ON COMMUTING RINGS OF ENDOMORPHISMS

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1. Introduction. Various problems concerning the general theory of centralizers of modules which are not assumed to be completely reducible have been discussed by Fitting **(3)**, Brauer **(2)**, and Nakayama. In this paper we present a new approach to some of these questions, which has its origin in Weyl's discussion **(15)** of the centralizer of a finite group of collineations.

Let $\mathfrak{B}$ be a ring with an identity element, and let $\mathfrak{M}'$ and $\mathfrak{M}$ be unital[1] left and right $\mathfrak{B}$-modules, respectively. We assume that there exists a function $\tau(\psi, x)$ on $\mathfrak{M}' \times \mathfrak{M} \rightarrow \mathfrak{B}$ which is bilinear with respect to $\mathfrak{B}$, and non-degenerate. The set $\mathfrak{b}$ of all finite sums $\sum \tau(\psi_i, x_i)$ is a two-sided ideal in $\mathfrak{B}$, called the nucleus of the pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$. Let $\mathfrak{E}$ be the ring of all $\mathfrak{B}$-endomorphisms of $\mathfrak{M}$. Then $\mathfrak{E}$ contains the right ideal $\mathfrak{M}' \odot \mathfrak{M}$ consisting of all finite sums of the endomorphisms $\psi \odot u$ of $\mathfrak{M}$, where $x(\psi \odot u) = u\tau(\psi, x)$, $x \in \mathfrak{M}$. By a centralizer $\mathfrak{C}$ of $\mathfrak{M}$ relative to $\mathfrak{B}$ we mean a subring $\mathfrak{C}$ of $\mathfrak{E}$ containing the right ideal $\mathfrak{M}' \odot \mathfrak{M}$.

Our basic assumption is that the nucleus $\mathfrak{b}$ contain a two-sided identity element. Then it is proved in §5 that the ring of $\mathfrak{C}$-endomorphisms of $\mathfrak{M}$ is precisely the set of endomorphisms $R_b\colon x \rightarrow xb$ determined by the elements of $\mathfrak{B}$. Let $\mathfrak{N}$ be a $\mathfrak{C}$-direct summand of $\mathfrak{M}$; then $\tau(\mathfrak{M}', \mathfrak{N})$ is a left ideal in $\mathfrak{b}$, and the mapping $\mathfrak{N} \rightarrow \tau(\mathfrak{M}', \mathfrak{N})$ is a (1–1) mapping, preserving direct sums, intersections, and isomorphism relations, between the set of $\mathfrak{C}$-direct summands of $\mathfrak{M}$ and the set of left ideal direct components of $\mathfrak{b}$. Dually, if $\mathfrak{M}' \odot \mathfrak{M}$ contains the identity operator on $\mathfrak{M}$, and if the pairing $\psi \odot u$ is non-degenerate, then the mapping $\mathfrak{N} \rightarrow \mathfrak{M}' \odot \mathfrak{N}$ defines a (1–1) mapping between the set of $\mathfrak{B}$-direct summands of $\mathfrak{M}$ and the set of left ideal direct components of the centralizer $\mathfrak{C}$. If $\mathfrak{B}$ satisfies the minimum condition for left ideals, then every indecomposable $\mathfrak{C}$-direct summand $\mathfrak{N}$ of $\mathfrak{M}$ contains a unique maximal $\mathfrak{C}$-submodule, and if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are indecomposable $\mathfrak{C}$-direct summands, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are $\mathfrak{C}$-isomorphic if and only if $\mathfrak{N}_1/\mathfrak{S}_1$ and $\mathfrak{N}_2/\mathfrak{S}_2$ are $\mathfrak{C}$-isomorphic, where $\mathfrak{S}_i$ is the unique maximal submodule of $\mathfrak{N}_i$, $i = 1, 2$.

The principal application of this theory is to projective (or ray) representations of a finite group $\mathfrak{G}$ by s.l.t. (semi-linear transformations) of a vector space $\mathfrak{M}$ over a division ring Δ. If $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ is the crossed product associated with the projective representation, then it is proved in §2 that a space $\mathfrak{M}'$, and a pairing τ of $\mathfrak{M}' \times \mathfrak{M} \rightarrow \mathfrak{B}$ which satisfies our hypotheses,

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[1]A left or right $\mathfrak{B}$-module $\mathfrak{M}$ is called *unital* if the identity element of $\mathfrak{B}$ acts as identity operator on $\mathfrak{M}$.

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can be constructed if and only if the normalized factor set ρ satisfies the condition $\rho_{s,s^{-1}} = 1$ for all s in $\mathfrak{G}$. In §3 the pairing considered by Weyl **(15)** is defined, and shown to satisfy our hypotheses, so that Weyl's results are consequences of the theorems proved in §5. In §4 and §8 some special results are derived which concern the pairings obtained in §2 from projective representations of finite groups. A few remarks are included in §9 on the applications of the results on projective representations to the Galois theory of primitive rings with minimal ideals. A direct proof is given in §10 of the fact that the centralizer of a symmetric algebra $\mathfrak{A}$ of l.t. in a finite dimensional vector space $\mathfrak{M}$ which is a projective $\mathfrak{A}$-module is a symmetric algebra.

2. Projective representations of finite groups[2]**.** Let $\mathfrak{M}$ be a commutative group, and Δ a division ring consisting of endomorphisms $\xi: x \to x\xi$ of $\mathfrak{M}$, such that Δ contains the identity mapping. Then $\mathfrak{M}$ is a right vector space over Δ. Two non-singular s.l.t. T_1 and T_2 in $\mathfrak{M}$ over Δ are said to be equivalent if $T_1 = T_2\mu$, where μ is a non-zero element of Δ. An equivalence class $\{T\}$ of non-singular s.l.t. is called a *projective transformation*. Multiplication of projective transformations is defined in the obvious way, and the projective transformations form a group $\mathfrak{P}(\mathfrak{M}, \Delta)$.

Now let $\mathfrak{G} = \{1, s, t, \ldots\}$ be a finite group. A homomorphism of $\mathfrak{G}$ into $\mathfrak{P}(\mathfrak{M}, \Delta)$ is called a *projective representation* of $\mathfrak{G}$. Evidently a projective representation is determined by a mapping $s \to T_s$ of $\mathfrak{G}$ into the set of non-singular s.l.t. of $\mathfrak{M}$ such that

$$(1) \qquad T_sT_t = T_{st}\,\rho_{s,t},$$

where the $\rho_{s,t}$ are certain non-zero elements of Δ. From the associative law and (1) we obtain

$$(2) \qquad \xi^{\bar{s}\,\bar{t}} = \rho_{s,t}^{-1}\,\xi^{\overline{st}}\,\rho_{s,t},$$

where $\bar{s}: \xi \to \xi^{\bar{s}}$ and $\bar{t}$ are the automorphisms of Δ determined by the s.l.t. T_s and T_t, and

$$(3) \qquad \rho_{s,tu}\,\rho_{t,u} = \rho_{st,u}\,\rho_{s,t}^{\bar{u}}.$$

If we denote the inner automorphism $\xi \to \rho_{s,t}^{-1}\xi\rho_{s,t}$ by $\bar{\rho}_{s,t}$, then (2) becomes

$$(2') \qquad \bar{s}\,\bar{t} = \overline{st}\,\bar{\rho}_{s,t}.$$

A set $\{\rho_{s,t}; \bar{u}\}$, where the $\rho_{s,t}$ are non-zero elements of Δ, and the $\bar{u}$ are automorphisms of Δ, is called a *factor set* of $\mathfrak{G}$ (in Δ) if the equations (2) and (3) hold. Thus the transformation T_s satisfying (1) determine a factor set $\{\rho_{s,t}; \bar{u}\}$. If we replace the representatives T_s of the projective transformations corresponding to the elements of $\mathfrak{G}$ by new representatives $T'_s = T_s\mu_s$, then we obtain

$$T'_sT'_t = T'_{st}\rho'_{s,t}$$

[2]For the terminology introduced in the first part of this section, see **(6,** Chap. 4, §17, 18**)**.

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where the automorphisms $\bar{s}'$, associated with T'_s satisfy

(4) $$\bar{s}' = \bar{s}\,\bar{m}_s,$$

where $\bar{m}_s$ is the inner automorphism $\xi \to \mu_s^{-1}\xi\mu_s$, and

(5) $$\rho'_{s,t} = \mu_{st}^{-1}\,\rho_{s,t}\,\mu_s^{\bar{t}}\,\mu_t.$$

Thus it is natural to say that two factor sets $\{\rho_{s,t};\bar{u}\}$ and $\{\rho'_{s,t};\bar{u}'\}$ are equivalent if there exist elements $\mu_s \neq 0$ such that (4) and (5) hold. Then a projective representation determines a class of equivalent factor sets.

Now let $\{\rho_{s,t};\bar{u}\}$ be a factor set, and let $\{b_s\}$ be a set of elements in (1–1) correspondence with the elements in $\mathfrak{G}$. The set $\mathfrak{B}$ of formal expressions $\sum b_s\xi_s$, $\xi_s \in \Delta$, $s \in \mathfrak{G}$ becomes an associative ring if we define two expressions to be equal if and only if they have the same coefficients, and if addition is defined componentwise, and multiplication using the distributive laws and the rules

$$b_s b_t = b_{st}\,\rho_{s,t},$$

$$\xi b_s = b_s\,\xi^{\bar{s}}.$$

Then $\mathfrak{B}$ is called a *crossed product* $\Delta(\mathfrak{G}, H, \rho)$ with correspondence $s \to \bar{s} = s^H$, and factor set ρ. If $\{\rho'_{s,t};\bar{u}'\}$ is a factor set equivalent to $\{\rho_{s,t};\bar{u}\}$, and if $\mathfrak{B}'$ is a crossed product $\Delta(\mathfrak{G}, H', \rho')$ with correspondence

$$s \to s^{H'} = \bar{s}$$

and factor set ρ', then it is easily verified that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic.

As Jacobson observes (**6**, p. 82), the element $b_1\rho_{1,1}^{-1}$ is an identity 1 for $\mathfrak{B}$, and if we identify Δ with the division subring 1Δ of $\mathfrak{B}$, then every element of $\mathfrak{B}$ can be expressed uniquely in the form $\sum b_s\xi_s$, where the term $b_s\xi_s$ is now the product of $b_s = b_s1$ with ξ_s. It follows that $\mathfrak{B}$ is a two-sided vector space of finite left and right dimension over Δ, and consequently $\mathfrak{B}$ satisfies both chain conditions for left and right ideals.

If $s \to T_s$ defines a projective representation of $\mathfrak{G}$ with correspondence H and factor set ρ, then

$$\sum b_s\xi_s \to \sum T_s\xi_s$$

defines a representation of $\mathfrak{B}$ by endomorphisms of the representation space $\mathfrak{M}$ such that the identity element of $\mathfrak{B}$ is mapped onto the identity mapping in $\mathfrak{M}$, while conversely any such representation of $\mathfrak{B}$ by endomorphisms of $\mathfrak{M}$ gives rise to a projective representation of $\mathfrak{G}$ with the same correspondence and factor set.

Now let $\mathfrak{M}$ be a unital right $\mathfrak{B}$-module, and hence, in particular, a right vector space over Δ. Let $\mathfrak{M}'$ be a left vector space dual to $\mathfrak{M}$ with respect to a non-degenerate bilinear form $\langle\psi, x\rangle$ on $\mathfrak{M}' \times \mathfrak{M} \to \Delta$, such that the s.l.t. R_s: $x \to xb_s$ determined by the elements of $\mathfrak{G}$ all have transposes R_s^* relative to the form $\langle\psi, x\rangle$. Thus R_s^* is a s.l.t. of $\mathfrak{M}'$ with automorphism $\bar{s}^{-1}$ such that (if we write operators on $\mathfrak{M}'$ to the left),

$$\langle \psi, xR_s\rangle^{\bar{s}^{-1}} = \langle R_s^{\ *}\ \psi, x\rangle \tag{6}$$

for all ψ and x.

