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ON PIC($D[\alpha]$) FOR A PRINCIPAL IDEAL DOMAIN D

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ABSTRACT. Let D be a PID with infinitely many maximal ideals. J. W. Brewer has asked whether some simple ring extension $D[\alpha]$ of D must have nontrivial Picard group. We show that this question has a negative answer.

Let D be a PID containing a field F such that $|F| > |MSpec(D)|$, where $|\cdot|$ denotes cardinality. We show that each simple ring extension of D has trivial Picard group. This answers a question raised by J. W. Brewer (personal communication). Brewer's question was motivated by the problem (from algebraic control theory) of determining, for an integral domain T, conditions under which the polynomial ring $T[X]$ is a BCS-ring. Recall that the notion (but not the terminology) of a BCS-ring arises from [7, Th. B], was touched on briefly in [2, Sect. 4], and studied extensively in [11]; also, see [1] for general motivation. In particular, Proposition 1.8 of [11] shows that $T[X]$ is a BCS-ring if T is a semilocal PID. Moreover, Theorem 2.3 of [11] shows that the natural map $Pic(R) \longrightarrow Pic(R/I)$ is surjective if R is a BCS-ring, so an affirmative answer to Brewer's question would have implied, for a PID T, that T is semilocal if $T[X]$ is a BCS-ring.

We remark that our use of the sets S in the proof of Theorem 1 is a modification of a technique used by Claborn [4] (see also [5, Section 13]) in determining conditions under which a Dedekind domain is a principal ideal domain.

THEOREM 1. *Suppose D is a PID with infinitely many maximal ideals, and assume that D contains a field F such that* $|F| > |MSpec(D)|$. *Then each simple ring extension* $D[\alpha]$ *of D has trivial Picard group.*

PROOF. We frequently consider $D[\alpha]$ as $D[X]/I$, where $I \cap D = (0)$. If $I = (0)$, it is well known that $Pic(D[\alpha]) = (0)$. Moreover, since D is a Noetherian Hilbert domain, the condition $I \cap D = (0)$ implies that $\dim(D[X]/I) > 0$. Hence we consider the case where dim $(D[\alpha]) = 1$. Since $D[\alpha]$ is Noetherian, we can show that $Pic(D[\alpha]) = (0)$ by showing that each proper invertible ideal J of $D[\alpha]$ is principal. Since J contains a regular element, J is contained in no height-zero prime of $D[\alpha]$, so each of the minimal

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primes of J is also maximal. Consequently, the primary decomposition of J has the form $Q_1 \cap Q_2 \cap \ldots \cap Q_n = Q_1 Q_2 \ldots Q_n$, where Q_i is primary for a maximal ideal M_i. Each Q_i is invertible, and to show that J is principal, it suffices to show that each Q_i is principal. Moreover, since $Pic\, D[\alpha] = Pic(D[\alpha]_{red})$, there is no loss of generality in assuming that I is a radical ideal of $D[X]$. Thus I has the form $(a) \cap (f(X)) \cap B$, where $a = \pi_1 \pi_2 \ldots \pi_s$ is a product of distinct prime elements of D, $f = f_1 f_2 \ldots f_t$ is a product of distinct irreducible polynomials in $D[X] \backslash D$, and $B = H_1 \cap H_2 \cap \ldots \cap H_u$ is an intersection of distinct maximal ideals of $D[X]$. We note that $t \geqq 1$ since $I \cap D = (0)$.

We change notation to assume that $J = Q$ is primary for a maximal ideal M and we prove that Q is principal. To do so, it suffices to prove that $Q > \cup\{QP^* | P^* \in MSpec(D[\alpha])\}$[6, Rem. 1]. Let $M \cap D = \pi D$ and assume that $q = \pi^v \in Q$. We consider two cases.

Case 1. $(q, a) = D$. Since q is regular in $D[\alpha]$ in this case, there exists $h \in Q$ such that $Q = (q, h)$ [9, Prop. 4.2], [8, p. 372]. Moreover, since a is a unit modulo q, $Q = (q, ah)$ as well. Consider the set $S = \{q + \mu ah | \mu \in F\}$. If P^* is a maximal ideal of $D[\alpha]$, then $P^* = P/I$ for some maximal ideal P of $D[X]$ such that either $\pi_i \in P$ for some i, $f_j \in P$ for some j, or $P = H_k$ for some k. Since $D[X]/(f_j)$ is a simple domain extension of D that is algebraic over D, the Krull-Akizuki Theorem [10, Thm. 33.2] implies that $|\{P \in MSpec(D[X]) \mid f_j \in P\}| < |F|$. Consequently, if $U = \{P^* \in MSpecD[\alpha] \mid a \notin P^*\}$, then $|U| < |F|$. Let $V = MSpecD[\alpha] \backslash U$. If $P^* \in V$, then $a \in P^*, q \notin P^*$, and hence $q + \mu ah \notin P^*$ for each $\mu \in F$. On the other hand if $P^* \in U$, then $QP^* < Q$, and hence QP^* contains $q + \mu ah$ for at most one element μ of F since $Q = (q + \mu_1 ah, q + \mu_2 ah)$ for $\mu_1 \neq \mu_2$. Since $|U| < |F|$, it follows that there exists $s \in S \subseteq Q$ such that $s \notin \cup\{QP^* | P^* \in MSpecD[\alpha]\}$. Therefore $Q > \cup QP^*$, as we wished to show.

