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GENERALIZED AFFINE KAC-MOODY LIE ALGEBRAS OVER LOCALIZATIONS OF THE POLYNOMIAL RING IN ONE VARIABLE

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ABSTRACT. We consider simple complex Lie algebras extended over the commutative ring $C[z,(z-a_1)^{-1},\ldots,(z-a_n)^{-1}]$ where $a_1,\ldots,a_n \in C$. We compute the universal central extensions of these Lie algebras and present explicit commutation relations for these extensions. These algebras generalize the untwisted affine Kac-Moody Lie algebras, which correspond to the case $n=1, a_1=0$.

1. **Introduction.** An untwisted affine Kac-Moody Lie algebra may be defined in two ways: either by generators and relations in terms of the data in a generalized Cartan matrix, or as the universal central extension of a loop algebra. By a loop algebra we mean a Lie algebra of the form $L = \mathbf{C}[z,z^{-1}] \otimes_{\mathbf{C}} \mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra; the commutation relations are $[f\otimes x, g\otimes y] = fg\otimes[xy]$. It is well-known that the homology group $H_2(L,\mathbf{C})$ is one-dimensional, and hence the center of the universal central extension $\hat{L}$ of L is also one-dimensional. All of this material, together with the theory of general Kac-Moody algebras and their representations, is explained in detail in [Kac, 1990].

The loop algebra construction suggests a natural generalization. We let A be any commutative associative $\mathbf{C}$-algebra; we then form the Lie algebra $L = A\otimes_{\mathbf{C}}\mathfrak{g}$, with Lie brackets defined by the formula given above. The theory of affine Kac-Moody algebras leads us to expect that the most interesting representations of L will be projective; that is, they will be ordinary representations of the universal central extension $\hat{L}$ of L. The homology group $H_2(L,\mathbf{C})$, and the commutation relations for $\hat{L}$, can be computed in terms of Kähler differentials of A, following [Kassel, 1984].

The ring of Laurent polynomials $\mathbf{C}[z,z^{-1}]$ may be regarded as the ring of rational functions on the projective line $\mathbf{C}\cup\{\infty\}$ which have poles only at $z\in\{\infty,0\}$. One of the simplest generalizations of this picture is to allow poles at an arbitrary finite set of points $\{\infty,a_1,a_2,\ldots,a_n\}$. This gives the ring

$$A = \mathbf{C}[z,(z-a_1)^{-1},\ldots,(z-a_n)^{-1}];$$

for the rest of this paper this will be the definition of A. The universal central extensions $\hat{L}$ of the Lie algebras $L = A\otimes_{\mathbf{C}}\mathfrak{g}$ are the Lie algebras to which the title of this paper

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refers. (Perhaps it is more informative and convenient to call these Lie algebras *N-point affine algebras*, where $N = n + 1$.)

Since the automorphism group $\mathrm{PGL}_2(\mathbf{C})$ of the projective line is (simply) 3-transitive, we lose no generality by assuming that ∞ is one of the points where a pole is allowed; indeed we could even assume that $a_1 = 0$ and $a_2 = 1$. For $n \in \{1, 2\}$ there is thus a unique isomorphism class of rings A. For $n \geq 3$ there will be parametrized families of non-isomorphic rings.

In principle, there is nothing to prevent us from letting A be the localization of a finite algebraic extension B of $\mathbf{C}[z]$. That is, we let K be a finite extension of the function field $\mathbf{C}(z)$, and let B be the subring of K consisting of the elements which are integral over $\mathbf{C}[z]$. This corresponds to replacing the projective line with an algebraic curve of positive genus. (Some results for the case of genus 1 appear in [Sheinman, 1990].) However in what follows we will consider only the case of genus zero.

Another generalization of the affine Kac-Moody Lie algebras has been studied in [Moody *et al.*, 1990]. That paper considers the ring $B = \mathbf{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ and the universal central extension of $B \otimes_{\mathbf{C}} \mathfrak{g}$, called a *toroidal Lie algebra*. The toroidal algebras have a $\mathbf{Z}^n$-grading (with finite dimensional subspaces), whereas the N-point affine algebras appear to have no grading by any finite Abelian group except when $N = 2$. The centre of a toroidal algebra is infinite dimensional, whereas the centre of an N-point affine algebra is finite dimensional (see Theorem 2 below). We have a surjective ring homomorphism $B \longrightarrow A$ given by $t_i \longmapsto z - a_i$ for $1 \leq i \leq n$. This induces a surjective linear map $H_2(B \otimes_{\mathbf{C}} \mathfrak{g}, \mathbf{C}) \longrightarrow H_2(A \otimes_{\mathbf{C}} \mathfrak{g}, \mathbf{C})$, which shows that the N-point affine algebra is a homomorphic image of the toroidal algebra.

