$\curlywedge$-DISTRIBUTIVE BOOLEAN MATRICES

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(Received 14 July, 1964; and in revised form 21 December, 1964)

In this paper we shall be concerned with the set $M_n(B)$ of $n \times n$ matrices whose elements belong to a given Boolean algebra $B(\leqslant, \cap, \cup, ')$.

It is well known that $M_n(B)$ also forms a Boolean algebra with respect to the partial ordering $\preccurlyeq$ defined by

$$X \preccurlyeq Y \Leftrightarrow x_{ij} \leqslant y_{ij} \quad (i, j = 1, 2, \ldots, n),$$

in which union ($\curlyvee$), intersection ($\curlywedge$) and complementation (*) are given by

$$Z = X \curlyvee Y \Leftrightarrow z_{ij} = x_{ij} \cup y_{ij} \quad (i, j = 1, 2, \ldots, n);$$

$$Z = X \curlywedge Y \Leftrightarrow z_{ij} = x_{ij} \cap y_{ij} \quad (i, j = 1, 2, \ldots, n);$$

$$Z^* = [z'_{ij}].$$

Multiplication in $M_n(B)$ is defined by

$$Z = XY \Leftrightarrow z_{ik} = \bigcup_j (x_{ij} \cap y_{jk}) \quad (i, k = 1, 2, \ldots, n).$$

It is an easy matter to show that this multiplication is associative and is, moreover, distributive with respect to $\curlyvee$ [i.e., for all $X, Y, Z \in M_n(B)$, we have $X(Y \curlyvee Z) = XY \curlyvee XZ$ and $(Y \curlyvee Z)X = YX \curlyvee ZX$]. In this way, $M_n(B)$ forms what is termed a $\curlyvee$-semireticulated semigroup.

It is not in general true, however, that this multiplication is distributive with respect to $\curlywedge$. For example, given any Boolean algebra B, consider the following matrices in $M_2(B)$:

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is readily verified that

$$X(Y \curlywedge Z) = X0 = 0 \neq X = X \curlywedge X = XY \curlywedge XZ.$$

However, the isotone property [namely, $X \preccurlyeq Y \Rightarrow XZ \preccurlyeq YZ$ and $ZX \preccurlyeq ZY, \forall Z \in M_n(B)$] implies that, for all $X, Y, Z \in M_n(B)$,

$$X(Y \curlywedge Z) \preccurlyeq XY \curlywedge XZ \quad \text{and} \quad (Y \curlywedge Z)X \preccurlyeq YX \curlywedge ZX,$$

and in this paper, we wish to find those matrices $X \in M_n(B)$ for which equality holds in either or both of the above for all choices of $Y, Z \in M_n(B)$.

It should be observed that, for given $X, Y, Z \in M_n(B)$, equality may hold in one of these without this being the case in the other. For example, for the particular matrices in $M_2(B)$ cited above, it is readily verified that

$$(Y \curlywedge Z)X = 0X = 0,$$

and that

$$YX \curlywedge ZX = X \curlywedge X^* = 0.$$

Hence in this case we have $(Y \curlywedge Z)X = YX \curlywedge ZX$ though, as we have seen above,

$$X(Y \curlywedge Z) \prec XY \curlywedge XZ.$$

We are thus led to make the following definition.

DEFINITION. $A \in M_n(B)$ will be called *left* $\curlywedge$*-distributive* if it satisfies the equality $A(X \curlywedge Y) = AX \curlywedge AY$, $\forall X, Y \in M_n(B)$; and *right* $\curlywedge$*-distributive* if it satisfies the equality $(X \curlywedge Y)A = XA \curlywedge YA$, $\forall X, Y \in M_n(B)$. A matrix which is both left and right $\curlywedge$-distributive will be called simply $\curlywedge$-distributive.

The left $\curlywedge$-distributive matrices are characterised by the following result.

THEOREM 1. *$A \in M_n(B)$ is left $\curlywedge$-distributive if and only if, for all i,*

$$a_{ij} \cap a_{ik} = 0 \quad (j \neq k).$$

Proof. Suppose that $A(X \curlywedge Y) = AX \curlywedge AY$, $\forall X, Y \in M_n(B)$. Choose in particular $X = I^* = [\delta'_{ij}]$ and $Y = I = [\delta_{ij}]$, where, as usual,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have on the one hand

$$A(X \curlywedge Y) = A(I^* \curlywedge I) = A0 = 0,$$

and on the other

$$\begin{aligned} [AX \curlywedge AY]_{ik} &= \left\{ \bigcup_j (a_{ij} \cap x_{jk}) \right\} \cap \left\{ \bigcup_m (a_{im} \cap y_{mk}) \right\} \\ &= \left\{ \bigcup_j (a_{ij} \cap \delta'_{jk}) \right\} \cap \left\{ \bigcup_m (a_{im} \cap \delta_{mk}) \right\} \\ &= \left(\bigcup_{j \neq k} a_{ij} \right) \cap a_{ik}. \end{aligned}$$

