MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIALS

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1. Introduction

It is well known that every monic polynomial of degree n with coefficients in a field Φ is the characteristic polynomial of some $n \times n$ matrix A with elements in Φ. However, it is clear that this result is an extremely weak one, and that it should be possible to impose considerable restrictions upon the matrix A. In this note we prove two results in this direction. In section 2, we show that it is possible to prescribe all but one of the diagonal elements of A. This result was first proved by Mirsky (2) when the ground field Φ is the field of complex numbers. In section 3, we see that we can require A to have any prescribed non-derogatory $n-1 \times n-1$ matrix in the top left-hand corner.

2. Theorem (2.1)

Let $f(x)$ be any monic polynomial of degree n with coefficients in a field Φ, and suppose that $a_1, a_2, \ldots, a_{n-1}$ are elements of Φ. Then there exists an $n \times n$ matrix A, having $a_1, a_2, \ldots, a_{n-1}$ on its main diagonal, whose characteristic polynomial is $f(x)$.

Proof. Let

$$f(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n$$

be the given polynomial, and define a_n by

$$c_1 = -(a_1 + a_2 + \ldots + a_{n-1} + a_n).$$

The $n+1$ polynomials

$$u_0 = 1,\ u_1 = x - a_1,\ u_2 = (x-a_1)(x-a_2),\ \ldots,\ u_n = (x-a_1)(x-a_2)\ldots(x-a_n),$$

clearly form a basis of the vector space of polynomials over Φ of degree less than $n+1$. Our polynomial $f(x)$ can therefore be written uniquely in the form

$$f(x) = u_n + e_1 u_{n-1} + e_2 u_{n-2} + \ldots + e_n u_0.$$

Comparing coefficients of x^{n-1}, we find that

$$e_1 = c_1 + a_1 + a_2 + \ldots + a_n = 0.$$

It follows that the matrix

$$A = \begin{bmatrix} a_1 & 1 & 0 & & & \\ 0 & a_2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 0 & a_{n-1} & 1 \\ -e_n & -e_{n-1} & \cdot & \cdot & -e_2 & a_n \end{bmatrix}$$

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has characteristic polynomial $f(x)$, as is readily verified by expanding *det* $(xI_n - A)$ with respect to the last row.

3. We denote by $\Phi_{p,q}$ the set of all $p \times q$ matrices over the field Φ. The transpose of any matrix A is denoted by A^T and its adjugate by adj A. A square $m \times m$ matrix A is said to be non-derogatory if its characteristic polynomial is also its minimum polynomial. It is well known (1) that A is non-derogatory if and only if there exists $P \in \Phi_{m,1}$ such that the vectors

$$P, AP, A^2P, \ldots, A^{m-1}P$$

form a basis of $\Phi_{m,1}$.

Lemma (3.1). *Let B be an $m \times m$ matrix over Φ. Then the set of all $1 \times m$ matrices of the form*

$$(Q^TP, Q^TBP, Q^TB^2P, \ldots, Q^TB^{m-1}P)$$

with $Q, P \in \Phi_{m,1}$ is the whole set $\Phi_{1,m}$ if and only if B is non-derogatory.

Remark : The " only if " part is not actually required for the proof of our theorem (3.4).

Proof. Let Γ be the set of all $(Q^TP, Q^TBP, \ldots, Q^TB^{m-1}P)$. If B is non-derogatory then we can find $P \in \Phi_{m,1}$ such that the $m \times m$ matrix

$$K = (P, BP, B^2P, \ldots, B^{m-1}P)$$

is non-singular. Since for every $X \in \Phi_{1,m}$ we have

$$X = (XK^{-1})K = (Q^TP, Q^TBP, \ldots, Q^TB^{m-1}P)$$

where $Q^T = XK^{-1}$, it follows that $\Gamma = \Phi_{1,m}$.

Conversely, suppose that $\Gamma = \Phi_{1,m}$. Then we can find $Q_1, \ldots, Q_m$; $P_1, \ldots, P_m \in \Phi_{m,1}$ such that the square matrix

$$\begin{bmatrix} Q_1^TP_1, & Q_1^TBP_1, & \ldots, & Q_1^TB^{m-1}P_1 \\ Q_2^TP_2, & Q_2^TBP_2, & \ldots, & Q_2^TB^{m-1}P_2 \\ \ldots & \ldots & \ldots, & \ldots \\ Q_m^TP_m, & \ldots & \ldots, & Q_m^TB^{m-1}P_m \end{bmatrix}$$

is non-singular. The columns of such a matrix are then linearly independent, and consequently the matrices $I_m, B, B^2, \ldots, B^{m-1}$ are linearly independent. The minimum polynomial of B is therefore of degree m and hence B is non-derogatory.