Case 2. $(q, a) \neq D$. In this case $(\pi) = (\pi_i)$ for some i and $a^v \in Q$. We show that $Pic(D[\alpha]/(a^v)) = (0)$. Note that $D[\alpha]/(a^v) \simeq D[X]/[(a^v) + I]$, and hence it suffices to prove that $D[X]/\sqrt{[(a^v) + I]}$ has trivial Picard group. But $\sqrt{[(a^v) + I]} = \sqrt{[\sqrt{(a^v)} + \sqrt{I}]} = \sqrt{[(a) + I]} = \sqrt{(a)} = (a)$, and

$$D[X]/(a) \simeq \oplus_{i=1}^{s} K_i[X],$$

where $K_i \simeq D/(\pi_i)$. Hence $D[X]/(a)$ is a PIR and $Pic(D[\alpha]/(a^v)) = (0)$. We note that $A/(a^v)$ is invertible in $D[\alpha]/(a^v)$ since this ideal is locally principal and is not contained in $\cup_{i=1}^{s}[(\pi_i)/(a^v)]$, the set of zero divisors of $D[\alpha]/(a^v)$. Therefore $Q/(a^v)$ is principal, say $Q = (b, a^v)$. The argument that $Q > \cup QP^*$ is now completed essentially as in Case 1. To wit, let $S = \{b + \mu a^v | \mu \in F\}$. If $P^* \in V, P^* \neq \sqrt{Q}$, then $b + \mu a^v \notin P^*$ for each $\mu \in F$ since $a^v \in P^*$ and $b \notin P^*$. Moreover, if $P^* \in U$ or if $P^* = \sqrt{Q}$, then there exists at most one element $\mu \in F$ such that $b + \mu a^v \in QP^*$. Hence $S \backslash \cup \{QP^* \mid P^* \in MSpecD[\alpha]\})$ is nonempty since $|U + \{\sqrt{Q}\}| < |F|$, and again we conclude that $Q > \cup QP^*$. This completes the proof of Theorem 1.

In connection with the proof of Theorem 1, it seems reasonable to ask whether a one-dimensional Noetherian ring R has trivial Picard group if $Pic(R/P_i) = (0)$ for each minimal prime P_i of R. David Lantz showed us the following example, which shows that this questions has a negative answer. Let $R = \{(a,b) \in Z \times Z : a \equiv b \pmod 5\}$. Then R has two minimal primes P_1, P_2 and $R/P_i \simeq Z$, but $Pic(R)$ is a cyclic group of order 2. For the case where R is the coordinate ring of an affine curve over an algebraically closed field, Theorem 3.6 of [12] provides a rich source of examples where $Pic(R) \neq (0)$, but $Pic(R/P_i) = (0)$ for each minimal prime P_i of R.

We remark that Brewer, Klingler and Minnaar [3] have recently and independently proved a result that also answers Brewer's question. More precisely, Theorem 6 of [3] shows that if E is a PID containing an uncountable field and having only countably many maximal ideals, then $E[X]$ is a BCS-ring. As noted in the introductory paragraph of this paper, it then follows that each simple ring extension of E has trivial Picard group.

References

1. J. Brewer, J. Bunce, and F. Van Vleck, *Linear systems over commutative rings*, Dekker, New York, 1986.
2. J. Brewer, D. Katz, and W. Ullery, *Pole assignability in polynomial rings, power series rings and Prüfer domains*, J. Algebra **106** (1987), 265– 286.
3. J. Brewer, L. Klingler, and F. Minnaar, *Polynomial rings which are BCS-rings*, Comm. in Algebra (to appear).
4. L. Claborn, *A generalized approximation theorem for Dedekind domains*, Proc. Amer. Math. Soc. **18** (1967), 378–380.
5. R. Fossum, *The divisor class group of a Krull domain*, Springer-Verlag, New York, 1973.
6. R. Gilmer and W. Heinzer, *On the number of generators of an invertible ideal*, J. Algebra **14** (1970), 139–151.
7. M. Hautus and E. Sontag, *New results on pole-shifting for parameterized families of systems*, J. Pure Appl. Algebra **40** (1986), 229– 244.
8. R. C. Heitmann and L. S. Levy, *1 1/2 and 2 generator ideals in Prüfer domains*, Rocky Mt. J. Math. **5** (1975), 361–373.
9. D. Lazard and P. Huet, *Dominions des anneaux commutatifs*, Bull Sci. Math. **94** (1970), 193–199.
10. M. Nagata, *Local rings*, Interscience, New York, 1962.
11. W. Vasconcelos and C. Weibel, *BCS-Rings*, J. Pure Appl. Algebra, to appear.
12. R. Wiegand and S. Wiegand, *Decomposition of torsion-free modules over affine curves*, Proc. Bowdoin Conf. on Algebraic Geometry (to appear).

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