spaces on spaces of homogeneous type and fractals*, Studia Math., **156** (2003), 15–30.

School of Mathematical Sciences
Beijing Normal University
Beijing 100875
People's Republic of China
dcyang@bnu.edu.cn