We prove first that if we set $(\sum b_s\xi_s)\,\psi = \sum R_s^*(\xi_s\psi)$, then $\mathfrak{M}'$ becomes a unital left $\mathfrak{B}$-module. For all x and ψ, we have, since $1 = b_1\rho_{1,1}{}^{-1}$,

$$\langle 1\psi, x\rangle = \langle R_1^{\ *}(\rho_{1,1}{}^{-1}\psi), x\rangle = \langle \rho_{1,1}{}^{-1}\psi, xR_1\rangle^{\bar{1}^{-1}} = \langle \psi, x\rangle$$

by (2′), and hence $1\psi = \psi$. In order to prove that $\mathfrak{M}'$ is a left $\mathfrak{B}$-module, it is sufficient to prove that $(ab)\psi = a(b\psi)$. For all x and ψ, we have

$$\begin{aligned}\langle (b_s\xi b_t\eta)\ \psi, x\rangle &= \langle R_{st}^{\ *}(\rho_{s,t}\xi^{\bar{t}}\eta\psi), x\rangle \\ &= \langle \rho_{s,t}\xi^{\bar{t}}\eta\psi, xR_{st}\rangle^{\overline{st}^{-1}} = \langle \xi^{\bar{t}}\eta\psi, xR_sR_t\rangle^{\bar{\rho}_{s,t}{}^{-1}\overline{st}^{-1}} \\ &= \langle R_s^{\ *}(R_t^{\ *}(\xi^{\bar{t}}\eta\psi)), x\rangle^{\bar{s}\,\bar{t}\bar{\rho}_{s,t}{}^{-1}\overline{st}^{-1}} = \langle b_s\xi(b_t\eta\psi), x\rangle\end{aligned}$$

by (2′), and the conclusion follows from the non-degeneracy of the form.

We wish to study the centralizer of $\mathfrak{M}$ relative to $\mathfrak{B}$. Neither the centralizer, nor the projective representation corresponding to $\mathfrak{M}$, nor the crossed product $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ is changed if we change the basis (b_s) of $\mathfrak{B}$ to $(b_s\mu_s)$, where the μ_s are non-zero elements of Δ. In particular, if we set $\mu_1 = \rho_{1,1}$ and $\mu_s = 1$ if $s \neq 1$, then the equivalent factor set $\{\rho'_{s,t};\bar{u}'\}$ corresponding to the new basis $(b_s\mu_s)$ has the property that $\rho'_{1,1} = 1$, and by an application of (3) (see [6]) it follows that $\rho'_{1,s} = \rho'_{s,1} = 1$. There is no loss of generality in assuming that our original factor set is normalized in this way, and in the rest of the paper, this normalization will be tacitly assumed.

Proposition 1. *The mapping*

$$\tau(\psi, x) = \sum_s b_s\langle \psi, xR_{s^{-1}}\rangle^{\bar{s}}, \tag{7}$$

on $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{B}$ is homogeneous, in the sense that the equations

$$\tau(b\psi, x) = b\tau(\psi, x) \text{ and } \tau(\psi, xb) = \tau(\psi, x)b, \qquad b \in \mathfrak{B}, \tag{8}$$

hold, if and only if the (normalized) factor set of $\mathfrak{B}$ satisfies the condition $\rho_{s,s^{-1}} = 1$ for all s in $\mathfrak{G}$.

Proof. In the proof of this result, we shall use the abbreviation u' for u^{-1}, $u \in \mathfrak{G}$. It is an easy matter to verify that the equations (8) hold if b is an element of Δ. From (7) it follows that $\tau(\psi, x)$ is biadditive, and consequently the homogeneity is equivalent to the equations

$$\tau(b_u\psi, x) = b_u\tau(\psi, x),\ \ \tau(\psi, xb_u) = \tau(\psi, x)b_u, \qquad u \in \mathfrak{G}. \tag{8′}$$

The coefficient of b_t in $\tau(b_u\psi, x)$ is

$$\langle R_u^{\ *}\psi, xR_{t'}\rangle^{\bar{t}} = \langle \psi, xR_{t'u}\rho_{t',u}\rangle^{\bar{u}'\bar{t}} = \langle \psi, xR_{t'u}\rangle^{\bar{u}'\bar{t}}\rho_{t',u}^{\bar{u}'\bar{t}}.$$

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The coefficient of b_t in $b_u\tau(\psi, x)$ is

$$\rho_{u,u'\,t}\langle\psi, xR_{t'u}\rangle^{\overline{u't}} = \rho_{u,u'\,t}\rho_{u',\,t}\langle\psi, xR_{t'u}\rangle^{\overline{u'}\,\bar{t}}\rho'_{u',\,t}$$
$$= \rho_{1,\,t}\rho^{\bar{t}}_{u,u'}\langle\psi, xR_{t'u}\rangle^{\bar{u}'\bar{p}_{u,u'}\bar{t}}\rho'_{u',\,t} = \langle\psi, xR_{t'u}\rangle^{\bar{u}'\bar{t}}\rho^{\bar{t}}_{u,u'}\rho'_{u',\,t},$$

by (2′) and (3), and the facts that $\rho_{1,\,t} = 1$, and $\overline{a'} = \bar{u}'\bar{p}_{u,u'}$ by (2′). Thus the first equation in (8′) holds if and only if

$$\rho^{\bar{u}'\bar{t}}_{t',u} = \rho^{\bar{t}}_{u,u'}\rho'_{u',\,t}. \tag{9}$$

The coefficient of b_t in $\tau(\psi, xb_u)$ is $\langle\psi, xR_{u\,t'}\rho_{u,\,t'}\rangle^{\bar{t}}$, while the coefficient of b_t in $\tau(\psi, x)b_u$ is

$$\rho_{tu',u}\langle\psi, xR_{u\,t'}\rangle^{\overline{tu'}\,\bar{u}} = \rho_{tu',u}\rho'_{tu',u}\langle\psi, xR_{u\,t'}\rangle^{\bar{t}}\rho_{tu',u}$$

by (2). Hence the second equation in (8′) holds if and only if

$$\rho^{\bar{t}}_{u,\,t'} = \rho_{tu'.u}. \tag{10}$$

Setting $t = u$ in (10) we obtain

$$\rho^{\bar{u}}_{u,u'} = \rho_{1,u} = 1,$$

and hence $\rho_{u,u'} = 1$, so that the condition is necessary.

Assume now that $\rho_{u,u'} = 1$ for all u. By (3) we have

$$1 = \rho_{u,1}\rho_{t',\,t} = \rho_{u\,t',\,t}\rho^{\bar{t}}_{u,\,t'} \tag{11}$$

and

$$1 = \rho_{1,\,v}\rho_{u'\bar{v},u} = \rho_{u',uv}\rho_{u,\,v}.$$

Upon substituting $u \to tu'$ and $v \to u$ in the last equation we obtain

$$\rho_{u\,t',\,t}\rho_{tu',u} = 1, \tag{12}$$

and by comparing (11) and (12) we obtain (10). The condition implies that $\overline{u'} = \bar{u}'$, and we have

$$\rho^{\overline{u'}}_{t',u}\rho^{\overline{t'}}_{u',\,t} = \rho_{u'\,t,\,t'}\rho_{t'u,\,t'} = 1$$

by (10) and (12), proving (9). This completes the proof.

For an example of a projective representation whose factor set satisfies the condition of Proposition 1, but is not equivalent to one, see **(17**, p. 182**)**.

The pairing $\tau(\psi, x)$ defined in (7) is *non-degenerate in* the sense that $\tau(\mathfrak{M}', x) = 0$ implies $x = 0$, and $\tau(\psi, \mathfrak{M}) = 0$ implies $\psi = 0$. This remark follows from the fact that $\tau(\mathfrak{M}', x) = 0$ implies $\langle\mathfrak{M}', xR_1\rangle = \langle\mathfrak{M}', x\rangle = 0$ since R_1 is the identity operator, and the non-degeneracy of the form $\langle\psi, x\rangle$.

An endomorphism C of $\mathfrak{M}$ is said to belong to the *centralizer* $\mathfrak{C}$ of $\mathfrak{M}$ relative to $\mathfrak{B}$ if $(xb)C = (xC)b$ for all b in $\mathfrak{B}$, x in $\mathfrak{M}$, and if there exists an endomorphism C^* of $\mathfrak{M}'$ such that $\langle C^*\psi, x\rangle = \langle\psi, xC\rangle$ for all x and ψ. An element of $\mathfrak{C}$ is necessarily a l.t. in $\mathfrak{M}$ over Δ, and it follows that C^*, which is uniquely determined, is also a l.t. in $\mathfrak{M}'$ over Δ.

PROPOSITION 2. *An endomorphism C of $\mathfrak{M}$ is an element of $\mathfrak{C}$ if and only if there exists an endomorphism C^{**} on $\mathfrak{M}'$ such that $\tau(C^{**}\psi, x) = \tau(\psi, xC)$ for all x and ψ.*

Proof. If $C \in \mathfrak{C}$ then evidently the transpose C^* of C relative to the form $\langle \psi, x\rangle$ satisfies the equation $\tau(C^*\psi, x) = \tau(\psi, xC)$. Conversely, if C^{**} is given, then upon comparing the coefficients of b_1, we obtain $\langle C^{**}\psi, x\rangle = \langle \psi, xC\rangle$. For all $\xi \in \Delta$,

$$\tau(\psi, (x\xi)C) = \tau(C^{**}\psi, x)\, \xi = \tau(\psi, (xC)\xi),$$

and by the non-degeneracy of the form τ, C is linear. Similarly C^{**} is a l.t., and hence C^{**} is the uniquely determined transpose of C relative to the form $\langle \psi, x\rangle$. Then for all s in $\mathfrak{G}$, comparison of the coefficients of $b_{s^{-1}}$ yields $\langle C^{**}\psi, xR_s\rangle = \langle \psi, xCR_s\rangle$, and hence $\langle \psi, xCR_s\rangle = \langle \psi, xR_sC\rangle$ so that $CR_s = R_sC$ since $\mathfrak{M}'$ and $\mathfrak{M}$ are dual. It follows that C is a $\mathfrak{B}$-endomorphism of $\mathfrak{M}$, and the proof is complete.

We shall call the system $(\mathfrak{M}', \mathfrak{M}, \tau)$ a *pairing* in case τ is bilinear and non-degenerate (τ is bilinear if τ is biadditive and homogeneous relative to right and left multiplication by elements of $\mathfrak{B}$). Necessary and sufficient conditions for the bilinearity of τ are given in Proposition 1. From the bilinearity of τ it follows that

$$\mathfrak{b} = \tau(\mathfrak{M}', \mathfrak{M}) = \{\sum \tau(\psi_i, x_i)\,|\psi_i \in \mathfrak{M}', x_i \in \mathfrak{M}\}$$

is a two-sided ideal in $\mathfrak{B}$, which we shall call the *nucleus* of the pairing.

