NON-LINEAR WEAK SHOCK DIFFRACTION

N. J. DE MESTRE

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Abstract. Perturbation expansions are sought for the flow variables associated with the diffraction of a plane weak shock wave around convex-angled corners in a polytropic, inviscid, thermally-nonconducting gas. Lighthill's method of strained co-ordinates [4] produces a uniformly valid expansion for most of the diffracted front, while the remainder of this front is treated by a modification of the shock-ray theory of Whitham [6]. The solutions from these approaches are patched just inside the 'shadow' region yielding a plausible description of the entire diffracted shock front.

1. Introduction

For a gas in which viscosity, thermal conductivity and body forces are neglected, the flow pattern arising from the diffraction of a weak shock wave by a rigid obstacle is governed by partial differential equations that are non-linear. In previous literature this non-linearity has, in most cases, been disregarded and the resulting approximate solutions exhibit a number of unacceptable features particularly at the diffracted wave fronts. These include the prediction of singularities in some pressure derivatives and the incorrect placement of the incident and diffracted fronts. But there is also an inherent physical inconsistency — the presence of a discontinuous lateral jump in the pressure at the junction of the incident shock, the diffracted wave front and the 'shadow' boundary of geometrical acoustics.

Now the non-linear terms cannot be given such a secondary role near the wave fronts. It is the purpose of this paper to include them using a perturbation expansion procedure, and such an approach discloses that this is a singular perturbation problem.

The analysis is simplified without loss of essential features if a rigid convex-angled wedge and a plane incident weak shock wave are the chosen geometries. Further, the elimination of reflected waves, triple points and slip streams is effected if the incident shock is made to propagate towards the corner with its wave front perpendicular to the wedge face along which it is travelling.

In Section 2 the governing flow equations are indicated, the existence of a velocity potential is established and a similarity transformation, namely that originated by Buseman [1], is deduced. These are combined to produce the ultimate formulation of the problem in terms of a 'conical' potential function.

The 'linearized' approximate results are given in Section 3, and display all the inconsistencies hitherto mentioned. When higher order approximations are obtained, it is observed that the perturbation expansion is not uniformly valid. Lighthill's method of strained coordinates [4] is applied and provides a non-linear description of the diffracted front for most of the 'shadow' region — in fact a shock of higher order strength than the incident shock is fitted into the flow. However troubles occur if Lighthill's technique is used when the angular distance from the 'shadow' boundary is of the order of the square root of the incident shock strength. As this is also the order of the angular distance from the 'shadow' boundary to the upper limit of the disturbed part of the shock in the 'illuminated' region, this troublesome zone near the 'shadow' boundary is termed the interaction zone for it is here that the shock front is extensively modified by interactions between advancing and receding waves.

Professor J. J. Mahony and the late Professor H. C. Levey suggested to the author that approximate relations for the shock strength and shock front equation could be obtained in this interaction zone by modifying Whitham's shock-ray theory [6] so that the correct point of intersection of the upper limiting characteristic and the incident shock front is predicted for shocks of all strengths. The modified theory is developed in Section 4 and yields results which suggest that a 'boundary layer' approach could be attempted near the wave front in this zone. Unfortunately the derived equation and its boundary conditions are quite complicated, although a similarity solution is obtained by considering only the equation and the shock front conditions. It is remarkable that this solution predicts expressions for the shock strength and shock front position that are asymptotically equivalent to those given by the modified Whitham theory.

An endeavour is made, therefore, in Section 5 to patch these approximate shock front relations from the interaction zone with those obtained by the method of strained co-ordinates away from the 'shadow' boundary. An effective patching is accomplished after a slight alteration to the 'strained co-ordinate' results, and this makes possible a new description of the diffracted shock wave in which the shock strength, the shock front position and the slope of the shock front are continuous from the wall to the undisturbed part of the shock.

The author is indebted to the late Professor H. C. Levey for suggesting the problem and critically discussing the work in a most helpful manner. This research was carried out at the University of Western Australia while the author was in receipt of a Commonwealth Public Service Post-Graduate Scholarship.

2. Formulation of the problem

Consider two semi-infinite plane walls that meet to form an edge, with the region outside the corner (formed by the reflex angle) occupied by a non-viscous, non-thermally conducting, polytropic gas at rest. Suppose that a weak plane shock wave is generated so that it travels towards the corner with its plane wave front perpendicular to the wall along which it is travelling. Until the corner is reached it travels with a constant speed U_0 (determined by its strength), but after it passes the corner the flow becomes non-trivial because of diffraction.

The flow can obviously be treated as two-dimensional and unsteady, and Cartesian axes Ox, Oy are chosen with the corner as the origin O, the y-axis in the direction of propagation of the plane shock front, and the x-axis such that its positive direction points into the region corresponding to the 'shadow' region of geometrical acoustics. Let ω be the angle made by the wall in this 'shadow' region with the x-axis (Figure I).