The equality therefore gives

$$0 = \left(\bigcup_{j \neq k} a_{ij} \right) \cap a_{ik} = \bigcup_{j \neq k} (a_{ij} \cap a_{ik}),$$

whence it follows that $a_{ij} \cap a_{ik} = 0, j \neq k$.

Conversely, suppose that the condition is satisfied; then

$$[AX \curlywedge AY]_{ik} = \left\{\bigcup_j (a_{ij} \cap x_{jk})\right\} \cap \left\{\bigcup_m (a_{im} \cap y_{mk})\right\}$$

$$= \bigcup_{j,\,m} (a_{ij} \cap x_{jk} \cap a_{im} \cap y_{mk})$$

$$= \bigcup_j (a_{ij} \cap x_{jk} \cap y_{jk})$$

$$= [A(X \curlywedge Y)]_{ik}.$$

In an analogous way, we can establish:

THEOREM 1′. *$A \in M_n(B)$ is right ⋏-distributive if and only if, for all j,*

$$a_{ij} \cap a_{kj} = 0 \quad (i \neq k).$$

⋏-distributive matrices of especial interest are those matrices which possess an inverse. We recall [1] that if an $n \times n$ Boolean matrix $A = [a_{ij}]$ has a one-sided inverse, then that inverse is a two-sided inverse, is unique and is none other than A^T, the transpose of A. Moreover, for such an inverse to exist, it is necessary and sufficient that

$$\begin{cases} \bigcup_j a_{ij} = 1 & (i = 1, 2, \ldots, n), \\ a_{ij} \cap a_{kj} = 0 & (i \neq k), \end{cases} \tag{1}$$

or, equivalently, that

$$\begin{cases} \bigcup_i a_{ij} = 1 & (j = 1, 2, \ldots, n), \\ a_{ij} \cap a_{ik} = 0 & (j \neq k). \end{cases} \tag{2}$$

If now we denote by $H_n(B)$ the set of all left ⋏-distributive matrices in $M_n(B)$, we have that

(*a*) $A,\ C \in H_n(B) \Rightarrow AC \in H_n(B)$; in fact, since matrix multiplication is associative, $AC(X \curlywedge Y) = A(CX \curlywedge CY) = ACX \curlywedge ACY$.

(*b*) $A \in H_n(B),\ X \in M_n(B) \Rightarrow A \curlywedge X \in H_n(B)$; this is an immediate consequence of Theorem 1.

It follows from these results that $H_n(B)$ forms a subsemigroup and an ⋏-subsemilattice of $M_n(B)$. The same is true of $K_n(B)$, the set of all right ⋏-distributive matrices.

LEMMA 1. *If $X,\ Y \in H_n(B)$, then $X \curlyvee Y \in H_n(B)$ if and only if, for all i, $x_{ij} \cap y_{ik} = 0$ $(j \neq k)$. Correspondingly, if $X,\ Y \in K_n(B)$, then $X \curlyvee Y \in K_n(B)$ if and only if, for all j, $x_{ij} \cap y_{kj} = 0$ $(i \neq k)$.*

Proof. Let $X,\ Y \in H_n(B)$ and let $Z = X \curlyvee Y$; then $z_{ij} = x_{ij} \cup y_{ij}$ and by Theorem 1 we have that $Z \in H_n(B)$ if and only if, for all i,

$$(x_{ij} \cup y_{ij}) \cap (x_{ik} \cup y_{ik}) = 0 \quad (j \neq k),$$

https://doi.org/10.1017/S2040618500035243 Published online by Cambridge University Press

which, by virtue of the distributive law, is true if and only if

$$(x_{ij}\cap x_{ik})\cup(x_{ij}\cap y_{ik})\cup(y_{ij}\cap x_{ik})\cup(y_{ij}\cap y_{ik}) = 0 \quad (j \neq k),$$

and, since X, $Y \in H_n(B)$ by hypothesis, this is satisfied if and only if, for all i,

$$x_{ij}\cap y_{ik} = 0 \quad (j \neq k).$$

A similar proof applied to $K_n(B)$ gives the second result.