With a pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$, we shall associate a dual pairing $(\psi, u) \rightarrow \psi \odot u$ of $\mathfrak{M}' \times \mathfrak{M} \rightarrow \mathfrak{C}$, where $\psi \odot u$ is the endomorphism of $\mathfrak{M}$ defined by

(13) $$x(\psi \odot u) = u\tau(\psi, x), \qquad x \in \mathfrak{M}.$$

It is easily verified that if $(\psi \odot u)^*$ is the endomorphism of $\mathfrak{M}'$ defined by

(14) $$(\psi \odot u)^*\phi = \tau(\phi, u)\, \psi, \qquad \phi \in \mathfrak{M}',$$

then $\tau(\phi, x(\psi \odot u)) = \tau((\psi \odot u)^*\phi, x)$, and by Proposition 2, it follows that the mappings $\psi \odot u$ are in $\mathfrak{C}$. The action of $\mathfrak{C}$ upon $\mathfrak{M}$ makes $\mathfrak{M}$ a right $\mathfrak{C}$-module, while $\mathfrak{M}'$ becomes a left $\mathfrak{C}$-module if we set $C\psi = C^*\psi$, where C^* is the transpose of C relative to the forms τ, and $\langle \psi, x\rangle$. It is immediate that the pairing $\psi \odot u$ is bilinear, that is, it is biadditive, and

$$C(\psi \odot u) = (C\psi) \odot u; \qquad (\psi \odot u)C = \psi \odot uC, \qquad C \in \mathfrak{C}.$$

A sufficient condition that $\psi \odot u$ be non-degenerate is that $x \neq 0$, $\psi \neq 0$ imply $x\mathfrak{b} \neq 0$, $\mathfrak{b}\psi \neq 0$, where $\mathfrak{b} = \tau(\mathfrak{M}', \mathfrak{M})$ is the nucleus of the original pairing. Indeed, suppose that $\psi \odot \mathfrak{M} = 0$. Then

$$\mathfrak{M}(\psi \odot \mathfrak{M}) = \mathfrak{M}\tau(\psi, \mathfrak{M}) = 0.$$

Therefore $\tau(\mathfrak{b}\psi, \mathfrak{M}) = 0$, and $\mathfrak{b}\psi = 0$ by the non-degeneracy of τ. Therefore

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$\psi = 0$. Similarly $\mathfrak{M}' \odot x = 0$ implies $x = 0$. We have proved the following result.

PROPOSITION 3. *Let $(\mathfrak{M}', \mathfrak{M}, \tau)$ be a pairing. Then $(\psi, u) \to \psi \odot u$ defines a pairing of $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{C}$ which is bilinear. The pairing $\psi \odot u$ is non-degenerate if $x \neq 0, \psi \neq 0$ imply $x\mathfrak{b} \neq 0$ and $\mathfrak{b}\psi \neq 0$, where $\mathfrak{b}$ is the nucleus of the pairing τ. The set $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$ consisting of all finite sums $\sum \psi_i \odot u_i$ is a two-sided ideal in $\mathfrak{C}$.*

The mappings $\psi \odot u$ belonging to the nucleus of the pairing defined by (13) can be characterized quite simply if we use the formalism of finite valued l.t. (**8**, Chap. VIII). Every finite valued l.t. X in $\mathfrak{M}$ over Δ which possesses a transpose X^* relative to $\langle\psi, x\rangle$ can be expressed in the form

$$X = \sum \psi_i \times u_i, \qquad \psi_i \in \mathfrak{M}', \quad u_i \in \mathfrak{M},$$

where $x(\sum \psi_i \times u_i) = \sum u_i\langle\psi_i, x\rangle$, $x \in \mathfrak{M}$. We wish to prove the formula

$$\psi \odot u = \sum_s R_{s^{-1}}(\psi \times u)R_s. \tag{15}$$

We have for all x,

$$x\sum {}_sR_{s^{-1}}(\psi \times u)\, R_s = \sum {}_s u\langle\psi, xR_{s^{-1}}\rangle\, R_s = \sum {}_s uR_s\langle\psi, xR_{s^{-1}}\rangle^{\bar{s}}$$
$$= u\tau(\psi, x) = x(\psi \odot u).$$

Various special cases of the situation considered in this section are of importance. We should like to mention especially the applications to *affine representations* of finite groups (**6**, p. 81), where all $\rho_{s,t} = 1$, and consequently the pairing τ is bilinear in all cases, by Proposition 1; and to ordinary representations of groups, where all $\rho_{s,t} = 1$, all $\bar{s} = 1$, and Δ is a field.

3. A pairing constructed by Weyl. We shall discuss a pairing introduced by Weyl (**15**) which differs from the one we have defined in §2 in that its bilinearity depends upon the existence of an involution in the crossed product $\mathfrak{B}$. We consider an affine representation $s \to U_s$ of a finite group $\mathfrak{G}$ by s.l.t. in a vector space $\mathfrak{M}$ over a field Φ; then all $\rho_{s,t} = 1$, and $s \to \bar{s}$ is a homomorphism. Let $\mathfrak{B} = \Phi(\mathfrak{G}, H, 1)$ be the crossed product constructed as in §2. In this case we have $b_sb_t = b_{st}$, and $\xi b_s = b_s\xi^{\bar{s}}$, $\xi \in \Phi$. Since Φ is commutative it follows that the mapping

$$J: \sum b_s\xi_s \to \sum \xi_s b_{s^{-1}}$$

is an involution in $\mathfrak{B}$. We obtain a representation of $\mathfrak{B}$ by endomorphisms of $\mathfrak{M}$ by setting

$$xU(\textstyle\sum_s b_s\xi_s) = \sum_s (xU_s)\xi_s.$$

Then $\mathfrak{M}$ becomes a left $\mathfrak{B}$-module (and a left vector space over Φ) if we define $bx = xU(b^J)$, $x \in \mathfrak{M}$, $b \in \mathfrak{B}$. The right vector space $\mathfrak{M}^*$ of all linear functions on $\mathfrak{M}$ becomes a right $\mathfrak{B}$-module if we define

$$\psi(\sum b_s \xi_s) = \sum \psi U^*(b_{s^{-1}})\xi_s, \; \psi \in \mathfrak{M}^*,$$

where

$$U^*(b_{s^{-1}})$$

is the transpose of the s.l.t. $U(b_s)$.

We introduce a pairing σ on $\mathfrak{M} \times \mathfrak{M}^* \to \mathfrak{B}$ by means of the following formula:

$$\sigma(x, \psi) = \sum_s b_s \langle x U_s, \psi \rangle, \tag{16}$$

where $\langle x, \psi \rangle$ is the bilinear form on $\mathfrak{M} \times \mathfrak{M}^* \to \Phi$. It is not difficult to verify that σ is bilinear:

$$\begin{aligned} \sigma(x_1 + x_2, \psi) &= \sigma(x_1, \psi) + \sigma(x_2, \psi) \\ \sigma(x, \psi_1 + \psi_2) &= \sigma(x, \psi_1) + \sigma(x, \psi_2), \\ \sigma(bx, \psi) &= b\sigma(x, \psi), \quad \sigma(x, \psi b) = \sigma(x, \psi)b, \qquad b \in \mathfrak{B}, \end{aligned}$$

and that σ is non-degenerate: $\sigma(\mathfrak{M}, \psi) = 0$ implies $\psi = 0$, and $\sigma(x, \mathfrak{M}^*) = 0$ implies $x = 0$.

Let $\mathfrak{C}$ be the ring of $\mathfrak{B}$-endomorphisms of $\mathfrak{M}$. If $C \in \mathfrak{C}$, then C is a l.t. and, if C^* is the transpose of C with respect to the form $\langle x, \psi \rangle$, then

$$\sigma(xC, \psi) = \sigma(x, \psi C^*)$$

for all x and ψ. Conversely if C is a endomorphism of $\mathfrak{M}$, and if there exists an endomorphism C^{**} of $\mathfrak{M}^*$ such that $\sigma(xC, \psi) = \sigma(x, \psi C^{**})$ for all x and ψ, then C^{**} is also the transpose of C with respect to the form $\langle x, \psi \rangle$, and $C \in \mathfrak{C}$.

The endomorphisms $\psi * u$ defined by $x(\psi * u) = \sigma(x, \psi)u$ are elements of $\mathfrak{C}$. If we introduce the action of $\mathfrak{C}$ upon $\mathfrak{M}^*$ by means of the formula $C\psi = \psi C^*$, then $\mathfrak{M}^*$ becomes a left $\mathfrak{C}$-module, and $(\psi, u) \to \psi * u$ defines a bilinear pairing of $\mathfrak{M}^* \times \mathfrak{M} \to \mathfrak{C}$. Finally it is possible to verify, as in §2, that for all ψ and u,

$$\psi * u = \sum_s U_{s^{-1}}(\psi \times u)\, U_s.$$

4. Remarks on the structure and representation theory of crossed products. Let $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ be a crossed product. We shall prove that there exists a (1–1) order inverting correspondence between the lattices of left and right ideals of $\mathfrak{B}$. Let $r(\mathfrak{S})$ and $l(\mathfrak{S})$ denote the right and left annihilators, respectively, of an arbitrary subset $\mathfrak{S}$ of $\mathfrak{B}$. If $\mathfrak{r}$ and $\mathfrak{l}$ are left and right ideals, respectively, then $r(\mathfrak{l})$ and $l(\mathfrak{r})$ are right and left ideals, respectively.

PROPOSITION 4. *If $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$, then the correspondences $\mathfrak{r} \to l(\mathfrak{r})$ and $\mathfrak{l} \to r(\mathfrak{l})$, where $\mathfrak{r}$ and $\mathfrak{l}$ are right and left ideals, respectively, are inverses of each other: $r(l(\mathfrak{r})) = \mathfrak{r}$ and $l(r(\mathfrak{l})) = \mathfrak{l}$. Moreover, every indecomposable right or left ideal direct component of $\mathfrak{B}$ contains a unique minimal non-zero subideal.*

Proof. Since $\Delta \subseteq \mathfrak{B}$, $\mathfrak{B}$ is a two-sided vector space over Δ, and the elements $\{b_1, b_s, \ldots\}$ corresponding to the elements of $\mathfrak{G}$ form both a left and right basis of $\mathfrak{B}$ over Δ. If $b = \sum b_s\xi_s$ is an arbitrary element of $\mathfrak{B}$, then the mapping

$$b \rightarrow \lambda(b) = \xi_1$$

is both a left and right Δ-linear function. It is easy to prove that the kernel of λ contains no non-zero left or right ideal of $\mathfrak{B}$ (**12**, p. 658). Therefore the associated bilinear form λ defined by

$$\lambda(b, b') = \lambda(bb'), \qquad b, b' \in \mathfrak{B}. \tag{17}$$

is non-degenerate. From these facts it follows that if $\mathfrak{r}$ and $\mathfrak{l}$ are right and left ideals, respectively, then

$$l(\mathfrak{r}) = \{b \mid b \in \mathfrak{B},\ \lambda(b, \mathfrak{r}) = 0\}, \qquad r(\mathfrak{l}) = \{b \mid b \in \mathfrak{B},\ \lambda(\mathfrak{l}, b) = 0\}.$$

Since $\mathfrak{B}$ is finite dimensional over Δ, a well-known property of dual vector spaces implies the first statement of the theorem.

Now let $e\mathfrak{B} \neq 0$ be an indecomposable right ideal, where e is an idempotent. Since $\mathfrak{B}$ satisfies the minimum condition for left and right ideals, $e\mathfrak{B}$ contains a unique maximal subideal. Moreover $\mathfrak{B}e$ is an indecomposable left ideal which also contains a unique maximal subideal (**1**, Chap. IX). Clearly $l(e\mathfrak{B}) = \mathfrak{B}(1-e)$. Suppose that for some $x \in \mathfrak{B}e$, $\lambda(x, e\mathfrak{B}) = 0$. Then, since $xe\mathfrak{B}$ is a right ideal, we have $xe\mathfrak{B} = 0$, and $x \in \mathfrak{B}(1 - e)$. Therefore $x = 0$, and it follows that the restriction of λ to $\mathfrak{B}e \times e\mathfrak{B}$ is non-degenerate. Because of the order inverting property of the annihilator correspondence, we conclude that both $\mathfrak{B}e$ and $e\mathfrak{B}$ possess unique minimal non-zero subideals.

We remark that $\mathfrak{B}$ is a quasi-Frobenius ring (**10**, p. 8) by Theorem 6 of (**10**).

Now we consider a pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$ of $\mathfrak{M}' \times \mathfrak{M} \rightarrow \mathfrak{B}$ (see §2), together with the associated pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ defined by (13) on $\mathfrak{M}' \times \mathfrak{M}$ to the centralizer $\mathfrak{C}$ of $\mathfrak{M}$ relative to $\mathfrak{B}$. Let $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$ be the nucleus of the pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$; then $\mathfrak{c}$ is a two-sided ideal in $\mathfrak{C}$. We shall prove that the statement $\mathfrak{c} = \mathfrak{C}$ is equivalent to certain structural properties of $\mathfrak{M}$ viewed as a $\mathfrak{B}$-module. Later, in §8, we shall show how, when $\mathfrak{C} = \mathfrak{c}$, these properties of $\mathfrak{M}$ can be used to prove certain ideal theoretic results concerning the ring $\mathfrak{C}$.