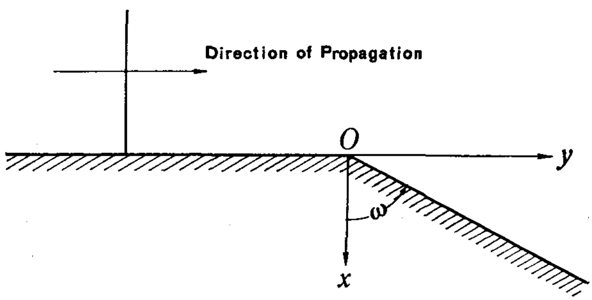

Figure 1

If u, v, ρ, S, p respectively denote the x-component of velocity, the y-component of velocity, the density, the specific entropy and the pressure at a general point (x, y) at time t, where $t = 0$ is the instant when the shock wave reaches the corner, then since body forces may be neglected the conservation laws and the equation of state yield

$$\rho_t + u\rho_x + v\rho_y + \rho u_x + \rho v_y = 0, \tag{2.1}$$

$$u_t + uu_x + vu_y + \frac{1}{\rho} p_x = 0, \tag{2.2}$$

$$v_t + uv_x + vv_y + \frac{1}{\rho} p_y = 0, \tag{2.3}$$

$$p = p(\rho, S), \tag{2.4}$$

and

$$S_t + uS_x + vS_y = 0, \tag{2.5}$$

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where lettered subscripts denote partial derivatives.

Now, if d/dt denotes the 'derivative following the motion',

$$\frac{d\rho}{dt} = \left(\frac{\partial\rho}{\partial p}\right)_S \frac{dp}{dt}$$

using (2.4) and (2.5), and as

$$a^2 = \left(\frac{\partial p}{\partial\rho}\right)_S, \tag{2.6}$$

where a is the sound speed for the gas, then

$$\rho_t+u\rho_x+v\rho_y = \frac{1}{a^2}[p_t+up_x+vp_y].$$

Hence (2.1) becomes

$$p_t+up_x+vp_y+\rho a^2[u_x+v_y] = 0. \tag{2.7}$$

Since the gas ahead of the shock wave is at rest, then by (2.1)—(2.4) and (2.6) the other flow quantities will be constant and are denoted respectively by p_0, a_0, ρ_0, S_0. At any time $t < 0$, the gas behind the shock wave has velocity components $u = 0$, $v = v_1$ (a constant), therefore the equations that govern the flow again show that the other dependent variables are constant. If they are denoted by p_1, a_1, ρ_1, S_1 then certain relations exist between these and p_0, a_0, ρ_0, S_0 by virtue of the conservation laws at the shock front.

As the incident shock is weak, its shock strength (denoted by ε) may be used as a suitable parameter for a perturbation expansion of the flow equations. Thus

$$\varepsilon \equiv \frac{p_1-p_0}{\gamma p_0} = \frac{2}{\gamma+1}(M_0^2-1)$$

where γ is the adiabatic index of the gas and

$$M_0 = \frac{U_0}{a_0},$$

and so

$$M_0 \sim 1+\varepsilon\frac{(\gamma+1)}{4}+O(\varepsilon^2).$$

Now the initial conditions are

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$$\frac{p-p_0}{\gamma p_0} = \varepsilon H(-y)$$

and

$$\left(\frac{p-p_0}{\gamma p_0}\right)_t = 0,$$

where H denotes the Heaviside unit step-function.

The boundary conditions are

$$u = 0, \qquad (x = 0, y < 0, \text{all } t)$$

$$v \cos \omega - u \sin \omega = 0, \qquad (y = x \tan \omega, x > 0, \text{all } t)$$

and across any shock waves formed the Rankine-Hugoniot shock jump conditions must hold.

At this stage it is found convenient to scale velocities with respect to a_0. With $u = a_0 \bar{u}$, $v = a_0 \bar{v}$, $a = a_0 \bar{a}$, $U_0 = a_0 \bar{U}_0$, $p = a_0^2 \bar{p}$ and $t = \bar{t}/a_0$, the equations (2.2), (2.3), (2.6) and (2.7) together with the conditions governing the flow remain unchanged with u, v, a, U_0, p, t replaced by $\bar{u}$, $\bar{v}$, $\bar{a}$, $\bar{U}_0$, $\bar{p}$, $\bar{t}$. The bars on these six variables are now dropped.

It is reasonable to assume that any additional shocks that may occur from interactions after $t = 0$ should only have strengths at most of order ε. The flow is therefore homentropic to order ε^2, and since the initial circulation is zero everywhere the flow may be treated as irrotational to this order. Thus to order ε^2 there exists a velocity potential denoted by ϕ such that $u = \phi_x$ and $v = \phi_y$.

Since no fundamental length- or time-scale can be produced by a combination of the physical constants defining the problem, the flow may be treated as 'conical' in the sense of Busemann [1]. Thus if

$$\frac{x}{t} = r \cos \theta,$$

$$\frac{y}{t} = r \sin \theta,$$

and

$$\phi = tf(r, \theta),$$

the flow equations (2.2), (2.3) and (2.7) with the aid of Bernoulli's equation

$$a^2 = 1-(\gamma-1)\left\{f - rf_r + \tfrac{1}{2}f_r^2 + \frac{1}{2r^2}f_\theta^2\right\} \tag{2.8}$$