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2. **Calculating $H_2(L, \mathbf{C})$.**

We start with an elementary fact.

LEMMA 1. *The subset* $\{1\} \cup \{z^k, (z - a_j)^{-k} \mid k \in Z_+, 1 \leq j \leq n\}$ *is a basis of* A *over* $\mathbf{C}$.

PROOF. This subset spans A over $\mathbf{C}$ by the theory of partial fractions. To show that this subset is linearly independent, suppose that the linear combination

$$\sum_{k=0}^{m_0} b_k^0 z^k + \sum_{j=1}^{n} \sum_{k=1}^{m_j} b_k^j (z - a_j)^{-k}$$

equals 0 in A. If $m_j = 0$ for all j, $1 \leq j \leq n$, then this expression is a polynomial, and so all its coefficients (*i.e.* the b_k^0 for $0 \leq k \leq m_0$) must be 0. In the other case, $m_j \geq 1$ for some j, $1 \leq j \leq n$, and $b_{m_j}^j \neq 0$. Multiplying the expression by $(z - a_j)^{m_j}$ then gives an element of A in which $z - a_j$ never occurs with negative exponent, and in which every term, except that with coefficient $b_{m_j}^j$, has $z - a_j$ as a factor. This element of A is clearly still 0 in A, and so its value at $z = a_j$ is also 0. But this gives $b_{m_j}^j = 0$, a contradiction. Thus all the coefficients of the expression must be 0. ∎

Using the results of [Kassel, 1984] (see also the special case worked out in [Moody *et al.*, 1990]), it is not hard to determine the dimension of the center of the universal central extension $\hat{L}$ of $A \otimes_{\mathbf{C}} \mathfrak{g}$. The center of $\hat{L}$ is linearly isomorphic to the homology group $H_2(L, \mathbf{C})$.

THEOREM 2. *We have* $H_2(L, \mathbf{C}) \cong \mathbf{C}^n$. *Hence the center of* $\hat{L}$ *has dimension* n.

PROOF. Let $\{f_i \mid i \in I\}$, for some index set I, denote the basis for A described in the lemma. Let F be the free left A-module on the generators $\{\tilde{d}f_i\}$, where the $\tilde{d}f_i$ are formal symbols in bijective correspondence with the f_i. By setting $(\tilde{d}f_i)g = g(\tilde{d}f_i)$ for any $g \in A$ we may regard F as a two-sided A-module. We define a $\mathbf{C}$-linear map $\tilde{d}: A \longrightarrow F$ by $\tilde{d}(\sum_i c_i f_i) = \sum_i c_i(\tilde{d}f_i)$. We let K denote the submodule of F spanned by the elements $\tilde{d}(gh) - \{(\tilde{d}g)h + g(\tilde{d}h)\}$ for any $g, h \in A$. In F/K the following elements are zero:

$$\tilde{d}1, \quad \tilde{d}(z^k) - kz^{k-1}\tilde{d}z, \quad \tilde{d}\big((z-a_i)^{-k}\big) + k(z-a_i)^{-k-1}\tilde{d}(z-a_i),$$

for $k \geq 1, 1 \leq i \leq n$. Note also that $\tilde{d}(z-a_i) = \tilde{d}z$. Write $\Omega_A = F/K$. The differential map $d: A \longrightarrow \Omega_A$ is defined by $dg = \tilde{d}g + K$. A basis of Ω_A consists of the elements

$$z^k\, dz \quad (k \geq 0), \ (z-a_i)^{-k}\, dz \quad (k \geq 1, 1 \leq i \leq n).$$

Write $C = \Omega_A / dA$. Then since

$$d(z^k) = kz^{k-1}\, dz, \quad d\big((z-a_i)^{-k}\big) = -k(z-a_i)^{-k-1}\, dz,$$

we see that a basis of C consists of the cosets of the n elements

$$c_i = (z-a_i)^{-1}\, dz, \quad 1 \leq i \leq n.$$

The result of [Kassel, 1984] states that the center of $\hat{L}$ is linearly isomorphic to C. ∎