The results we require have been established recently by several authors (**4**; **5**; **9**), and it is unnecessary to include the details here. Let us assume that the (right) dimension of $\mathfrak{M}$ over Δ is finite; then every l.t. X in $\mathfrak{M}$ over Δ has the form $X = \sum \psi_i \times u_i$, for some ψ_i in $\mathfrak{M}'$ and u_i in $\mathfrak{M}$. Our starting point is the observation ((15), §2) that $\mathfrak{c} = \mathfrak{C}$ if and only if there exists a l.t. X in $\mathfrak{M}$ over Δ such that

$$\sum_s R_{s^{-1}} X R_s = 1, \qquad s \in \mathfrak{G}, \tag{18}$$

where 1 is the identity l.t., and R_s is the mapping $x \rightarrow xb_s$ in $\mathfrak{M}$.

Now we adopt some terminology due to Cartan and Eilenberg. A (right) $\mathfrak{B}$-module is called *projective*[3] (M_0 in the sense of Gaschutz (**4**; cf. also **5** and

[3]No connection between projective representations and projective modules is implied by this definition.

7) if whenever $\mathfrak{T}$ and $\mathfrak{U}$ are $\mathfrak{B}$-modules such that $\mathfrak{U} \subseteq \mathfrak{T}$ and $\mathfrak{T}/\mathfrak{U} \cong \mathfrak{M}$, then there exists a $\mathfrak{B}$-submodule $\mathfrak{U}^*$ of $\mathfrak{T}$ such that $\mathfrak{T} = \mathfrak{U} \oplus \mathfrak{U}^*$. $\mathfrak{M}$ is called *injective* (M_u in **(4, 5, 9)**) if whenever $\mathfrak{M}$ is $\mathfrak{B}$-isomorphic to a submodule $\mathfrak{V}$ of $\mathfrak{T}$, then there exists a $\mathfrak{B}$-submodule $\mathfrak{V}^*$ of $\mathfrak{T}$ such that $\mathfrak{T} = \mathfrak{V} \oplus \mathfrak{V}^*$.

PROPOSITION 5. *Let* $(\mathfrak{M}', \mathfrak{M}, \tau)$ *be a pairing of* $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$, *and let the (right) dimension of* $\mathfrak{M}$ *over* Δ *be finite. Then the following statements are equivalent.*

(i) $\mathfrak{c}$ *contains the identity l.t.;*

(ii) $\mathfrak{M}$ *is a projective* $\mathfrak{B}$*-module;*

(iii) $\mathfrak{M}$ *is an injective* $\mathfrak{B}$*-module;*

(iv) $\mathfrak{M}$ *is a direct sum of indecomposable* $\mathfrak{B}$*-submodules which are* $\mathfrak{B}$*-isomorphic to right ideal direct components of* $\mathfrak{B}$.

Proof. Theorem 1 of **(9)** states that (ii) and (iv) are equivalent (see also the remark on p. 107 of **(9)**). The equivalence of (i), (ii), and (iii) has been proved by Kasch **(5**, Theorem 12**)**. To verify this statement, the following remarks may be helpful. We should observe first that $\mathfrak{B}$ is a Frobenius extension of Δ with Frobenius homomorphism $b \to \lambda(b)$ **(5**, p. 462**)**. Then statement (i) is equivalent to the statement that (18) holds for some l.t. X, where we note that $\{b_1, b_s, b_t, \ldots\}$ and $\{b_1, b_{s^{-1}}, b_{t^{-1}}, \ldots\}$ are orthogonal left and right bases of $\mathfrak{B}$ over Δ **(5**, p. 457**)** with respect to the bilinear form $\lambda(b, b')$ defined by (17). We now see that Kasch's theorem is indeed applicable to our situation.

Remark 1. If it is not assumed that the dimension of $\mathfrak{M}$ over Δ is finite, then not every l.t. X in $\mathfrak{M}$ over Δ has the form $\sum \psi_i x u_i$. The following implications remain valid: (i) $\to$ (18) $\to$ [(ii) and (iii)] $\to$ (iv).

Remark 2. Assume (i); then from (18) we obtain

$$\sum L_{s^{-1}} X^* L_s = 1,$$

where 1 is now the identity mapping on $\mathfrak{M}'$, X^* is a l.t. on $\mathfrak{M}'$, and L_s is the mapping $\psi \to b_s\psi = R_s^*\psi$ in $\mathfrak{M}'$. Therefore we have the implications (i) $\to$ (ii)$'$ $\to$ (iv)$'$, where (ii)$'$ and (iv)$'$ are obtained from (ii) and (iv) by replacing $\mathfrak{M}$ by $\mathfrak{M}'$, and "right" by "left" in (iv).

Remark 3. It follows from the considerations of §3 that a result analogous to Proposition 5 can be established for the pairing σ of $\mathfrak{M} \times \mathfrak{M}^* \to \mathfrak{B}$ which was constructed in §3. We shall not include the details of this discussion.

5. Abstract theory of regular pairings. Let $\mathfrak{B}$ be an arbitrary ring with identity element 1, and let $\mathfrak{B}$ admit a set of Ω of (left) operators. We shall assume that 1 acts as the identity operator on all $\mathfrak{B}$-modules which we shall consider. Let $\mathfrak{M}'$ and $\mathfrak{M}$ be left and right $\mathfrak{B}$-Ω-modules, which are paired to $\mathfrak{B}$ by a function $\tau(\psi, x)$. We assume that τ is bilinear, relative to both $\mathfrak{B}$ and Ω, in the sense that the equations

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$$\tau(\psi_1 + \psi_2, x) = \tau(\psi_1, x) + \tau(\psi_2, x),\ \tau(\psi, x_1 + x_2) = \tau(\psi, x_1) + \tau(\psi, x_2)$$
$$\tau(b\psi, x) = b\tau(\psi, x), \quad \tau(\psi, xb) = \tau(\psi, x)b,$$
$$\tau(\alpha\psi, x) = \alpha\tau(\psi, x), \quad \tau(\psi, \alpha x) = \alpha\tau(\psi, x)$$

hold for all x in $\mathfrak{M}$, ψ in $\mathfrak{M}'$, b in $\mathfrak{B}$, and α in Ω. Our second assumption is that τ is non-degenerate. If these conditions are satisfied, then we shall call the system $(\mathfrak{M}', \mathfrak{M}, \tau)$ an (*abstract*) *pairing*. The *nucleus* $\mathfrak{b} = \tau(\mathfrak{M}', \mathfrak{M})$ of the pairing is a two-sided ideal in $\mathfrak{B}$.

We let $\mathfrak{E}$ be the set of all $\mathfrak{B}$-Ω-endomorphisms of $\mathfrak{M}$. If $\bar{\alpha}$ denotes the endomorphism $x \to \alpha x$ of $\mathfrak{M}$ determined by an element of Ω, then $\bar{\alpha}E \in \mathfrak{E}$ for every E in $\mathfrak{E}$, and $\bar{\alpha}E = E\bar{\alpha}$, so that if we define $\alpha E = \bar{\alpha}E$, then $\mathfrak{E}$ becomes an Ω-ring.

The endomorphisms $\psi \odot u$ defined by (13) are elements of $\mathfrak{E}$, and possess transposes relative to the form τ. Let $\mathfrak{c}$ be the subgroup of $\mathfrak{E}$ consisting of all finite sums of the $\psi \odot u$. If the action of $\mathfrak{c}$ upon $\mathfrak{M}'$ is defined by the formula $E\psi = E^*\psi$, for E in $\mathfrak{c}$, then it follows that $(\psi, u) \to \psi \odot u$ is a $\mathfrak{c}$-Ω-bilinear mapping of $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{c}$. We shall denote this pairing by $(\mathfrak{M}', \mathfrak{M}, \odot)$, and observe that the nucleus $\mathfrak{c}$ is an Ω-subring of $\mathfrak{E}$. If $E \in \mathfrak{E}$, then $(\psi \odot u)E = \psi \odot uE$, and hence $\mathfrak{c}$ is a right Ω-ideal in $\mathfrak{E}$. We shall denote by $\mathfrak{C}$ an arbitrary Ω-subring of $\mathfrak{E}$ such that

$$\mathfrak{c} \subseteq \mathfrak{C} \subseteq \mathfrak{E}. \tag{19}$$

Then $\mathfrak{C}$ will be called a *centralizer* of $\mathfrak{M}$ relative to $\mathfrak{B}$, and will remain fixed throughout the discussion. Our aim is to establish relationships between the nuclei $\mathfrak{b}$ and $\mathfrak{c}$ of the rings $\mathfrak{B}$ and $\mathfrak{C}$, and the properties of $\mathfrak{M}$ and $\mathfrak{M}'$ as $\mathfrak{B}$ and $\mathfrak{C}$-modules.

In order to discuss the connection between the ring $\mathfrak{B}$ and the structure of $\mathfrak{M}$ (or $\mathfrak{M}'$) as a $\mathfrak{C}$-module, we shall assume that the pairing τ is *regular* in the sense that $\mathfrak{b}$ contains an element $e_0 = \sum\tau(\psi^*_i, x^*_i)$ such that $be_0 = e_0b = b$ for all $b \in \mathfrak{b}$. By the non-degeneracy of τ it follows that $xe_0 = x$ and $e_0\psi = \psi$ for all $x \in \mathfrak{M}$ and $\psi \in \mathfrak{M}'$.

It is always possible to construct a regular pairing from an arbitrary one. Let e_0 be any central idempotent contained in the nucleus $\mathfrak{b}$ of a pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$, or let $e_0 = 0$ if $\mathfrak{b}$ contains no central idempotent. Then

$$\mathfrak{M} = \mathfrak{M}e_0 \oplus \mathfrak{M}(1 - e_0), \qquad \mathfrak{M}' = e_0\mathfrak{M}' \oplus (1 - e_0)\mathfrak{M}',$$

where the direct summands are invariant relative to both $\mathfrak{B}$ and $\mathfrak{c}$. We define a new pairing τ_0 of $e_0\mathfrak{M}' \times \mathfrak{M}e_0 \to \mathfrak{B}$ by setting

$$\tau_0(e_0\psi, xe_0) = \tau(e_0\psi, xe_0)$$

for all ψ and x and we shall prove that τ_0 is a regular pairing. The nucleus b_0 of τ_0 contains e_0, for if $e_0 = \sum \tau(\psi_i, x_i)$, then $e_0 = \sum\tau(e_0\psi_i, x_ie_0)$. Obviously

$$e_0b = be_0 = b, \qquad b \in \mathfrak{b}_0.$$

The bilinearity of τ_0 is evident. It remains to prove that τ_0 is non-degenerate.

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Suppose $\tau_0(e_0\psi, xe_0) = 0$ for all $e_0\psi \in e_0\mathfrak{M}'$. If ψ is arbitrary in $\mathfrak{M}'$, we write

$$\psi = e_0\psi + (1 - e_0)\psi$$

and obtain

$$\begin{aligned}\tau(\psi, xe_0) &= \tau_0(e_0\psi, xe_0) + \tau((1 - e_0)\psi, xe_0)\\ &= \tau((1 - e_0)\psi, xe_0)e_0 = e_0\tau((1 - e_0)\psi, xe_0) = 0,\end{aligned}$$

so that $xe_0 = 0$ by the non-degeneracy of τ. Similarly $\tau_0(e_0\psi, \mathfrak{M}e_0) = 0$ implies $e_0\psi = 0$.