yield the equation

$$
\begin{aligned}
(1-r^2)f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta}
&= (\gamma-1)(f-rf_r)\left(f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta}\right)+\frac{2}{r^2}f_\theta(f_\theta-rf_{r\theta}) \\
&\quad -2rf_rf_{rr}+\frac{(\gamma-1)}{2}\left(f_r^2+\frac{1}{r^2}f_\theta^2\right)\left(f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta}\right) \\
&\quad +f_{rr}f_r^2+\frac{1}{r^4}f_{\theta\theta}f_\theta^2+\frac{2}{r^2}f_rf_\theta f_{r\theta}-\frac{1}{r^3}f_rf_\theta^2
\end{aligned} \tag{2.9}
$$

for the 'conical' potential. The wall boundary condition is now

$$
f_\theta = 0 \qquad \left(\text{on } \theta=\omega \text{ and } \frac{3\pi}{2}\right) \tag{2.10}
$$

while, if $r=\kappa(\theta)$ is the map of the shock wave in this pseudo-two-dimensional plane, the shock front conditions may be written in a form analogous to that of Lighthill [5] for a steady three-dimensional potential problem. For the unsteady problem these conditions are

$$
f = 0 \qquad (\text{on } r=\kappa(\theta)) \tag{2.11}
$$

and

$$
f_r = \frac{2}{\gamma+1}\left\{a^2(\kappa-f_r+f_\theta\kappa^{-2}\kappa')^{-1}-(\kappa-f_r+f_\theta\kappa^{-2}\kappa')(1+\kappa^{-2}(\kappa')^2)^{-1}\right\} \qquad (\text{on } r=\kappa(\theta)), \tag{2.12}
$$

where a prime denotes differentiation with respect to θ. Therefore the non-linear problem is to solve (2.9) under the boundary conditions (2.10)–(2.12).

3. The strained co-ordinate approach

The non-linear problem for the potential will be initially attacked through the tentative assumption of the existence of a perturbation expansion in the form

$$
f(r,\theta;\varepsilon) = \varepsilon f^{(1)}(r,\theta)+\varepsilon^2 f^{(2)}(r,\theta)+\varepsilon^3 f^{(3)}(r,\theta)+\cdots. \tag{3.1}
$$

As it is intended near $r=1$ to try to fit a shock $r=\kappa(\theta)$ into the description of the flow, let

$$
\kappa(\theta) = 1+\varepsilon\kappa^{(1)}(\theta)+\varepsilon^2\kappa^{(2)}(\theta)+\cdots. \tag{3.2}
$$

If (3.1) and (3.2) are substituted into (2.9)–(2.12), and terms with coefficients ε^2 or higher are neglected, the equation remaining is the first approximate or linear equation, namely

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$$(3.3) \qquad (1-r^2)f_{rr}^{(1)}+\frac{1}{r}f_r^{(1)}+\frac{1}{r^2}f_{\theta\theta}^{(1)}=0,$$

with the boundary conditions

$$(3.4) \qquad f_\theta^{(1)}=0 \qquad \left(\theta=\omega \text{ and } \frac{3\pi}{2}\right),$$

while on $r=1$ $(\omega \leqq \theta < \pi/2)$

$$(3.5) \qquad f^{(1)}=0$$

and

$$(3.6) \qquad f_r^{(1)}=-\frac{2}{\gamma+1}[(\gamma-1)(f^{(1)}-rf_r^{(1)})+2(\kappa^{(1)}-f_r^{(1)})].$$

The boundary condition on $r=1$ $(\pi/2 < \theta \leqq 3\pi/2)$ can be deduced to be

$$f^{(1)}=\sin\theta-1$$

and this completes the specifications on the closed boundary.

The solution for $f^{(1)}$ can be obtained via the method of Keller and Blank [2]. The problem reduces to finding a function $P^{(1)}(\sigma,\theta)$, where

$$\sigma=\frac{1-(1-r^2)^{\frac{1}{2}}}{r},$$

which is harmonic inside the sector bounded by the arc $\sigma=1$ and the radial lines $\theta=\omega$ and $\theta=3\pi/2$, and satisfies the boundary conditions

$$P^{(1)}=0 \qquad \left(\sigma=1,\ \omega\leqq\theta<\frac{\pi}{2}\right),$$

$$P^{(1)}=1 \qquad \left(\sigma=1,\ \frac{\pi}{2}<\theta\leqq\frac{3\pi}{2}\right),$$

and

$$P_\theta^{(1)}=0 \qquad \left(0\leqq\sigma\leqq 1,\ \theta=\omega \text{ and } \frac{3\pi}{2}\right).$$