Write $g(dh) \longmapsto \overline{g(dh)}$ for the canonical quotient map $\Omega_A \longrightarrow C$. Kassel's paper shows, in addition, that $\hat{L}$ is linearly isomorphic to $(A \otimes_{\mathbf{C}} \mathfrak{g}) \oplus C$, with Lie brackets

$$[g \otimes x, h \otimes y] = gh \otimes [xy] + (x, y)\overline{(dg)h}, \quad [\hat{L}, C] = 0,$$

where (x, y) denotes the Killing form of $\mathfrak{g}$.

3. **Commutation relations for the universal central extension.** To give a more explicit form to the commutation relations for $\hat{L}$, we need a formula to express the product of two basis elements of A as a linear combination of basis elements. We start with a combinatorial result.

LEMMA 3. *For any non-negative integer* a*, and any real numbers* b, c*, we have the identity of binomial coefficients (where the sum is always finite):*

$$\binom{b-c}{a} = \sum_{i=0}^{\infty} (-1)^i \binom{b}{a-i} \binom{c+i-1}{i}.$$

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PROOF. From the formal power series expansion of the equation

$$(1+x)^b = (1+x)^c(1+x)^{b-c},$$

we derive the Vandermonde convolution formula

$$\binom{b}{a} = \sum_{i=0}^{\infty} \binom{b-c}{a-i}\binom{c}{i}.$$

(See [Riordan, 1968], p. 8.) Writing $b' = b-c$, replacing c by $-c$, and then using b instead of b', we obtain

$$\binom{b-c}{a} = \sum_{i=0}^{\infty} \binom{b}{a-i}\binom{-c}{i}.$$

Now using the relation

$$\binom{-c}{i} = (-1)^i \binom{c+i-1}{i},$$

we obtain the result. ■

There are two non-trivial cases for the product of basis elements of A.

PROPOSITION 4. *(a) We have*

$$z^k(z-a_j)^{-l} = \sum_{i=1}^{l} \binom{k}{l-i} a_j^{k-l+i}(z-a_j)^{-i} + \sum_{h=0}^{k-l} \binom{k-h-1}{l-1} a_j^{k-l-h} z^h.$$

(b) For $i \neq j$, we have

$$(z-a_i)^{-k}(z-a_j)^{-l} = \sum_{h=1}^{k} (-1)^l \binom{k+l-h-1}{l-1} (a_j-a_i)^{h-k-l}(z-a_i)^{-h}$$
$$+ \sum_{h=1}^{l} (-1)^{l+h} \binom{k+l-h-1}{k-1} (a_j-a_i)^{h-k-l}(z-a_j)^{-h}.$$

PROOF. (a) We first consider the special case $a_j = 1$. We set $w = z-1$ and obtain

$$z^k(z-1)^{-l} = (w+1)^k w^{-l} = \sum_{i=0}^{k} \binom{k}{i} w^{i-l}$$
$$= \sum_{i=0}^{l-1} \binom{k}{i} (z-1)^{i-l} + \sum_{i=l}^{k} \binom{k}{i} (z-1)^{i-l},$$

where the second summation must be rewritten, since the powers of $z-1$ are nonnegative. We have

$$\sum_{i=l}^{k} \binom{k}{i} \sum_{h=0}^{i-l} (-1)^{i-l-h} \binom{i-l}{h} z^h = \sum_{h=0}^{k-l} \left(\sum_{i=l+h}^{k} (-1)^{i-l-h} \binom{k}{i}\binom{i-l}{h} \right) z^h.$$

Replacing i by $i+l+h$ in the inner summation, and using the lemma with $a=k-l-h$, $b=k$, $c=h+1$, gives

$$\sum_{i=0}^{k-l-h}(-1)^i\binom{k}{i+l+h}\binom{i+h}{h}=\sum_{i=0}^{k-l-h}(-1)^i\binom{k}{k-l-h-i}\binom{i+h}{i}=\binom{k-h-1}{k-l-h}.$$

We now conclude that

$$z^k(z-1)^{-l}=\sum_{i=1}^{l}\binom{k}{l-i}(z-1)^{-i}+\sum_{h=0}^{k-l}\binom{k-h-1}{l-1}z^h,$$

where we have replaced i by $l-i$ in the first summation.