We return now to our assumption that the pairing is regular. If $\mathfrak{S}$ is any subset of $\mathfrak{B}$, we shall write $\mathfrak{S}_r$ (resp. $\mathfrak{S}_l$) for the set of endomorphisms R_s: $x \rightarrow xs$ (resp. L_s: $\psi \rightarrow s\psi$) of $\mathfrak{M}$ (resp. $\mathfrak{M}'$) determined by the elements of $\mathfrak{S}$. We are in a position to prove the following result:

THEOREM 1. *Let $\overline{\mathfrak{B}}$ be the set of all $\mathfrak{C}$-endomorphisms of $\mathfrak{M}$. Then $\mathfrak{b}_r = \mathfrak{B}_r = \overline{\mathfrak{B}}$.*

Proof. Obviously $\mathfrak{b}_r \subseteq \mathfrak{B}_r \subseteq \overline{\mathfrak{B}}$. Conversely let $B \in \overline{\mathfrak{B}}$; then $B(\psi \odot u) = (\psi \odot u)B$ for all ψ and u. Consequently

$$u\tau(\psi, xB) = (u\tau(\psi, x))B$$

for all x, ψ, and u. Let $b = \sum\tau(\psi^*_i, x^*_iB)$; then for all $u \in \mathfrak{M}$ we have

$$ub = \sum u\tau(\psi^*_i, x^*_iB) = (u\sum\tau(\psi^*_i, x^*_i))B = (ue_0)B = uB,$$

and $R_b = B$. This completes the proof.

If $\mathfrak{N}$ is a $\mathfrak{C}$-Ω-submodule of $\mathfrak{M}$, then

$$\tau(\mathfrak{M}', \mathfrak{N}) = \{\sum \tau(\psi_i, x_i) \,|\psi_i \in \mathfrak{M}', x_i \in \mathfrak{N}\}$$

is a left Ω-ideal contained in $\mathfrak{b}$. If $\mathfrak{l}$ is a left Ω-ideal in $\mathfrak{B}$, then $\mathfrak{M}\mathfrak{l}$ is a $\mathfrak{C}$-Ω-submodule of $\mathfrak{M}$. We have, for all $\mathfrak{N}$ and $\mathfrak{l}$,

$$(20) \qquad \mathfrak{M}\tau(\mathfrak{M}', \mathfrak{N}) \subseteq \mathfrak{N}; \qquad \tau(\mathfrak{M}', \mathfrak{M}\mathfrak{l}) \subseteq \mathfrak{l}:$$

the first since $\mathfrak{M}\tau(\mathfrak{M}', \mathfrak{N}) \subseteq \mathfrak{N}(\mathfrak{M}' \odot \mathfrak{M}) \subseteq \mathfrak{N}\mathfrak{C} \subseteq \mathfrak{N}$ by (19) and the fact that $\mathfrak{N}$ is a $\mathfrak{C}$-submodule[4] of $\mathfrak{M}$; the second, obvious. For later use we observe also that

$$(21) \qquad \tau(\mathfrak{M}', \sum \mathfrak{N}_\nu) = \sum \tau(\mathfrak{M}', \mathfrak{N}_\nu), \quad \mathfrak{M}(\sum \mathfrak{l}_\nu) = \sum (\mathfrak{M}\mathfrak{l}_\nu),$$

and

$$(22) \qquad \tau(\mathfrak{M}', \mathfrak{N}\mathfrak{l}) = \tau(\mathfrak{M}', \mathfrak{N})\mathfrak{l}, \qquad \mathfrak{M}(\mathfrak{l}_1\mathfrak{l}_2) = (\mathfrak{M}\mathfrak{l}_1)\mathfrak{l}_2.$$

LEMMA 1. *Let $\mathfrak{N}$ be a $\mathfrak{C}$-direct summand of $\mathfrak{M}$. Then there exists an idempotent $e \in \mathfrak{B}$ such that $\tau(\mathfrak{M}', \mathfrak{N}) = \mathfrak{B}e$.*

[4]For the rest of §5, 6, and 7, we shall omit explicit reference to the set Ω. Thus by submodule, ideal, etc. we shall mean Ω-submodule, Ω-ideal, etc.

Proof. Let E be a projection of $\mathfrak{M}$ upon $\mathfrak{N}$ such that $E \in \overline{\mathfrak{B}}$. By Theorem 1, $E = R_e$, where

$$e = \sum \tau(\psi^*_i, x^*_i E) \in \tau(\mathfrak{M}', \mathfrak{N}).$$

If $b = \sum \tau(\psi_i, x_i)$ is an arbitrary element of $\tau(\mathfrak{M}', \mathfrak{N})$, then $be = b$ since the restriction of E to $\mathfrak{N}$ is the identity mapping. Therefore $\tau(\mathfrak{M}', \mathfrak{N}) = \mathfrak{B}e$.

LEMMA 2. *Let $\mathfrak{N}$ be a $\mathfrak{C}$-submodule such that $\tau(\mathfrak{M}', \mathfrak{N}) = \mathfrak{B}e$, where e is an idempotent in $\mathfrak{B}$. Then $\mathfrak{M}\tau(\mathfrak{M}', \mathfrak{N}) = \mathfrak{N}$.*

Proof. By the non-degeneracy of τ we have $x = xe \in \mathfrak{M}\tau(\mathfrak{M}', \mathfrak{N})$ for all $x \in \mathfrak{N}$, and together with (20), this proves the Lemma.

LEMMA 3. *Let $\mathfrak{l} = \mathfrak{B}e$, where $e^2 = e \in \mathfrak{b}$. Then $\mathfrak{M}\mathfrak{l} = \mathfrak{M}e$ is a $\mathfrak{C}$-direct summand of $\mathfrak{M}$, and $\tau(\mathfrak{M}', \mathfrak{M}\mathfrak{l}) = \mathfrak{l}$.*

Proof. We have $\mathfrak{M}\mathfrak{l} = \mathfrak{M}\mathfrak{B}e = \mathfrak{M}e$, and $\mathfrak{M} = \mathfrak{M}e \oplus \mathfrak{M}(1 - e)$, proving the first statement. For the second, $b \in \mathfrak{l}$, $b = \sum \tau(\psi_i, x_i)$, implies

$$b = be = \sum \tau(\psi_i, x_i e) \in \tau(\mathfrak{M}', \mathfrak{M}\mathfrak{l}),$$

and by (20) we infer that $\mathfrak{l} = \tau(\mathfrak{M}', \mathfrak{M}\mathfrak{l})$.

THEOREM 2. *Let $(\mathfrak{M}', \mathfrak{M}, \tau)$ be a regular pairing with nucleus $\mathfrak{b}$. The mappings $\mathfrak{l} \to \mathfrak{M}\mathfrak{l}$ and $\mathfrak{N} \to \tau(\mathfrak{M}', \mathfrak{N})$ between the set of left ideal direct components of $\mathfrak{b}$ and the $\mathfrak{C}$-direct summands of $\mathfrak{M}$ are inverses of each other. The mapping $\mathfrak{l} \to \mathfrak{M}\mathfrak{l}$ preserves sums of arbitrary ideals, and intersections of left ideal direct components of $\mathfrak{B}$. Two left ideal direct components $\mathfrak{l}_1$ and $\mathfrak{l}_2$ of $\mathfrak{b}$ are $\mathfrak{B}$-isomorphic if and only if $\mathfrak{M}\mathfrak{l}_1$ and $\mathfrak{M}\mathfrak{l}_2$ are $\mathfrak{C}$-isomorphic.*

Proof. The first statement follows from Lemmas 1–3. By (21) the mapping $\mathfrak{l} \to \mathfrak{M}\mathfrak{l}$ preserves sums. The statement concerning intersections is an immediate consequence of the fact to be proved next, that if $\mathfrak{l}$ is a left ideal direct component of $\mathfrak{b}$ then

$$\mathfrak{M}\mathfrak{l} = \{x \mid \tau(\mathfrak{M}', x) \subseteq \mathfrak{l}\}.$$

Let $\mathfrak{l} = \mathfrak{B}e$, where $e^2 = e \in \mathfrak{b}$. Then $\tau(\mathfrak{M}', x) \subseteq \mathfrak{l}$ implies $\tau(\psi, xe) = \tau(\psi, x)$ for all $\psi \in \mathfrak{M}'$, and by the non-degeneracy of τ, $x = xe \in \mathfrak{M}\mathfrak{l}$. Conversely $x \in \mathfrak{M}\mathfrak{l}$ implies $xe = x$, and

$$\tau(\mathfrak{M}', x) = \tau(\mathfrak{M}', x)e \subseteq \mathfrak{l}.$$

Let $\mathfrak{B}e_1$ and $\mathfrak{B}e_2$ be $\mathfrak{B}$-isomorphic; then there exist elements a and b such that

$$\mathfrak{B}e_1 a = \mathfrak{B}e_2, \quad \mathfrak{B}e_2 b = \mathfrak{B}e_1, \quad cab = c, \qquad c \in \mathfrak{B}e_1,$$

$dba = d$ for all $d \in \mathfrak{B}e_2$. One verifies easily that $xe_1 \to xe_1 a$ and $xe_2 \to xe_2 b$ are $\mathfrak{C}$-homomorphisms between $\mathfrak{M}e_1$ and $\mathfrak{M}e_2$ which are inverses of each other, and consequently $\mathfrak{M}e_1$ and $\mathfrak{M}e_2$ are $\mathfrak{C}$-isomorphic.

Conversely let $x \to x^h$ be a $\mathfrak{C}$-isomorphism of $\mathfrak{N}_1$ onto $\mathfrak{N}_2$. Define

$$\bar{h}\colon \sum \tau(\psi_i, x_i) \to \sum \tau(\psi_i, x_i^h)$$

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of $\tau(\mathfrak{M}', \mathfrak{N}_1)$ into $\tau(\mathfrak{M}', \mathfrak{N}_2)$. In order to prove that $\tau(\mathfrak{M}', \mathfrak{N}_1)$ and $\tau(\mathfrak{M}', \mathfrak{N}_2)$ are $\mathfrak{B}$-isomorphic, it is clearly sufficient to prove that h is a $\mathfrak{B}$-homomorphism onto. If

$$\sum\tau(\psi_i, x_i) = 0, \quad x_i \in \mathfrak{N}_1,$$

then $0 = \mathfrak{M}(\sum\tau(\psi_i, x_i)) = \sum x_i(\psi_i \odot \mathfrak{M})$, and since h is a $\mathfrak{C}$-isomorphism,

$$\sum x_i^h(\psi_i \odot \mathfrak{M}) = \sum \mathfrak{M}\tau(\psi_i, x_i^h) = 0.$$

Since $e_0 = \sum\tau(\psi^*_i, x^*_i)$ is a left identity element in $\mathfrak{b}$, we have

$$\sum\tau(\psi_i, x_i^h) = \sum e_0\tau(\psi_i, x_i^h) = 0.$$

Thus $\bar{h}$ is single valued. The fact that it is onto, and is a $\mathfrak{B}$-homomorphism can be checked in a similar way using the properties of τ. This completes the proof.

COROLLARY. *A left ideal direct component* $\mathfrak{l}$ *of* $\mathfrak{b}$ *is indecomposable if and only if* $\mathfrak{M}\mathfrak{l}$ *is an indecomposable direct summand of* $\mathfrak{M}$.

Proof. Let $\mathfrak{l}$ be a decomposable direct component of $\mathfrak{B}$: $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$. Then by the theorem $\mathfrak{M}\mathfrak{l} = \mathfrak{M}\mathfrak{l}_1 \oplus \mathfrak{M}\mathfrak{l}_2$, where neither component is zero. The converse is proved similarly.

Let us denote by $\mathfrak{c}^*$ the set of transposes relative to τ of the elements of $\mathfrak{c}$, and write $\mathfrak{B}_l$ and $\mathfrak{b}_l$, respectively, for the sets of endomorphisms $\psi \to b\psi$ determined in $\mathfrak{M}'$ by the elements of $\mathfrak{B}$ and $\mathfrak{b}$. Let $\mathfrak{E}^*$ be the set of all $\mathfrak{B}$-Ω-endomorphisms of $\mathfrak{M}'$, and let $\mathfrak{C}^*$ be an arbitrary ring of Ω-endomorphisms of $\mathfrak{M}'$ such that $\mathfrak{c}^* \subseteq \mathfrak{C}^* \subseteq \mathfrak{E}^*$. We shall write $\mathfrak{B}'$ for the set of $\mathfrak{C}^*$-endomorphisms of $\mathfrak{M}'$. Then we may state the following duals to Theorems 1 and 2.