A conformal transformation of this sector into a semi-circle and an application of the Riemann-Schwarz principle of reflection enables the unique solution for $P^{(1)}$ to be obtained directly by the Poisson Integral Formula. Then for $\omega\leqq\theta<\pi/2$,

$$\pi P^{(1)}(\sigma,\theta)=\operatorname{artan}\left\{\left(\frac{1-\sigma^\lambda}{1+\sigma^\lambda}\right)\cot\left(\frac{\lambda}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right\}$$
$$-\operatorname{artan}\left\{\left(\frac{1-\sigma^\lambda}{1+\sigma^\lambda}\right)\cot\left(-\frac{\lambda}{2}\left(\frac{\pi}{2}+\theta-2\omega\right)\right)\right\},$$

while for $\pi/2<\theta\leqq 3\pi/2$,

$$\pi P^{(1)}(\sigma, \theta) = \pi + \operatorname{artan}\left\{\left(\frac{1-\sigma^\lambda}{1+\sigma^\lambda}\right)\cot\left(\frac{\lambda}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right\}$$
$$-\operatorname{artan}\left\{\left(\frac{1-\sigma^\lambda}{1+\sigma^\lambda}\right)\cot\left(-\frac{\lambda}{2}\left(\frac{\pi}{2}+\theta-2\omega\right)\right)\right\}$$

where principal values of the inverse tangents are to be taken and

$$\lambda = \frac{2\pi}{3\pi-2\omega}.$$

Now $P^{(1)}(r, \theta)$ obtained from Keller and Blank's approach corresponds to the first order approximation to the 'reduced' pressure $P(r, \theta)$, where

$$\frac{p-p_0}{\gamma p_0} = P(r, \theta)$$
$$= \varepsilon P^{(1)}(r, \theta)+\varepsilon^2 P^{(2)}(r, \theta)+\cdots,$$

hence

$$1+\gamma P = \frac{p}{p_0}$$
$$= \left\{1-(\gamma-1)\left[f-rf_r+\tfrac{1}{2}f_r^2+\frac{1}{2r^2}f_\theta^2\right]\right\}^{\gamma/(\gamma-1)}$$

by Bernoulli's equation. It should be noted that the value of the 'reduced' pressure at any point on the shock front predicts the strength of the shock at that point.

If coefficients of ε on either side of this last equation are equated, it is seen that

$$P^{(1)} = rf_r^{(1)}-f^{(1)},$$

and so

$$f^{(1)} = \begin{cases} r\displaystyle\int_1^r \frac{P^{(1)}(r, \theta)}{r^2}\,dr, & \omega \leqq \theta < \dfrac{\pi}{2}, \\ r(\sin\theta-1)+r\displaystyle\int_1^r \frac{P^{(1)}(r, \theta)}{r^2}\,dr, & \dfrac{\pi}{2} < \theta \leqq \dfrac{3\pi}{2}. \end{cases}$$

This solution predicts a lateral discontinuity in $f_r^{(1)}$ at $r = 1$, $\theta = \pi/2$ as well as an algebraic singularity in $f_{rr}^{(1)}$ on $r = 1$, $\omega \leqq \theta \leqq 3\pi/2$ (except $\theta = \pi/2$).

For the purpose of trying to fit a shock near $r = 1$ it will only be necessary to obtain local expansions of $f^{(1)}$ and its partial derivatives of first and second order near $r = 1$. Thus for $\omega \leqq \theta < \pi/2$, as $r \to 1$

$$f^{(1)}(r, \theta) = -\frac{2^{\frac{1}{2}}\lambda}{3\pi}Q(\theta)(1-r)^{\frac{3}{2}}+O(1-r)^2 \tag{3.7}$$

where

$$Q(\theta) = \cot\left\{\frac{\lambda}{2}\left(\frac{\pi}{2}-\theta\right)\right\} - \cot\left\{-\frac{\lambda}{2}\left(\frac{\pi}{2}+\theta-2\omega\right)\right\} \tag{3.8}$$

provided that

$$(1-r)^{\frac{1}{2}} \ll \tan\left(\frac{\lambda}{2}\left(\frac{\pi}{2}-\theta\right)\right)$$

for only then can the inverse tangent expressions in $P^{(1)}$ be expanded uniformly in powers of $(1-r)$. The local solution (3.7) obviously satisfies (3.5) and substitution of it into (3.6) yields

$$\kappa^{(1)} \equiv 0,$$

since $f_r^{(1)}$ vanishes on $r = 1$.

The consideration of terms in ε^2 in (2.9) leads to the second approximate equation

$$\begin{aligned}(1-r^2)f_{rr}^{(2)}+\frac{1}{r}f_r^{(2)}+\frac{1}{r^2}f_{\theta\theta}^{(2)} &= (\gamma-1)(f^{(1)}-rf_r^{(1)})\left(f_{rr}^{(1)}+\frac{1}{r}f_r^{(1)}+\frac{1}{r^2}f_{\theta\theta}^{(1)}\right)\\ &\quad -2rf_r^{(1)}f_{rr}^{(1)}+\frac{2}{r^2}f_\theta^{(1)}(f_\theta^{(1)}-rf_{r\theta}^{(1)})\end{aligned} \tag{3.9}$$

with boundary conditions

$$f_\theta^{(2)} = 0 \qquad \left(\theta = \omega \text{ and } \frac{3\pi}{2}\right),$$

$$f^{(2)}(1,\theta) = 0$$

and the second-order shock wave condition. The solution for $f^{(2)}$ is

$$f^{(2)} = -\frac{(\gamma+1)\lambda^2}{4\pi^2}\{Q(\theta)\}^2(1-r)+O[(1-r)^{\frac{3}{2}}],$$

which is no more singular than $f^{(1)}$, although $f_r^{(2)}$ will contribute to the pressure.