THEOREM 2. *Given $A \in H_n(B)$, the matrix M defined by*

$$\begin{cases} m_{ij} = a_{ij} & (j \neq i), \\ m_{ii} = \bigcap_{k \neq i} a'_{ik}, \end{cases}$$

is a maximal element of $H_n(B)$ containing A.

Proof. To show that $A \preccurlyeq M$, all we need verify is that $a_{ii} \leqslant m_{ii}$ for all i. Now, since $A \in H_n(B)$ by hypothesis, it follows from Theorem 1 that

$$a_{ij}\cap\left(\bigcup_{k \neq j} a_{ik}\right) = 0,$$

so that

$$a_{ij} \leqslant \left(\bigcup_{k \neq j} a_{ik}\right)' = \bigcap_{k \neq j} a'_{ik}.$$

Choosing $j = i$, we then have $a_{ii} \leqslant m_{ii}$.

To prove that $M \in H_n(B)$, we observe that, for i, j, k all different,

$$m_{ij}\cap a_{ik} = a_{ij}\cap a_{ik} = 0, \tag{3}$$

whilst for $k \neq i$,

$$m_{ii}\cap a_{ik} = \left(\bigcap_{j \neq i} a'_{ij}\right)\cap a_{ik} = \left\{\bigcap_{j \neq i, k} a'_{ij}\right\}\cap a'_{ik}\cap a_{ik} = 0. \tag{4}$$

The equations (3) and (4) taken along with Lemma 1 show that $M = A \curlyvee M \in H_n(B)$.

To prove that M is a maximal element of $H_n(B)$, consider any $X \in H_n(B)$ such that $M \preccurlyeq X$. Since $x_{ij}\cap x_{ik} = 0$ $(j \neq k)$, we have that

$$\bigcup_{j \neq k} x_{ij} \leqslant x'_{ik},$$

so that, for all i and k,

$$m'_{ik} = \bigcup_{j \neq k} m_{ij} \leqslant \bigcup_{j \neq k} x_{ij} \leqslant x'_{ik}.$$

But clearly from $M \preccurlyeq X$ we have that $x'_{ik} \leqslant m'_{ik}$ for all i, k. It follows, therefore, that $X = M$ and consequently M is maximal in $H_n(B)$.

COROLLARY 1. *$A \in H_n(B)$ is maximal in $H_n(B)$ if and only if $\bigcup_j a_{ij} = 1$.*

Proof. If A is maximal in $H_n(B)$, then, by the above theorem, we have $a_{ii} = \bigcap_{j \neq i} a'_{ij}$, so that

$$\bigcup_j a_{ij} = a_{ii} \cup \bigcup_{j \neq i} a_{ij} = a_{ii} \cup \left(\bigcap_{j \neq i} a'_{ij} \right)' = a_{ii} \cup a'_{ii} = 1.$$

Conversely, if $A \in H_n(B)$ is such that $\bigcup_j a_{ij} = 1$, then clearly

$$\left(\bigcup_{j \neq i} a_{ij} \right) \cap a_{ii} = 0 \quad \text{and} \quad \left(\bigcup_{j \neq i} a_{ij} \right) \cup a_{ii} = 1,$$

from which it follows that

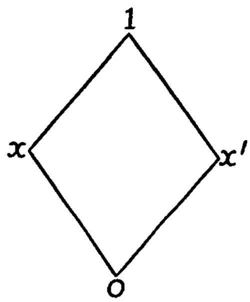
$$a_{ii} = \left(\bigcup_{j \neq i} a_{ij} \right)' = \bigcap_{j \neq i} a'_{ij}.$$

Hence, by the theorem, A is maximal in $H_n(B)$.

COROLLARY 2. *$A \in M_n(B)$ is a maximal element of both $H_n(B)$ and $K_n(B)$ if and only if A has an inverse.*

Proof. This follows immediately from the Wedderburn-Rutherford conditions (1) and (2), the above results and their analogues.

It should be observed that A may be maximal in $H_n(B)$ without being maximal in $K_n(B)$. For example, choosing the Boolean algebra whose Hasse diagram is



and considering matrices in $M_2(B)$, we see that the matrix

$$X = \begin{bmatrix} x & x' \\ 1 & 0 \end{bmatrix}$$

belongs to and is a maximal element of $H_2(B)$. X is not, however, invertible. (In fact, the only invertible matrices in $M_2(B)$ are of the form

$$\begin{bmatrix} y & y' \\ y' & y \end{bmatrix}$$

where $y = 0, x, x'$ or 1.)

We now establish the following characterisation of ⋏-distributive Boolean matrices.