THEOREM 1′. *Let* $(\mathfrak{M}', \mathfrak{M}, \tau)$ *be a regular pairing with nucleus* $\mathfrak{b}$, *and let* $\mathfrak{C}^*$ *be an* Ω*-subring of* $\mathfrak{E}^*$ *containing* $\mathfrak{c}^*$. *Then* $\mathfrak{b}_l = \mathfrak{B}_l = \mathfrak{B}'$.

THEOREM 2′. *Let* $(\mathfrak{M}', \mathfrak{M}, \tau)$ *be a regular pairing with nucleus* $\mathfrak{b}$. *The mappings* $\mathfrak{r} \to \mathfrak{r}\mathfrak{M}'$, $\mathfrak{N}' \to \tau(\mathfrak{N}', \mathfrak{M})$ *between the sets of right ideal direct components of* $\mathfrak{b}$ *and the* $\mathfrak{C}^*$*-direct summands of* $\mathfrak{M}'$ *are inverses of each other, and possess the properties stated in Theorem* 2.

THEOREM 3 (Weyl).[5] *Let* $(\mathfrak{M}', \mathfrak{M}, \tau)$ *be a pairing of* $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{B}$, *where* $\mathfrak{B}$ *is a semi-simple* Ω*-ring satisfying the minimum condition for left ideals. Then the pairing is regular. The mappings* $\mathfrak{l} \to \mathfrak{M}\mathfrak{l}$ *and* $\mathfrak{N} \to \tau(\mathfrak{M}', \mathfrak{N})$ *are inverses of each other, and establish a* (1–1) *inclusion preserving correspondence between the set of all left ideals of* $\mathfrak{B}$ *which are contained in the nucleus* $\mathfrak{b}$, *and the set of all* $\mathfrak{C}$*-submodules of* $\mathfrak{M}$. *If*

$$\mathfrak{l}_1 \leftrightarrow \mathfrak{N}_1 = \mathfrak{M}\mathfrak{l}_1, \quad \mathfrak{l}_2 \leftrightarrow \mathfrak{N}_2 = \mathfrak{M}\mathfrak{l}_2,$$

[5]This result, and Theorem 2 in its essentials, have been proved by Weyl for pairings of the type considered in §3 (**16**, Chap. 5; **15**; **17**, Chap. 3).

then

$$\mathfrak{l}_1 + \mathfrak{l}_2 \leftrightarrow \mathfrak{N}_1 + \mathfrak{N}_2, \qquad \mathfrak{l}_1 \cap \mathfrak{l}_2 \leftrightarrow \mathfrak{N}_1 \cap \mathfrak{N}_2,$$

and $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are* $\mathfrak{B}$*-isomorphic if and only if* $\mathfrak{N}_1$ *and* $\mathfrak{N}_2$ *are* $\mathfrak{C}$*-isomorphic.*

Proof. The structure theory of semi-simple rings implies that the pairing is regular, and that every left ideal in $\mathfrak{b}$ is a direct component of $\mathfrak{b}$. By Theorem 2, $\mathfrak{l} \rightarrow \mathfrak{M}\mathfrak{l}$ is a (1–1) inclusion preserving correspondence between the set of all left ideal direct components of $\mathfrak{b}$ and the set of all $\mathfrak{C}$-submodules of $\mathfrak{M}$. By a principle of lattice theory, the mapping preserves the lattice operations. That it preserves isomorphism relations has been proved in Theorem 2.

Example. Let $b \rightarrow U(b)$ be an ordinary representation of the group algebra $\mathfrak{B}$ of a finite group $\mathfrak{G}$ by l.t. in a finite dimensional vector space $\mathfrak{M}$ over a field, and let $\mathfrak{C}$ be the set of all l.t. commuting with the l.t. $U(b)$, $b \in \mathfrak{B}$. Let $\mathfrak{b}_\tau$ and $\mathfrak{b}_\sigma$ be the nuclei of the pairings constructed in **2** and **3** respectively. Finally let us assume that both pairings are regular. Then by Theorems 2 and 2′, a left ideal $\mathfrak{B}e$ of $\mathfrak{b}_\tau$ is matched against the $\mathfrak{C}$-submodule $\mathfrak{M}U(e)$ of $\mathfrak{M}$, while a right ideal $f\mathfrak{B}$ of $\mathfrak{b}_\sigma$ generated by an idempotent f is matched against the $\mathfrak{C}$-submodule $\mathfrak{M}U(f^J)$. We remark finally that $\mathfrak{b}_\sigma = \mathfrak{b}_\tau{}^J$.

6. Maximal submodules of indecomposable $\mathfrak{C}$-direct summands. We adhere to the assumptions and notation of §5, and make the additional assumption that $\mathfrak{B}$ satisfies the minimum condition for left ideals, and hence also the maximum condition, since $\mathfrak{B}$ has an identity element. Let $\mathfrak{N}$ be the radical of $\mathfrak{B}$; then every indecomposable left ideal direct component $\mathfrak{B}e$ of $\mathfrak{B}$ has a unique maximal subideal $\mathfrak{N}e$. Every proper subideal of $\mathfrak{B}e$ is nilpotent, and $\mathfrak{B}e$ and $\mathfrak{B}e'$ are $\mathfrak{B}$-isomorphic if and only if $\mathfrak{B}e/\mathfrak{N}e$ and $\mathfrak{B}e'/\mathfrak{N}e'$ are $\mathfrak{B}$-isomorphic (**1**, Chap. IX).

LEMMA 4. *Let* $\mathfrak{R} = \mathfrak{M}e$ *be an indecomposable* $\mathfrak{C}$*-direct summand of* $\mathfrak{M}$. *Then* $\mathfrak{R}$ *has a unique maximal* $\mathfrak{C}$*-submodule* $\mathfrak{S}$, *and*

$$\tau(\mathfrak{M}', \mathfrak{S}) \subseteq \mathfrak{N}e, \quad \mathfrak{M}(\mathfrak{N}e) \subseteq \mathfrak{S}. \tag{23}$$

Proof. By the Corollary to Theorem 2, $\mathfrak{B}e = \tau(\mathfrak{M}', \mathfrak{R})$ is an indecomposable left ideal. Let $\mathfrak{S} = \sum \mathfrak{R}_\nu$, where $\{\mathfrak{R}_\nu\}$ is the set of all proper $\mathfrak{C}$-submodules of $\mathfrak{R}$. By (21) and the fact that $\mathfrak{B}$ satisfies the maximum condition for left ideals, we have

$$\tau(\mathfrak{M}', \mathfrak{S}) = \sum \tau(\mathfrak{M}', \mathfrak{R}_\nu),$$

which in turn can be expressed as a finite sum

$$\textstyle\sum_{i=1}^{m} \tau(\mathfrak{M}', \mathfrak{R}_{\nu_i}).$$

No $\tau(\mathfrak{M}', \mathfrak{R}_\nu) = \mathfrak{B}e$, otherwise, by Lemma 2,

$$\mathfrak{R}_\nu = \mathfrak{M}\tau(\mathfrak{M}', \mathfrak{R}_\nu) = \mathfrak{M}e = \mathfrak{R}.$$

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Hence each $\tau(\mathfrak{M}', \mathfrak{R}_\nu)$ is nilpotent, and since the sum is finite, $\tau(\mathfrak{M}', \mathfrak{S})$ is nilpotent. This proves (i) $\mathfrak{S} \neq \mathfrak{R}$ (for if $\mathfrak{S} = \mathfrak{R}$ then $\tau(\mathfrak{M}', \mathfrak{S})$ contains an idempotent $\neq 0$) and (ii), $\tau(\mathfrak{M}', \mathfrak{S}) \subseteq \mathfrak{N}e$. For the other inclusion of (23) it is sufficient to prove that $\mathfrak{M}(\mathfrak{N}e) \neq \mathfrak{R}$. If, however, $\mathfrak{M}(\mathfrak{N}e) = \mathfrak{R}$, then by Lemma 3 and (22) we have

$$\mathfrak{B}e = \tau(\mathfrak{M}', \mathfrak{R}) = \tau(\mathfrak{M}', \mathfrak{M})\mathfrak{N}e \subseteq \mathfrak{N},$$

contrary to our assumption that $e^2 = e \neq 0$. This completes the proof.

THEOREM 4. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be indecomposable $\mathfrak{C}$-direct summands of $\mathfrak{M}$ with maximal $\mathfrak{C}$-submodules $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Then $\mathfrak{R}_1/\mathfrak{S}_1$ and $\mathfrak{R}_2/\mathfrak{S}_2$ are $\mathfrak{C}$-isomorphic if and only if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are $\mathfrak{C}$-isomorphic.*

Proof. We prove the result by throwing the argument back to the known results concerning the ideals in $\mathfrak{B}$. Using Lemma 4, it is easy to prove that the $\mathfrak{C}$-isomorphism of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ induces a $\mathfrak{C}$-isomorphism of $\mathfrak{R}_1/\mathfrak{S}_1$ onto $\mathfrak{R}_2/\mathfrak{S}_2$. For the proof of the converse it is enough to show, by Theorem 2, that $\mathfrak{B}e_1 = \tau(\mathfrak{M}', \mathfrak{R}_1)$ and $\mathfrak{B}e_2 = \tau(\mathfrak{M}', \mathfrak{R}_2)$ are $\mathfrak{B}$-isomorphic. This we prove by showing that $\mathfrak{B}e_1/\mathfrak{N}e_1$ and $\mathfrak{B}e_2/\mathfrak{N}e_2$ are $\mathfrak{B}$-isomorphic, assuming that $\mathfrak{R}_1/\mathfrak{S}_1$ and $\mathfrak{R}_2/\mathfrak{S}_2$ are $\mathfrak{C}$-isomorphic.

Let ζ be a $\mathfrak{C}$-isomorphism of $\mathfrak{R}_1/\mathfrak{S}_1$ onto $\mathfrak{R}_2/\mathfrak{S}_2$, and let $\theta = \zeta^{-1}$. In both $\mathfrak{R}_1$ and $\mathfrak{R}_2$ select a fixed system of representatives of the cosets in $\mathfrak{R}_1/\mathfrak{S}_1$ and $\mathfrak{R}_2/\mathfrak{S}_2$ respectively, and for each $x_1 \in \mathfrak{R}_1$, let $x_1\tilde{\zeta}$ be the representative of the coset $(x_1 + \mathfrak{S}_1)\zeta$; that is

$$x_1\tilde{\zeta} + \mathfrak{S}_2 = (x_1 + \mathfrak{S}_1)\zeta.$$

Similarly we define a map $\tilde{\theta}$ of $\mathfrak{R}_2$ into $\mathfrak{R}_1$. We have

$$x_1 \equiv x_1\tilde{\zeta}\tilde{\theta} \pmod{\mathfrak{S}_1}, \quad x_2 \equiv x_2\tilde{\theta}\tilde{\zeta} \pmod{\mathfrak{S}_2}, \tag{24}$$

for all $x_1 \in \mathfrak{R}_1$, $x_2 \in \mathfrak{R}_2$.

Now define a mapping μ of $\tau(\mathfrak{M}', \mathfrak{R}_1)$ into $\tau(\mathfrak{M}', \mathfrak{R}_2)$, namely

$$\mu: \sum \tau(\psi_i, x_{1i}) \to \sum \tau(\psi_i, x_{1i}\tilde{\zeta}),$$

where $x_{1i} \in \mathfrak{R}_1$, $\psi_i \in \mathfrak{M}'$ for all i. We contend that the induced mapping

$$\bar{\mu}: \sum \tau(\psi_i, x_{1i}) + \mathfrak{N}e_1 \to \sum \tau(\psi_i, x_{1i}\tilde{\zeta}) + \mathfrak{N}e_2$$

is a $\mathfrak{B}$-isomorphism of $\mathfrak{B}e_1/\mathfrak{N}e_1$ onto $\mathfrak{B}e_2/\mathfrak{N}e_2$.