When the third approximate differential equation is considered the solution $f^{(3)}$ is seen to be $O[(1-r)^{\frac{1}{2}}]$, and this is more singular than $f^{(1)}$ since its first derivative possesses a square root singularity at $r = 1$, which leads to infinite speeds as $r \to 1$. It is easily shown that higher approximate terms in the perturbation expansion (3.1) possess progressively worse singularities at $r = 1$, so that within a certain neighbourhood of $r = 1$ later terms in (3.1) are just as important as the earlier ones. This means that the perturbation expansion is not uniformly valid near $r = 1$.

The non-uniformities are removed by applying the method of strained co-ordinates [4], which yields

$$f(r,\theta;\varepsilon) = -\varepsilon\frac{2^{\frac{1}{2}}\lambda}{3\pi}Q(\theta)(1-R)^{\frac{3}{2}} \tag{3.10}$$
$$-\varepsilon^2\frac{(\gamma+1)\lambda^2}{4\pi^2}\{Q(\theta)\}^2(1-R)+O[\varepsilon^3(1-R)]$$

where

$$r = R+\varepsilon^2\frac{(\gamma+1)^2\lambda^2}{8\pi^2}\{Q(\theta)\}^2+O(\varepsilon^3).$$

When these are used in the second shock condition (2.12) it is seen that

$$\kappa^{(2)}(\theta) = \frac{3(\gamma+1)^2\lambda^2}{32\pi^2}\{Q(\theta)\}^2,$$

and so the equation of the shock front is

$$r = 1+\varepsilon^2\frac{3(\gamma+1)^2\lambda^2}{32\pi^2}\{Q(\theta)\}^2+O(\varepsilon^3) \tag{3.11}$$

which is equivalent to

$$R = 1-\varepsilon^2\frac{(\gamma+1)^2\lambda^2}{32\pi^2}\{Q(\theta)\}^2+O(\varepsilon^3). \tag{3.12}$$

Thus the shock lies inside $R = 1$ (which shows by (3.10) that no awkward pressure derivative singularities occur in the flow behind this diffracted front) and outside $r = 1$ (which confirms that the shock speed is greater than the sonic speed in the fluid at rest). The strength of the fitted shock is

$$\varepsilon^2\frac{3(\gamma+1)\lambda^2}{8\pi^2}\{Q(\theta)\}^2+O(\varepsilon^3), \tag{3.13}$$

and the above results are seen to be similar to those obtained by Lighthill [5] and Levey [3]. However neither of these authors pursued the strained co-ordinate approach to see whether or not solutions from the higher approximate equations would contribute appreciably to the expansion for $f(r, \theta)$. The position is resolved simply by making each of the higher order straining coefficients zero identically, so that

$$r = R+\varepsilon^2\frac{(\gamma+1)^2\lambda^2}{8\pi^2}\{Q(\theta)\}^2.$$

The n^{th} approximate solution $f^{(n)}$ is then $O[(1-R)^{(5-n)/2}]$ $(n = 4, 5, 6\cdots)$. But by (3.12), for all points in the neighbourhood behind the shock front

$$\varepsilon^n(1-R)^{(5-n)/2} = o[\varepsilon(1-R)^{\frac{3}{2}}],$$

so in the neighbourhood of $r = 1$ the leading two terms in the expansion

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for $f(r, \theta)$ represent a fairly good approximation to the required velocity potential.

For the 'shadow' region, the above results are valid near the shock front provided that

$$\varepsilon Q(\theta) \ll \tan\left(\frac{\lambda}{2}\left(\frac{\pi}{2}-\theta\right)\right),$$

which indicates that different solutions will have to be sought in this region as soon as $\pi/2-\theta = O(\varepsilon^{\frac{1}{2}})$. But in the 'illuminated' region it is easily shown that the junction of the undisturbed part of the shock front with the diffracted part is at

$$\theta = \frac{\pi}{2}+\varepsilon^{\frac{1}{2}}2^{-\frac{1}{2}}(\gamma+1)^{\frac{1}{2}}+o(\varepsilon^{\frac{1}{2}}),$$

therefore the region $\pi/2-\theta = O(\varepsilon^{\frac{1}{2}})$ is termed the interaction zone for it appears that this is the region where secondary effects — such as the interaction of advancing and receding waves — play a major role in the determination of relatively rapid changes along the shock. It should be noted that the shock strength is $O(\varepsilon)$ at one end of this zone and only $O(\varepsilon^2)$ at the other end.

4. The interaction zone solutions

An application of the method of strained co-ordinates to the interaction zone has so far proved to be unsuccessful, therefore some other approach is needed.

A theory which could be applied — among other things — to the diffraction of a shock wave around a finite-angled corner was developed by Whitham [6]. Although the results predicted for strong shocks were in reasonable agreement with those obtained by alternate methods, Whitham recognized the unsuitability of his theory for weak shock diffraction as it predicted that the junction of the undisturbed part of the shock front with the diffracted part is at

$$\theta = \frac{\pi}{2}+\varepsilon^{\frac{1}{2}}2^{-\frac{3}{2}}(\gamma+1)^{\frac{1}{2}}+o(\varepsilon^{\frac{1}{2}}),$$

that is, the lateral speed of disturbance propagation is only half of what it should be.