For the case of arbitrary nonzero a_j, we set $z=a_jw$ and obtain $z^k(z-a_j)^{-l}=a_j^{k-l}w^k(w-1)^{-l}$. We conclude that

$$\begin{aligned}z^k(z-a_j)^{-l}&=a_j^{k-l}\sum_{i=1}^{l}\binom{k}{l-i}(w-1)^{-i}+a_j^{k-l}\sum_{h=0}^{k-l}\binom{k-h-1}{l-1}w^h\\&=\sum_{i=1}^{l}\binom{k}{l-i}a_j^{k-l+i}(z-a_j)^{-i}+\sum_{h=0}^{k-l}\binom{k-h-1}{l-1}a_j^{k-l-h}z^h.\end{aligned}$$

(b) We first consider the special case $a_i=0$, $a_j=1$. We set $w=1/z$ and obtain

$$\begin{aligned}z^{-k}(z-1)^{-l}&=(-1)^l w^{k+l}(w-1)^{-l}\\&=(-1)^l\sum_{i=0}^{l}\binom{k+l}{l-i}(w-1)^{-i}-(-1)^l\binom{k+l}{l}\\&\quad+(-1)^l\sum_{h=0}^{k}\binom{k+l-h-1}{l-1}w^h,\end{aligned}$$

where the first summation must be rewritten. Using $1/(w-1)=-\big(1+1/(z-1)\big)$ we have

$$\begin{aligned}\sum_{i=0}^{l}\binom{k+l}{l-i}(-1)^{-i}\Big(1+\frac{1}{z-1}\Big)^i&=\sum_{i=0}^{l}\binom{k+l}{l-i}(-1)^{-i}\sum_{h=0}^{i}\binom{i}{h}(z-1)^{-h}\\&=\sum_{h=0}^{l}\left(\sum_{i=h}^{l}(-1)^{-i}\binom{k+l}{l-i}\binom{i}{h}\right)(z-1)^{-h}.\end{aligned}$$

Replacing i by $i+h$ in the inner sum, and using the lemma with $a=l-h$, $b=k+l$, $c=h+1$, gives

$$(-1)^h\sum_{i=0}^{l-h}(-1)^i\binom{k+l}{l-h-i}\binom{h+i}{h}=\binom{k+l-h-1}{l-h}=\binom{k+l-h-1}{k-1}.$$

We conclude that

$$z^{-k}(z-1)^{-l}=\sum_{h=1}^{l}(-1)^{l+h}\binom{k+l-h-1}{k-1}(z-1)^{-h}+\sum_{h=1}^{k}(-1)^l\binom{k+l-h-1}{l-1}z^{-h},$$

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where the three constant terms have cancelled.

For the case of general a_i, a_j, we set $z = (a_j - a_i)w + a_i$, and obtain

$$\begin{aligned}(z-a_i)^{-k}(z-a_j)^{-l} &= (a_j-a_i)^{-k-l}w^{-k}(w-1)^{-l}\\ &= \sum_{h=1}^{l}(-1)^{l+h}\binom{k+l-h-1}{k-1}(a_j-a_i)^{-k-l}(w-1)^{-h}\\ &\quad+\sum_{h=1}^{k}(-1)^{l}\binom{k+l-h-1}{l-1}(a_j-a_i)^{-k-l}w^{-h}.\end{aligned}$$

Since $w = (z-a_i)/(a_j-a_i)$ and $w-1 = (z-a_j)/(a_j-a_i)$, we conclude that

$$\begin{aligned}(z-a_i)^{-k}(z-a_j)^{-l} &= \sum_{h=1}^{l}(-1)^{l+h}\binom{k+l-h-1}{k-1}(a_j-a_i)^{h-k-l}(z-a_j)^{-h}\\ &\quad+\sum_{h=1}^{k}(-1)^{l}\binom{k+l-h-1}{l-1}(a_j-a_i)^{h-k-l}(z-a_i)^{-h}.\end{aligned}$$ ∎

In order to write down the explicit commutation relations for $\hat{L}$ we also need to work out $\overline{(dg)h}$ for any two elements g, h of the basis of A given in the first lemma. In each case the result will lie in the n-dimensional vector space C spanned by the central basis elements $c_i = (z-a_i)^{-1}\,dz$ for $1 \le i \le n$.