THEOREM 3. *$X \in M_n(B)$ is ⋏-distributive if and only if there exists an invertible $A \in M_n(B)$ such that $X \preccurlyeq A$.*

https://doi.org/10.1017/S2040618500035243 Published online by Cambridge University Press

Proof. If $X \preccurlyeq A$ where A is invertible, then, by the result (*b*) preceding Lemma 1, we have that $X = A \curlywedge X \in H_n(B)$ and similarly $X \in K_n(B)$.

Conversely, given that X is an $n \times n$ $\curlywedge$-distributive matrix, we wish to show that X is contained in some invertible matrix Y.

We build up systematically a sequence of matrices

$$X \preccurlyeq M_1^{(n)} \preccurlyeq M_2^{(n)} \preccurlyeq \dots \preccurlyeq M_n^{(n)} \tag{5}$$

in which each $M_i^{(n)}$ is $\curlywedge$-distributive and $M_n^{(n)}$ is invertible. By hypothesis, X satisfies the conditions

$$\begin{cases} x_{ij} \cap x_{ik} = 0 & (j \neq k), \\ x_{ij} \cap x_{kj} = 0 & (i \neq k), \end{cases} \tag{6}$$

from which it follows that

$$x_{ij} \leqslant \bigcap_{k \neq j} x'_{ik} \cap \bigcap_{k \neq i} x'_{kj}. \tag{7}$$

Now we observe that the relations (6) remain unaltered if, for a given x_{ij}, we replace this x_{ij} by the right-hand side of (7). We use this fact repeatedly in building up the sequence (5) in the following way. We begin by replacing the leading element of X, then proceed along the first row and then down the first column. At this stage, we will have the matrix $M_1^{(n)}$ of (5) which is $\curlywedge$-distributive, contains X and is such that its first row and column satisfy the conditions (1) and (2).

We begin, therefore, with the matrix $P_1^{(1)}$ defined from X by

$$[P_1^{(1)}]_{ij} = \begin{cases} \bigcap_{k>1} x'_{1k} \cap \bigcap_{k>1} x'_{k1} & \text{if } i = 1, j = 1, \\ x_{ij} & \text{otherwise.} \end{cases}$$

We now proceed along the first row, defining recursively the sequence

$$X \preccurlyeq P_1^{(1)} \preccurlyeq P_1^{(2)} \preccurlyeq \dots \preccurlyeq P_1^{(n)}$$

in the following way:

$$[P_1^{(r)}]_{ij} = \begin{cases} [P_1^{(j)}]_{1j} & \text{if } i = 1, j < r, \\ \bigcap_{k<r} [P_1^{(k)}]'_{1k} \cap \bigcap_{k>r} x'_{1k} \cap \bigcap_{k>1} x'_{kr} & \text{if } i = 1, j = r, \\ x_{ij} & \text{otherwise.} \end{cases} \tag{8}$$

Denoting for convenience $P_1^{(n)}$ by $M_1^{(1)}$, we now proceed down the first column, thus defining the sequence

$$M_1^{(1)} \preccurlyeq M_1^{(2)} \preccurlyeq \dots \preccurlyeq M_1^{(n)}$$

in the following recursive way:

$$[M_1^{(r)}]_{ij} = \begin{cases} [M_1^{(i)}]_{i1} & \text{if } i < r,\ j = 1, \\ \bigcap_{k<r} [M_1^{(k)}]'_{k1} \cap \bigcap_{k>1} x'_{rk} \cap \bigcap_{k>r} x'_{k1} & \text{if } i = r,\ j = 1, \\ x_{ij} & \text{otherwise.} \end{cases}$$

At this stage, we have the matrix $M_1^{(n)}$ of the sequence (5), and by its construction it satisfies the conditions (6).

Consider now the first row of $M_1^{(n)}$; using the formula

$$x \cup (x' \cap y) = x \cup y, \tag{9}$$

we have

$$[M_1^{(n)}]_{1,n-1} \cup [M_1^{(n)}]_{1,n} = [P_1^{(n)}]_{1,n-1} \cup \left\{ \bigcap_{k<n} [P_1^{(k)}]'_{1k} \cap \bigcap_{k>1} x'_{kn} \right\}$$

$$= [P_1^{(n)}]_{1,n-1} \cup \left\{ \bigcap_{k<n-1} [P_1^{(k)}]'_{1k} \cap \bigcap_{k>1} x'_{kn} \right\}$$

$$= \bigcap_{k<n-1} [P_1^{(k)}]'_{1k} \cap \left\{ \left(x'_{1n} \cap \bigcap_{k>1} x'_{k,n-1} \right) \cup \bigcap_{k>1} x'_{kn} \right\},$$

by (8) and the distributive law.