In Whitham's theory orthogonal curvilinear co-ordinates α and β are introduced such that the (x, y)-plane is mapped by a network of curves $\alpha =$ constant and $\beta =$ constant. The curves $\alpha =$ constant are chosen to represent the successive positions of the advancing shock, while the $\beta =$ constant curves represent the orthogonal trajectories which Whitham calls

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rays. At any general time t the position of the shock is given by $\alpha = a_0 t$ and so the distance along the trajectory through (α, β) between the shock positions given by α and $\alpha+\delta\alpha$ is $M(\alpha, \beta)\delta\alpha$, where M is the Mach number of the shock at (α, β). The corresponding distance along the shock front through this point between the trajectories β and $\beta+\delta\beta$ is written as $A(\alpha, \beta)\delta\beta$. From this the differential equation

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{M}\frac{\partial A}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{A}\frac{\partial M}{\partial\beta}\right) = 0$$

for the problem is obtained, and if $A = A(M)$ where $dA/dM < 0$ this is a hyperbolic type and is solvable by the method of characteristics. The crux of Whitham's approach is the choice of the relation between A and M. By approximating the orthogonal trajectories by the instantaneous stream-lines of the flow, Whitham made use of the results for shocks travelling down a slowly varying channel, but the relation between A and M given by this approach predicted an incorrect lateral speed of disturbance propagation for weak shocks.

As a consequence of the suggestion by Levey and Mahony no attempt is made to identify the stream-lines with the orthogonal trajectories, and instead a relation is proposed between A and M based on a 'first principles' approach to the spread of a disturbance generated at a point by the passage of a plane shock. Thus if

$$\frac{cA}{M} = \frac{(M^2-1)^{\frac{1}{2}}\left(\frac{\gamma-1}{\gamma+1}M^2+\frac{2}{\gamma+1}\right)^{\frac{1}{2}}}{M^2}$$

where

$$c = \sqrt{-\frac{M}{A}\frac{dM}{dA}}$$

the correct rate of lateral spread of the disturbances is predicted for shocks of all strengths (including in particular weak shocks). The Whitham shock-ray theory with this new $A-M$ relation will from here on be referred to as the modified Whitham theory.

When the modified Whitham theory is applied to the diffraction of a weak shock around the convex-angle corner of this problem the position of the shock front is given by

$$r = 1+\tfrac{1}{4}\left\{\frac{\pi}{2}-\theta-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}^2+O(\varepsilon^2), \tag{4.1}$$

while the shock strength is

$$P = \frac{1}{2(\gamma+1)}\left\{\frac{\pi}{2}-\theta-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}^2+O(\varepsilon^2). \tag{4.2}$$

The orders of magnitude involved in (4.1) and (4.2) led the author to consider the possibility of obtaining a 'boundary layer' type solution in this interaction zone as this should give a more accurate description of the flow. If the dependent and independent variables of the potential problem are stretched such that

$$r-1 = \varepsilon(\gamma+1)s,$$

$$\frac{\pi}{2}-\theta = \varepsilon^{\frac{1}{2}}(\gamma+1)^{\frac{1}{2}}\eta$$

and

$$f(r,\theta) = \varepsilon^2(\gamma+1)F(s,\eta)$$
$$= \varepsilon^2(\gamma+1)[F^{(1)}(s,\eta)+\varepsilon F^{(2)}(s,\eta)+\cdots],$$

then terms of order ε in the differential equation (2.9) yield

$$(F_s^{(1)}-2s)F_{ss}^{(1)}+F_s^{(1)}+F_{\eta\eta}^{(1)} = 0, \tag{4.3}$$

while terms of the same order in the shock front boundary conditions give

$$F^{(1)} = 0 \tag{4.4}$$

and

$$F_s^{(1)} = 4W-2\left(\frac{dW}{d\eta}\right)^2 \tag{4.5}$$

on the shock front $s = W(\eta)$.

Additional conditions will have to be satisfied on the remaining boundaries of this small region, and the problem is relatively intractable in this form. If these other conditions are neglected for the present, the problem given by (4.3)—(4.5) can be solved in terms of a similarity variable

$$\mu = \frac{s}{(\eta+\eta_0)^2}$$

where η_0 is a constant, provided that

$$F^{(1)}(s,\eta) = (\eta+\eta_0)^4G(\mu)$$

and

$$W(\eta) = \mu_0(\eta+\eta_0)^2$$

where μ_0 is also a constant. Surprisingly, when μ_0 and η_0 are determined, the results for the shock position and shock strength are those given by (4.1) and (4.2). However the solutions away from the shock front are different from the 'modified Whitham' solutions.

From (4.1) and (4.2) it is seen that $r = 1+O(\varepsilon^2)$ and $P = O(\varepsilon^2)$ when

$$\theta = \frac{\pi}{2} - \varepsilon^{\frac{1}{2}} \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}},$$

which indicates that this theory alone will not suffice to give reasona diffraction results for corners with a large angle. Although the modil Whitham theory gives a fairly correct analysis of the flow near the junct of the undisturbed and diffracted parts of the shock front, it predict far too rapid attenuation of the shock strength as θ decreases. Since exact derivations are available for the whole interaction zone, the suppo given to the 'modified Whitham' solutions by the results of the 'bounda layer' similarity approach lends plausibility to the use of the 'modifi Whitham' results in at least the upper part of the interaction zone.