First we prove that $\bar{\mu}$ is single valued. Let $a = \sum \tau(\psi_i, x_{1i}) \in \mathfrak{N}e_1$; then $\mathfrak{M}a \subseteq \mathfrak{M}(\mathfrak{N}e_1) \subseteq \mathfrak{S}_1$ by (23). Thus for all $u \in \mathfrak{M}$,

$$ua + \mathfrak{S}_1 = \sum x_{1i}(\psi_i \odot u) + \mathfrak{S}_1 = 0.$$

Applying ζ we have $\sum (x_{1i} + \mathfrak{S}_1)\zeta(\psi_i \odot u) = 0$. Then

$$\sum x_{1i}\tilde{\zeta}(\psi_i \odot u) \in \mathfrak{S}_2, \quad \sum \mathfrak{M}\tau(\psi_i, x_{1i}\tilde{\zeta}) \subseteq \mathfrak{S}_2.$$

If $e_0 = \sum \tau(\psi^*_i, x^*_i)$ is the identity element in $\mathfrak{b}$, then $a = e_0 a$ implies

$$a\mu = e_0(a\mu) \in \tau(\mathfrak{M}', \mathfrak{S}_2) \subseteq \mathfrak{N}e_2$$

by (23), and $\bar{\mu}$ is single valued. That $\bar{\mu}$ is a $\mathfrak{B}$-homomorphism follows from the bilinearity of τ, and the ontoness from (24) and (23). To prove that $\bar{\sigma}$ is (1–1), let

$$\sum \tau(\psi_i, x_{1i}\xi) \in \mathfrak{N}e_2, \qquad \psi_i \in \mathfrak{M}', \qquad x_{1i} \in \mathfrak{N}_1.$$

Then by (23), $\sum \mathfrak{M}\tau(\psi_i, x_{1i}\xi) \subseteq \mathfrak{S}_2$; as in the first part of the proof we now verify that $\sum \tau(\psi_i, x_{1i}\xi\theta) \in \mathfrak{N}e_1$, and that

$$\sum \tau(\psi_i, x_{1i}) - \sum \tau(\psi_i, x_{1i}\xi\theta) = \sum \tau(\psi_i, x_{1i} - x_{1i}\xi\theta) \in \tau(\mathfrak{M}', \mathfrak{S}_1) \subseteq \mathfrak{N}e_1$$

by (23). Thus $\sum \tau(\psi_i, x_{1i}) \in \mathfrak{N}e_1$, and we have proved that $\bar{\mu}$ is (1–1). This completes the proof of the theorem.

7. The structure of $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$. Let $(\mathfrak{M}', \mathfrak{M}, \tau)$ be an abstract pairing. We shall assume that the nucleus $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$ of the associated pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ contains the identity mapping on $\mathfrak{M}$, and that the function $\psi \odot u$ is non-degenerate. Since $\mathfrak{c}$ is a right ideal in $\mathfrak{E}$, the first assumption implies that $\mathfrak{c} = \mathfrak{E}$, and the two assumptions combined imply that the dual pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ is regular in the sense of §5. Since the ring $\overline{\mathfrak{B}}$ of all $\mathfrak{c}$-endomorphisms of $\mathfrak{M}$ is a centralizer of $\mathfrak{M}$ relative to $\mathfrak{c}$, the methods of §5 yield a correspondence between the $\overline{\mathfrak{B}}$-direct summands of $\mathfrak{M}$ and the left ideal direct components of $\mathfrak{c}$: to a $\overline{\mathfrak{B}}$-direct summand $\mathfrak{N}$ corresponds

$$\mathfrak{M}' \odot \mathfrak{N} = \{\psi_i \odot u_i \mid \psi_i \in \mathfrak{M}', \quad u_i \in \mathfrak{N}\},$$

while to a left ideal direct component $\mathfrak{l}$ of $\mathfrak{c}$ corresponds the $\overline{\mathfrak{B}}$-direct summand $\mathfrak{M}\mathfrak{l}$.

THEOREM 5. *Let $(\mathfrak{M}', \mathfrak{M}, \odot)$ be a regular pairing of $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{c}$, which is dual to an abstract pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$. Then the mappings $\mathfrak{N} \to \mathfrak{M}' \odot \mathfrak{N}$ and $\mathfrak{l} \to \mathfrak{M}\mathfrak{l}$ between the set of $\mathfrak{B}$-direct summands of $\mathfrak{M}$ and the set of left ideal direct components of $\mathfrak{c}$ are inverses of each other. These mappings preserve direct sums and intersections whenever all modules concerned are direct summands. Two $\mathfrak{B}$-direct summands $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are $\mathfrak{B}$-isomorphic if and only if $\mathfrak{M}' \odot \mathfrak{N}_1$ and $\mathfrak{M}' \odot \mathfrak{N}_2$ are $\mathfrak{c}$-isomorphic. $\mathfrak{N}$ is an indecomposable $\mathfrak{B}$-direct summand of $\mathfrak{M}$ if and only if $\mathfrak{M}' \odot \mathfrak{N}$ is an indecomposable left ideal in $\mathfrak{c}$.*

Proof. The first part of the theorem follows from Theorem 2, if we observe that a $\mathfrak{B}$-direct summand of $\mathfrak{M}$ is necessarily a $\overline{\mathfrak{B}}$-direct summand. By Theorem 2, a $\mathfrak{c}$-isomorphism between $\mathfrak{M}' \odot \mathfrak{N}_1$ and $\mathfrak{M}' \odot \mathfrak{N}_2$ induces a $\overline{\mathfrak{B}}$-isomorphism between $\mathfrak{N}_1$ and $\mathfrak{N}_2$, and hence $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are $\mathfrak{B}$-isomorphic, since $\mathfrak{B}_r \subseteq \overline{\mathfrak{B}}$. Now let $x \to x^h$ be a $\mathfrak{B}$-isomorphism between $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We supply the first step in the proof that

$$\sum \psi_i \odot x_i \to \sum \psi_i \odot x_i^h$$

is a $\mathfrak{c}$-isomorphism of $\mathfrak{M}' \odot \mathfrak{N}_1$ onto $\mathfrak{M}' \odot \mathfrak{N}_2$. Let $\sum \psi_i \odot x_i = 0$; then

$$\sum x_i \tau(\psi_i, \mathfrak{M}) = 0.$$

Since h is a $\mathfrak{B}$-isomorphism and $\tau(\psi_i, \mathfrak{M}) \subseteq \mathfrak{B}$, we have $\sum x_i^h \tau(\psi_i, \mathfrak{M}) = 0$. Then $\sum \psi_i \odot x_i^h = 0$, and the mapping is single valued. The rest of the proof is left to the reader. The final statement of the Theorem follows from the proof of the Corollary to Theorem 2.

Dually, we may state the following result.

THEOREM 5′. *Let* $(\mathfrak{M}', \mathfrak{M}, \odot)$ *be a regular pairing, as in Theorem* 5. *Then the mappings* $\mathfrak{N}' \to \mathfrak{N}' \odot \mathfrak{M}$ *and* $\mathfrak{r} \to \mathfrak{r}\mathfrak{M}'$ *between the* $\mathfrak{B}$*-direct summands of* $\mathfrak{M}'$ *and the right ideal direct components of* $\mathfrak{c}$ *have the properties stated in Theorem* 5.

We shall omit the proof of Theorem 5′.

8. Further results on the structure of $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$. Using the results of §4, we shall establish a further theorem on the structure of the ring $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$, in case the pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$ is constructed from a projective representation of a finite group according to §2. In this case $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ is a crossed product, and the set Ω is vacuous. We shall assume that the dual pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ is regular, so that the results of §7 are available.

THEOREM 7. *Let* $(\mathfrak{M}', \mathfrak{M}, \tau)$ *be a regular pairing of* $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ *as defined in* §2. *Let the dual pairing* $(\mathfrak{M}', \mathfrak{M}, \odot)$ *be regular. Then every indecomposable left or right ideal direct component of* $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$ *contains a unique minimal subideal.*

Proof. First let $\mathfrak{l}$ be an indecomposable left ideal direct component of $\mathfrak{c}$. By Theorem 5, $\mathfrak{l} = \mathfrak{M}' \odot \mathfrak{N}$, where $\mathfrak{N}$ is an indecomposable $\mathfrak{B}$-direct summand of $\mathfrak{M}$. Our assumption that the pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ is regular implies that $\mathfrak{M}$ is a projective $\mathfrak{B}$-module, by Proposition 5 and the first remark thereafter. Therefore $\mathfrak{N}$ is $\mathfrak{B}$-isomorphic to an indecomposable right ideal direct component of $\mathfrak{B}$, and by Proposition 4, it follows that $\mathfrak{N}$ contains a unique minimal $\mathfrak{B}$-submodule $\mathfrak{m} \neq 0$. Since the pairing $(\mathfrak{M}', \mathfrak{M}, \odot)$ is non-degenerate, $\mathfrak{M}' \odot \mathfrak{m} \neq 0$. Now let $\mathfrak{l}' \neq 0$ be any left ideal contained in $\mathfrak{l}$. The fact that the pairing $(\mathfrak{M}', \mathfrak{M}, \tau)$ is regular implies that $\mathfrak{M}\mathfrak{l}' \neq 0$. By (20) we have

$$\mathfrak{l}' \supseteq \mathfrak{M}' \odot \mathfrak{M}\mathfrak{l}' \supseteq \mathfrak{M}' \odot \mathfrak{m},$$

and we have proved that $\mathfrak{M}' \odot \mathfrak{m}$ is the unique minimal subideal of $\mathfrak{l}$.

Now let $\mathfrak{r}$ be an indecomposable right ideal direct component of $\mathfrak{c}$. By Theorem 5′, $\mathfrak{r} = \mathfrak{N}' \odot \mathfrak{M}$, where $\mathfrak{N}'$ is an indecomposable $\mathfrak{B}$-direct summand of $\mathfrak{M}'$. By the second remark following Proposition 5, $\mathfrak{M}'$ is a projective $\mathfrak{B}$-module, and the argument given in the first part of the proof can be applied to prove that $\mathfrak{r}$ has a unique minimal subideal, as required.

COROLLARY. *Let* Δ *be a field, and let* E *be the subfield of* Δ *consisting of those elements of* Δ *left fixed by the automorphisms* $\bar{s}$, $s \in \mathfrak{G}$. *Let the hypotheses of Theorem* 6 *be satisfied, and assume also that* $\mathfrak{M}$ *is finite dimensional over* Δ. *Then* $\mathfrak{c} = \mathfrak{M}' \odot \mathfrak{M}$ *is a* QF-2 *algebra*[6] *over the field* E.

[6]A finite-dimensional algebra $\mathfrak{A}$ over a field **E** is a QF-2 algebra **(14)** if every right or left ideal direct component of $\mathfrak{A}$ contains a unique minimal subideal.

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Proof. It suffices to prove that $\mathfrak{c}$ is finite dimensional over E. Since Δ is commutative, the automorphisms $\bar{s}$, $s \in \mathfrak{G}$, form a finite group, and from Galois theory it follows that Δ is a finite extension of E. Therefore $\mathfrak{M}$ is finite dimensional over E. The elements of $\mathfrak{c}$ are l.t. in $\mathfrak{M}$ over E, and $\mathfrak{c}$ contains the scalar multiplications by elements of E, so that $\mathfrak{c}$ is a finite dimensional algebra over E, and the Corollary is proved.

Thrall's paper **(14)** contains a number of results concerning QF-2 algebras, all of which are directly applicable to $\mathfrak{c}$. We refer the reader to that paper for the details.