Lemma (3.2). *Let $g(x)$ be a monic polynomial of degree m over Φ and let λ be an indeterminate. Then the rational function*

$$\phi(x, \lambda) = [g(x) - g(\lambda)]/(x - \lambda)$$

is a polynomial in x, λ over Φ. If, furthermore

$$\phi(x, \lambda) = \sum_{0}^{m-1} u_r(x)\lambda^r$$

then the polynomials $u_0(x), \ldots, u_{m-1}(x)$ form a basis of the space of all polynomials in x over Φ of degree at most $m-1$.

Proof. The first assertion is clear. To prove the second let

$$g(x)=x^m+d_1x^{m-1}+\ldots+d_m.$$

Then

$$\phi(x, \lambda)=\sum_{r=0}^{m-1} d_{m-r-1} \sum_{s=0}^{r} x^{r-s}\lambda^s \quad (d_0=1)$$

$$=\sum_{s=0}^{m-1} \lambda^s \sum_{r=s}^{m-1} d_{m-r-1}x^{r-s}.$$

Consequently, for $s=0, 1, \ldots, m-1$ we have

$$u_s(x)=x^{m-s-1}+d_1x^{m-s-2}+\ldots+d_{m-s-2}x+d_{m-s-1}.$$

This proves the assertion.

Lemma (3.3). (Frobenius.) *Let $g(\lambda)$ be the characteristic polynomial of the $m\times m$ matrix B over Φ and let $\phi(x, \lambda)=[g(x)-g(\lambda)]/(x-\lambda)$.*

Then $$\phi(x, B)=\text{adj }(xI_m-B).$$

Proof. We have

$$(x-\lambda).\phi(x, \lambda)=g(x)-g(\lambda).$$

Therefore, by the Cayley-Hamilton theorem,

$$(xI_m-B)\phi(x, B)=g(x)I_m,$$

$$\phi(x, B)=g(x)(xI_m-B)^{-1}=\text{adj}(xI_m-B).$$

Theorem (3.4). *Let B be a given $n-1\times n-1$ non-derogatory matrix over Φ and $f(x)$ a given monic polynomial of degree n over Φ. Then there exists an $n\times n$ matrix having B in the top left-hand corner, whose characteristic polynomial is $f(x)$.*

Proof. The result is trivial for $n=1$. Suppose then that $n>1$, and that

$$g(x)=x^{n-1}+t_1x^{n-2}+\ldots$$

is the characteristic polynomial of B. Let $P, Q\in\Phi_{m,1}$, $b\in\Phi$, and consider the $n\times n$ matrix

$$A=\begin{bmatrix} B & P \\ Q^T & b \end{bmatrix}.$$

We have, using lemmas (3.2) and (3.3),

$$\begin{aligned}\det (xI_n-A)&=(x-b)\det (xI_{n-1}-B)-Q^T[\text{adj }(xI_{n-1}-B)]P\\ &=(x-b)(x^{n-1}+t_1x^{n-2}+\ldots)-\sum_{r=0}^{n-2} u_r(x)Q^TB^rP\\ &=x^n+(t_1-b)x^{n-1}+h(x)-\sum_{r=0}^{n-2} u_r(x)Q^TB^rP,\end{aligned}$$

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where $h(x)$ has degree at most $n-2$. By lemmas (3.1) and (3.2) we can choose Q, P such that

$$h(x)-\sum_{r=0}^{n-2} u_r(x)Q^T B^r P.$$

is any polynomial of degree at most $n-2$. It follows that by choosing b suitably, we can make sure that det (xI_n-A) is any prescribed monic polynomial of degree n. This proves the result.

We conclude by pointing out that Theorem (2.1) also follows from Theorem (3.4) in virtue of the following

Lemma (3.5). *Let $a_1, a_2, \ldots, a_m$ be any elements of Φ. Then there exists a non-derogatory $m \times m$ matrix B over Φ with $a_1, a_2, \ldots, a_m$ (in that order) as diagonal elements.*

In fact the matrix $B=(b_{ij})$ defined by

$b_{ii}=a_i$, $(i=1, 2, \ldots, m)$,
$b_{ij}=1$ whenever $i<j$, $a_i=a_j$, but $a_i \neq a_k$ for $i<k<j$,
$b_{ij}=0$ in all other cases,

satisfies our requirements.

REFERENCES

(1) N. Jacobson, *Lectures on Abstract Algebra, II*, p. 69.

(2) L. Mirsky, Matrices with prescribed characteristic roots and diagonal elements, *J. London Math. Soc.*, **33**, 1 (1958), No. 129, pp. 14-21.

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