With this in mind an attempt is now made to 'patch' the shock fror results obtained from the application of the method of strained coordinate away from the 'shadow' boundary with the approximate shock front result from the modified Whitham theory.

5. Patching the shock front results

Provided that ω is not $\pi/2+O(\varepsilon^{\frac{1}{2}})$ it is deduced from (3.8) that

$$Q(\theta) \sim \frac{2}{\lambda\left(\frac{\pi}{2}-\theta\right)}$$

as $\theta \to \pi/2$. Hence the 'strained co-ordinate' results give $r = 1+O(\varepsilon)$ as the interaction region is entered (although they are not applicable inside this region), and this is the same order as that given by the modified Whitham theory. Physically there seems to be no reason for the presence of any form of discontinuity in the shock front equation inside the 'shadow' region, therefore the assumption is now made that (3.11) is the approximate shock front equation even as the region $\pi/2-\theta = O(\varepsilon^{\frac{1}{2}})$ is penetrated. Of course this assumption cannot be correct for $\pi/2-\theta = o(\varepsilon^{\frac{1}{2}})$ for it predicts that $r \to \infty$ as $\theta \to \pi/2$, but it should not lead to errors that are too large if a patching can be arranged between the 'strained co-ordinate' results and the 'modified Whitham' results at a value of $\theta < \pi/2$.

However in the region $\theta = \pi/2+O(\varepsilon^{\frac{1}{2}})$, as θ approaches $\pi/2$ from below, the shock front position co-ordinate r as given by (3.11) increases too quickly to enable a successful patching to take place with the corresponding result from the modified Whitham theory. Such a large value of $dr/d\theta$ near $\theta = \pi/2$ must arise from the discontinuity in $f_r^{(1)}$ which occurs on the sound circle $r = 1$.

If modifications are made such that (3.11) is significantly altered near

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$\theta = \pi/2$ but almost left unchanged in value for the remainder of the 'shadow' region, it is conceivable that a patching might now be attained.

The author tried various modifications and the most successful was that in which the discontinuity in $f_r^{(1)}$ is shifted along the sound circle into the 'illuminated' region. This new position of the singularity has no physical significance since the discontinuity itself is physically inconsistent. The principal asset of the modification is that the 'strained co-ordinate' results for the 'shadow' region have a smaller value of $dr/d\theta$ than before. This is accomplished by requiring that the boundary conditions on $\sigma = 1$ are now

$$P^{(1)} = 0 \qquad \left(\omega \leqq \theta < \frac{\pi}{2} + \tau\varepsilon^{\frac{1}{2}}\right),$$

$$P^{(1)} = 1 \qquad \left(\frac{\pi}{2} + \tau\varepsilon^{\frac{1}{2}} < \theta \leqq \frac{3\pi}{2}\right),$$

where τ is $O(1)$ and has to be determined. With this modification the first order approximate potential solution near $r = 1$ in the range $\omega \leqq \theta < \pi/2 + \tau\varepsilon^{\frac{1}{2}}$ is

$$f^{(1)}(r, \theta) = -\frac{2^{\frac{1}{2}}\lambda}{3\pi} Q^*(\theta)(1-r)^{\frac{3}{2}} + O(1-r)^2 \tag{5.1}$$

where

$$Q^*(\theta) = \cot\left\{\frac{\lambda}{2}\left(\frac{\pi}{2} + \tau\varepsilon^{\frac{1}{2}} - \theta\right)\right\} - \cot\left\{-\frac{\lambda}{2}\left(\frac{\pi}{2} + \tau\varepsilon^{\frac{1}{2}} + \theta - 2\omega\right)\right\}. \tag{5.2}$$

If the development of Section 3 is now followed using (5.1) instead of (3.7), the shock front results (3.11) and (3.13) are replaced by

$$r = 1 + \varepsilon^2 \frac{3(\gamma+1)^2\lambda^2}{32\pi^2} \{Q^*(\theta)\}^2 + O(\varepsilon^3) \tag{5.3}$$

and

$$P = \varepsilon^2 \frac{3(\gamma+1)\lambda^2}{8\pi^2} \{Q^*(\theta)\}^2 + O(\varepsilon^3). \tag{5.4}$$

Away from the neighbourhood of the shadow boundary the difference between each of these two modified results and the corresponding result of Section 3 is only of the same order as the errors already indicated.

The discussion and assumption concerning (3.11) at the beginning of this section now applies to (5.3), so that (5.2) yields

$$r = 1 + \varepsilon^2 \frac{3(\gamma+1)^2}{8\pi^2} \left\{\frac{\pi}{2} + \tau\varepsilon^{\frac{1}{2}} - \theta\right\}^{-2} + o(\varepsilon) \tag{5.5}$$

even as the region $\pi/2 - \theta = O(\varepsilon^{\frac{1}{2}})$ is penetrated. Again from (5.3) it is seen that

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$$(5.6)\qquad \frac{dr}{d\theta} = \varepsilon^2 \frac{3(\gamma+1)^2}{4\pi^2}\left\{\frac{\pi}{2}+\tau\varepsilon^{\frac{1}{2}}-\theta\right\}^{-3}+o(\varepsilon^{\frac{1}{2}})$$

and

$$(5.7)\qquad \frac{d^2r}{d\theta^2} = \varepsilon^2 \frac{9(\gamma+1)^2}{4\pi^2}\left\{\frac{\pi}{2}+\tau\varepsilon^{\frac{1}{2}}-\theta\right\}^{-4}+o(1)$$

when $\pi/2-\theta = O(\varepsilon^{\frac{1}{2}})$.