We add a final remark on the application of the theory to projective representations of groups. Let $(\mathfrak{M}', \mathfrak{M}, \tau)$ be a regular pairing of $\mathfrak{M}' \times \mathfrak{M} \to \mathfrak{B}$, constructed as in §2, and let $\mathfrak{C}$ be a centralizer of $\mathfrak{M}$ relative to $\mathfrak{B}$. Then Proposition 4, and the results of §5, can be applied to prove that every indecomposable $\mathfrak{C}$-direct summand of $\mathfrak{M}$ contains a unique minimal submodule. The proof is similar to the proof of Theorem 6, and will be omitted.

9. Applications to the Galois theory of primitive rings with minimal ideals. Let $\mathfrak{M}'$ and $\mathfrak{M}$ be left and right, respectively, vector spaces over a division ring Δ, which are dual relative to a non-degenerate bilinear form $\langle\psi, x\rangle$ on $\mathfrak{M}' \times \mathfrak{M} \to \Delta$. Let $\mathfrak{L}(\mathfrak{M}', \mathfrak{M})$ be the set of l.t. A on $\mathfrak{M}$ over Δ which possess transposes relative to the form $\langle\psi, x\rangle$, and let $\mathfrak{F}(\mathfrak{M}', \mathfrak{M})$ be the subset of $\mathfrak{L}(\mathfrak{M}', \mathfrak{M})$ consisting of finite valued l.t. We shall consider a ring $\mathfrak{A}$ of l.t. in $\mathfrak{M}$ over Δ such that **(7, 8)**

$$\mathfrak{F}(\mathfrak{M}', \mathfrak{M}) \subseteq \mathfrak{A} \subseteq \mathfrak{L}(\mathfrak{M}', \mathfrak{M}), \tag{25}$$

together with a finite group $\mathfrak{G}$ of automorphisms $A \to A^s$ of $\mathfrak{A}$. Then $\mathfrak{A}$ is a primitive ring with minimal ideals, and conversely, every primitive ring with minimal ideals is isomorphic to a dense ring of l.t. which satisfies (25). Let $\mathfrak{C}$ be the set of elements of $\mathfrak{A}$ which are left fixed by all the elements of $\mathfrak{G}$. We shall indicate how $\mathfrak{C}$ may be regarded as a centralizer of $\mathfrak{M}$ relative to a crossed product $\Delta(\mathfrak{G}, H, \rho)$, so that the results of §5–8 can be applied to discuss, for example, the subspaces of $\mathfrak{M}$ which are invariant relative to $\mathfrak{C}$.

For each element s in $\mathfrak{G}$, there exists a (1–1) s.l.t. U_s with associated automorphism $\bar{s}$ of $\mathfrak{M}$ onto itself, which possesses a transpose relative to the form, and which satisfies the equation

$$A^s = U_s^{-1} A U_s \tag{26}$$

for all $A \in \mathfrak{A}$. Since $(A^s)^t = A^{st}$, we obtain from (26),

$$U_t^{-1} U_s^{-1} A U_s U_t = U_{st}^{-1} A U_{st},$$

and

$$A U_s U_t U_{st}^{-1} = U_s U_t U_{st}^{-1} A,$$

for all A. Since $\mathfrak{A}$ is a dense ring of l.t., for each pair (s, t) there exists a scalar multiplication

$$\rho_{s,t}^{\overline{st}^{-1}}$$

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such that

$$U_s U_t U_{st}^{-1} = \rho_{s,t}^{\overline{st}^{-1}},$$

or

(27) $$U_s U_t = U_{st}\rho_{s,t}.$$

It is now easy to verify that $\{\rho_{s,t}; \bar{s}\}$ is a factor set, and that if $\mathfrak{B} = \Delta(\mathfrak{G}, H, \rho)$ is the corresponding crossed product, then the mappings U_s define a representation of $\mathfrak{B}$ by endomorphisms of $\mathfrak{M}$. Since (26) is unchanged if we replace U_s by $U_s\mu_s$, we may assume that $\rho_{1,1} = 1$. Then the condition

$$\rho_{s,s^{-1}} = 1$$

of Proposition 1 is satisfied if and only if $U_s U_{s^{-1}} = U_1$ for all s in $\mathfrak{G}$. We have to show finally that $\mathfrak{C}$ satisfies (19). The elements of $\mathfrak{C}$ are $\mathfrak{B}$-endomorphisms of $\mathfrak{M}$. On the other hand, by (15) it follows that $\mathfrak{M}' \odot \mathfrak{M}$ is precisely the set of l.t. $\sum_s A^s$, where A ranges throughout $\mathfrak{F}(\mathfrak{M}', \mathfrak{M}) \subseteq \mathfrak{A}$, so that $\mathfrak{M}' \odot \mathfrak{M} \subseteq \mathfrak{C}$, and (19) is proved.

10. On the centralizer of a projective module. It seems probable that more penetrating results than we have obtained in §7 and 8 can be proved concerning the structure of the centralizer of a projective module. To support this view we shall prove the following result.

THEOREM 7. *Let $\mathfrak{A}$ be a commutative symmetric algebra of l.t. on a finite dimensional space $\mathfrak{M}$ over a field Φ such that $\mathfrak{M}$ is a unital projective (right) $\mathfrak{A}$-module. Then the centralizer $\mathfrak{C}$ of $\mathfrak{M}$ relative to $\mathfrak{A}$ is a symmetric algebra.*

Proof. The only consequences which we shall require of the assumption that $\mathfrak{M}$ is a projective $\mathfrak{A}$-module are the following: (a) the indecomposable direct summands of the $\mathfrak{A}$-module $\mathfrak{M}$ are $\mathfrak{A}$-isomorphic to indecomposable right ideal direct components of $\mathfrak{A}$ (**9**, Theorem 1); and (b) if $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $\mathfrak{A}$-modules, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are projective $\mathfrak{A}$-modules (**5**, p. 473).

We recall that $\mathfrak{A}$ is symmetric if and only if there exists a hyperplane $\mu(a) = 0$, which contains all commutators $ab - ba$ but no non-zero right or left ideals. We shall require the result that if $\mathfrak{A}$ and $\mathfrak{B}$ are symmetric algebras, then the Kronecker product $\mathfrak{A} \otimes \mathfrak{B}$ is symmetric.

Now we begin the proof of the theorem. First assume that $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-zero ideals. If we set $\mathfrak{M}_i = \mathfrak{M}\mathfrak{A}_i$ $(i = 1, 2)$, then each $\mathfrak{M}_i$ is a faithful $\mathfrak{A}_i$-module, and $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$. The elements of the centralizer $\mathfrak{C}_i$ of $\mathfrak{M}_i$ relative to $\mathfrak{A}_i$ $(i = 1, 2,)$ may be viewed as elements of the centralizer $\mathfrak{C}$ of $\mathfrak{M}$ relative to $\mathfrak{A}$, and with this agreement, $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$. It follows that $\mathfrak{C}$ is symmetric if we can prove that the $\mathfrak{C}_i$ are symmetric. Furthermore, each $\mathfrak{M}_i$ is a projective $\mathfrak{A}$-module, and hence a projective $\mathfrak{A}_i$-module $(i = 1, 2)$. Thus we may assume, without loss of generality, that, in addition to the hypotheses stated in the theorem, $\mathfrak{A}$ is an indecomposable algebra. Now let

$$\mathfrak{M} = \mathfrak{M}_1 \oplus \ldots \oplus \mathfrak{M}_s,$$

where the $\mathfrak{M}_i$ are indecomposable $\mathfrak{A}$-modules. Since $\mathfrak{M}$ is projective, each $\mathfrak{M}_i$ is $\mathfrak{A}$-isomorphic to $\mathfrak{A}$, by the indecomposability of $\mathfrak{A}$, and hence the $\mathfrak{M}_i$ are isomorphic to each other. Evidently $\mathfrak{M}_i$ is a faithful cyclic $\mathfrak{A}$-module. Since $\mathfrak{A}$ is commutative, the centralizer of $\mathfrak{M}_1$ relative to $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$. The centralizer $\mathfrak{C}$ of $\mathfrak{M}$ is isomorphic to the full algebra of s by s matrices with coefficients in the centralizer of $\mathfrak{M}_1$ (**6**, p. 58), and hence

$$\mathfrak{C} \cong (\mathfrak{A})_s \cong \mathfrak{A} \otimes \Phi_s.$$

Since both $\mathfrak{A}$ and Φ_s are symmetric algebras, we conclude that $\mathfrak{C}$ is symmetric, and the theorem is proved.

11. Examples of regular pairings. We shall consider the pairing σ of §3, which has been studied by Weyl in connection with the representation theory of the full linear group. Let Φ be an arbitrary field of characteristic $p \geqslant 0$. Let $\mathfrak{M}$ be the m-fold Kronecker product with itself of an n-dimensional space $\mathfrak{V}$ over Φ. Let $\mathfrak{G} = \mathfrak{S}_m$ be the symmetric group on m letters, and let $b \rightarrow U(b)$ be the (ordinary) representation of the group algebra $\mathfrak{B}$ of $\mathfrak{G}$ by symmetry operators on $\mathfrak{M}$. Let $(\mathfrak{M}, \mathfrak{M}^*, \sigma)$ be the pairing defined in §3, and let $\mathfrak{b}$ be the nucleus $\sigma(\mathfrak{M}, \mathfrak{M}^*)$. We shall state without proof a few special results.

(a) $p > m$ or $p = 0$, n arbitrary. Then $\mathfrak{B}$ is semi-simple, and the pairing is regular. The centrally primitive idempotents of $\mathfrak{B}$ which are contained in $\mathfrak{b}$ have been determined explicitly by Weyl (**17**, Chap. IV).

(b) $m \leqslant n$, p arbitrary. Then $\mathfrak{b} = \mathfrak{B}$, and the pairing is regular.

(c) $m = 3$, $p = 3$, $n = 2$. Then $\mathfrak{b} = \mathfrak{B}$, and the pairing is regular. In this case the kernel $\mathfrak{K}$ of the representation U is different from zero, and $\mathfrak{b} \cap \mathfrak{K} = \mathfrak{K}$.

References

1. E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with minimum condition*, University of Michigan Publications in Mathematics, No. 1, 1944.

2. R. Brauer, *On sets of matrices with coefficients in a division ring*, Trans. Amer. Math. Soc., *49* (1951), 502–548.

3. H. Fitting, *Die Theorie der Automorphismenringe Abelscher Gruppen und ihr Analogen bei nicht kommutativen Gruppen*, Math. Ann., *107* (1932), 514–542.

4. W. Gaschütz, *Über der Fundamentalsatz von Maschke zur Darstellungstheorie der endlichen Gruppen*, Math. Z., *56* (1952), 376–387.

5. F. Kasch, *Grundlagen einer Theorie der Frobeniuserweiterungen*, Math. Ann., *127* (1954), 453–474.

6. N. Jacobson, *The theory of rings*, Mathematical Surveys, II (New York, 1943).

7. ———, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math., *67* (1945), 300–320.

8. ———, *Lectures in abstract algebra*, II (New York, 1953).

https://doi.org/10.4153/CJM-1956-032-7 Published online by Cambridge University Press

9. H. Nagao and T. Nakayama, *On the structure of* (M_o) *and* (M_u) *modules*, Math. Zeit., *59* (1953), 164–170.
10. T. Nakayama, *On Frobeniusean algebras*, I, Ann. Math., *40* (1939), 611–633.
11. ———, *On Frobeniusean algebras* II, Ann. Math., *42* (1941), 1–21.
12. C. Nesbitt, *On the regular representations of algebras*, Ann. Math., *39* (1938), 634–658.
13. C. Nesbitt and R. Thrall, *Some ring theorems with applications to modular representations*, Ann. Math., *47* (1946), 551–567.
14. R. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc., *64* (1948), 173–183.
15. H. Weyl, *Commutator algebra of a finite group of collineations*, Duke Math. J., *3* (1937), 200–212.
16. ———, *The theory of groups and quantum mechanics* (New York, 1931).
17. ———, *The classical groups* (Princeton, 1939).

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https://doi.org/10.4153/CJM-1956-032-7 Published online by Cambridge University Press