Taking the union of this in turn with $[M_1^{(n)}]_{1,n-2}$, $[M_1^{(n)}]_{1,n-3}$, ..., and using repeatedly the formula (9), we have

$$\bigcup_j [M_1^{(n)}]_{1j} = \left\{ \bigcap_{k>1} x'_{1k} \cap \bigcap_{k>1} x'_{k1} \right\} \cup \left\{ \bigcap_{k>2} x'_{1k} \cap \bigcap_{k>1} x'_{k2} \right\} \cup \dots \cup \left\{ \bigcap_{k>1} x'_{kn} \right\}$$

$$= \bigcup_j \left\{ \bigcap_{k>j} x'_{1k} \cap \bigcap_{k>1} x'_{kj} \right\} = 1,$$

since we may express this as an intersection of unions, each of which has the value 1 by virtue of the conditions (6). [E.g., for $n = 2$,

$$\bigcup_j [M_1^{(2)}]_{1j} = (x'_{12} \cap x'_{21}) \cup x'_{22} = (x'_{12} \cup x'_{22}) \cap (x'_{21} \cup x'_{22})$$

$$= (x_{12} \cap x_{22})' \cap (x_{21} \cap x_{22})'$$

$$= 1, \quad \text{by (6).]}$$

In an exactly analogous way, we can also show that

$$\bigcup_i [M_1^{(n)}]_{i1} = \bigcup_i \left\{ \bigcap_{k>1} x'_{ik} \cap \bigcap_{k>i} x'_{k1} \right\} = 1.$$

We may now re-start the process of substitution and build up the matrix $M_2^{(n)}$ from $M_1^{(n)}$ exactly as we built up $M_1^{(n)}$ from X, though in this case we leave the first row and column alone and deal with the second row and second column. We then build up $M_3^{(n)}$ from $M_2^{(n)}$ by concentrating on the third row and column of $M_2^{(n)}$, and so on.

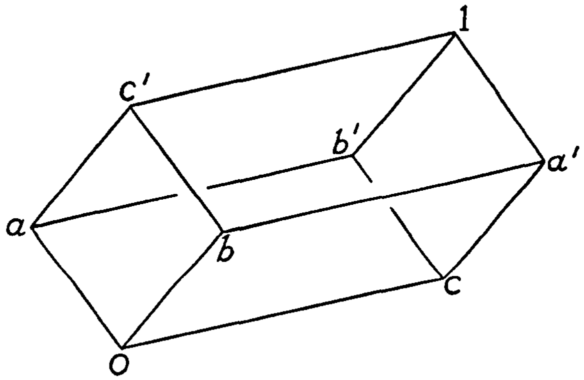
In this way, we construct the sequence (5) and eventually arrive at the matrix $M_n^{(n)}$ which is $\curlywedge$ -distributive, contains X and satisfies the condition

$$\bigcup_j [M_n^{(n)}]_{ij} = 1 = \bigcup_i [M_n^{(n)}]_{ij} \quad \text{for each } i, j.$$

Consequently, $M_n^{(n)}$ is invertible.

By way of illustration of the above process, consider the following $\curlywedge$ -distributive matrix over the Boolean algebra shown:

$$X = \begin{bmatrix} a & c & 0 \\ b & 0 & a \\ 0 & c' & 0 \end{bmatrix}$$



To exhibit the process, we write it as a sequence as follows, in which the replacement of each entry in turn, according to the process described in the proof, is entered in bold type:

$$X \to \underset{P_1^{(1)}}{\begin{bmatrix} \mathbf{a} & c & 0 \\ b & 0 & a \\ 0 & c' & 0 \end{bmatrix}} \to \underset{P_1^{(2)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & 0 \\ b & 0 & a \\ 0 & c' & 0 \end{bmatrix}} \to \underset{P_1^{(3)} = M_1^{(1)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ b & 0 & a \\ 0 & c' & 0 \end{bmatrix}} \to$$

$$\to \underset{M_1^{(2)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & 0 & a \\ 0 & c' & 0 \end{bmatrix}} \to \underset{M_1^{(3)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & 0 & a \\ \mathbf{0} & c' & 0 \end{bmatrix}} \to \begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & \mathbf{0} & a \\ \mathbf{0} & c' & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & c' & 0 \end{bmatrix} \to \underset{M_2^{(3)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{c}' & 0 \end{bmatrix}} \to \underset{M_3^{(3)}}{\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{a}' & \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{c}' & \mathbf{c} \end{bmatrix}}.$$

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