However the shock strength is not given in this region by (5.4) for the second shock wave condition produces an extra term in the dominant part of f_r as $\theta\to\pi/2$. Thus as the interaction region is entered from below the modified shock strength relation is

$$(5.8)\qquad P = \varepsilon^2 \frac{3(\gamma+1)}{2\pi^2}\left\{\frac{\pi}{2}+\tau\varepsilon^{\frac{1}{2}}-\theta\right\}^{-2} -\varepsilon^4 \frac{9(\gamma+1)^3}{8\pi^4}\left\{\frac{\pi}{2}+\tau\varepsilon^{\frac{1}{2}}-\theta\right\}^{-6}+o(\varepsilon).$$

It is now proposed to patch the 'modified strained co-ordinate' results (5.5)—(5.8) with the corresponding ones from the modified Whitham theory, which are

$$(5.9)\qquad r = 1+\tfrac{1}{4}\left\{\frac{\pi}{2}-\theta-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}^2+O(\varepsilon^2),$$

$$(5.10)\qquad \frac{dr}{d\theta} = -\tfrac{1}{2}\left\{\frac{\pi}{2}-\theta-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}+O(\varepsilon^{\frac{3}{2}}),$$

$$(5.11)\qquad \frac{d^2r}{d\theta^2} = \tfrac{1}{2}+O(\varepsilon)$$

and

$$(5.12)\qquad P = \frac{1}{2(\gamma+1)}\left\{\frac{\pi}{2}-\theta-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}^2+O(\varepsilon^2).$$

Because of the lower limit of the Whitham theory any angle at which patching takes place is governed by the condition

$$\frac{\pi}{2}-\varepsilon^{\frac{1}{2}}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}-\theta < 0.$$

Elementary calculations then show that corresponding r and P expressions agree at the patching angle $\theta^{(p)}$ given by

$$\theta^{(p)} = \frac{\pi}{2}-\left[1-\left(\frac{6}{\pi^2}\right)^{\frac{1}{4}}\right]\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$$
$$\approx \frac{\pi}{2}-0.12\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$$

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provided that

$$\tau = \left[\left(\frac{96}{\pi^2}\right)^{\frac{1}{4}} - 1\right]\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}$$

$$\approx 0.76\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}.$$

As well the two expressions for $dr/d\theta$ are equal at the patching angle, but the curvatures do not patch at these values of $\theta^{(p)}$ and τ since (5.7) gives

$$\frac{d^2 r}{d\theta^2} = \tfrac{3}{2} + o(1).$$

Thus it has been shown that it is possible to approximately join up the 'modified Whitham' results with the 'modified strained co-ordinate' results for all convex corners whose angles lie in the range

$$-\frac{\pi}{2} \leqq \omega < \frac{\pi}{2} - O(\varepsilon^{\frac{1}{2}}).$$

It may be plausible then that a reasonable description of the shock front is given by

$$r = \begin{cases} 1+\varepsilon^2 \dfrac{3(\gamma+1)^2\lambda^2}{32\pi^2}\{Q^*(\theta)\}^2 + o[\varepsilon^2\{Q^*(\theta)\}^2], & \omega \leqq \theta \leqq \theta^{(p)}, \\ 1+\tfrac{1}{4}\left\{\dfrac{\pi}{2} - \theta - \varepsilon^{\frac{1}{2}}\left(\dfrac{\gamma+1}{2}\right)^{\frac{1}{2}}\right\}^2 + O(\varepsilon^2), & \theta^{(p)} < \theta \leqq \dfrac{\pi}{2} + \left(\dfrac{\gamma+1}{2}\right)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}, \end{cases}$$

where

$$Q^*(\theta) = \cot\left\{\frac{\lambda}{2}\left(\frac{\pi}{2} + \varepsilon^{\frac{1}{2}}\left[\left(\frac{96}{\pi^2}\right)^{\frac{1}{4}} - 1\right]\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} - \theta\right)\right\}$$
$$- \cot\left\{-\frac{\lambda}{2}\left(\frac{\pi}{2} + \varepsilon^{\frac{1}{2}}\left[\left(\frac{96}{\pi^2}\right)^{\frac{1}{4}} - 1\right]\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} + \theta - 2\omega\right)\right\},$$

which is continuous up to second derivatives at the patching point, with the values of the 'reduced' pressure on this curve being taken as the approximate shock strength. It is recognized that this description can still only be qualitative as the 'modified Whitham' results are more and more unreliable as θ decreases away from its value at the junction of the straight and diffracted parts of the shock. On the other hand the relative success of the patching procedure means that a treatment has now been developed which gives a continuous shock strength along a front that is certainly closer to the true shock front than those predicted by 'linearized' theory.

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Royal Military College
Duntroon, A.C.T.

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