PROPOSITION 5. *(a) For $g = z^k$, $h = (z-a_j)^{-l}$, we have*

$$\overline{(dg)h} = k\binom{k-1}{l-1}a_j^{k-l}c_j.$$

(b) For $g = (z-a_i)^{-k}$, $h = (z-a_j)^{-l}$, we have

$$\overline{(dg)h} = (-1)^l k\binom{k+l-1}{k}(a_j-a_i)^{-k-l}(c_j-c_i).$$

PROOF. For (a), it is easy to see that the coefficient of c_m in $\overline{(dg)h}$ is just the coefficient of $(z-a_m)^{-1}$ in the expansion of $kz^{k-1}(z-a_j)^{-l}$ as a linear combination of basis elements of A. This coefficient can be easily read off from the formula in the previous proposition. For (b) we find the coefficient of $(z-a_m)^{-1}$ in the expansion of $-k(z-a_i)^{-k-1}(z-a_j)^{-l}$. Note that the correctness of the last proposition can be checked by verifying that in each case $\overline{(dg)h}$ and $\overline{g(dh)}$ give the same answer but with opposite sign. ∎

We can now write down the commutation relations for $\hat{L}$. We first introduce some shorthand notation for non-central elements of $\hat{L}$. For any $x \in \mathfrak{g}$, we set

$$x(0,0) = 1 \otimes x,\ x(k,0) = z^k \otimes x, \quad k \in Z_+,$$
$$x(k,i) = (z-a_i)^{-k} \otimes x, \quad 1 \le i \le n,\ k \in Z_+.$$

THEOREM 6. *The commutation relations for $\hat{L}$ are:*

$$[x(k,i),y(l,i)] = [xy](k+l,i), \quad k,l \in Z_+,\ 0 \le i \le n,$$

$$\begin{aligned}[x(k,0),y(l,j)] &= \sum_{i=1}^{l} \binom{k}{l-i} a_j^{k-l+i}[xy](-i,j) \\ &\quad + \sum_{h=0}^{k-l} \binom{k-h-1}{l-1} a_j^{k-l-h}[xy](h,0) \\ &\quad + k\binom{k-1}{l-1} a_j^{k-l}(x,y)c_j, \quad k,l \in Z_+,\ 1 \le j \le n,\end{aligned}$$

and

$$\begin{aligned}&[x(k,i),y(l,j)] \\ &= \sum_{h=1}^{k}(-1)^l \binom{k+l-h-1}{l-1}(a_j-a_i)^{h-k-l}[xy](-h,i) \\ &\quad + \sum_{h=1}^{l}(-1)^{l+h}\binom{k+l-h-1}{k-1}(a_j-a_i)^{h-k-l}[xy](-h,j) \\ &\quad + (-1)^l k\binom{k+l-1}{k}(a_j-a_i)^{-k-l}(x,y)(c_j-c_i), \quad k,l \in Z_+,\ 1 \le i,j \le n. \ \blacksquare\end{aligned}$$

We conclude with some brief remarks on representation theory. To define Verma modules over the Lie algebras $\hat{L}$, we consider the subalgebra $\hat{L}_0 \oplus \hat{L}_+$ where $\hat{L}_0 = (\mathbf{C} \otimes \mathfrak{g}_0) \oplus C$ (here $\mathfrak{g}_0$ is the Cartan subalgebra of $\mathfrak{g}$) and $\hat{L}_+ = z\mathbf{C}[z] \otimes \mathfrak{g}$. We let v be a symbol and consider the vector space $\mathbf{C}v$. We make this into a module over the Lie algebra $\hat{L}_0 \oplus \hat{L}_+$ by defining $\hat{L}_+ . v = \{0\}$ and $a . v = \lambda(a)v$ for all $a \in \hat{L}_0$, where $\lambda \in \hat{L}_0^*$ (the dual vector space). We then form the induced module

$$V(\lambda) = U(\hat{L}) \otimes_{U(\hat{L}_0 \oplus \hat{L}_+)} \mathbf{C}v,$$

and call this the Verma module with highest weight λ. The standard argument from Kac-Moody theory then shows that each $V(\lambda)$ has a unique maximal proper submodule, and hence a unique irreducible quotient $M(\lambda)$.

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