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HOMOGENEOUS C^*-ALGEBRAS WHOSE SPECTRA ARE TORI

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Abstract

By a theorem of Fell and Tomiyama-Takesaki, an N-homogeneous C^*-algebra with spectrum X has the form $\Gamma(E)$ for some bundle E over X with fibre $M_N(\mathbf{C})$, and its isomorphism class is determined by that of E and its pull-backs f^*E along homeomorphisms f of X. We describe the homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ or $\mathbf{T}^3$ by classifying the M_N-bundles over $\mathbf{T}^k$ using elementary homotopy theory. We then use our results to determine the isomorphism classes of a variety of transformation group C^*-algebras, twisted group C^*-algebras and more general crossed products.

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Let A be an N-homogeneous C^*-algebra with spectrum X. A well-known theorem of Fell [6] and Tomiyama-Takesaki [18] asserts that there is a locally trivial bundle E over X with fibre $M_N(\mathbf{C})$ and structure group $PU_N(\mathbf{C}) = \operatorname{Aut} M_N(\mathbf{C})$ such that A is isomorphic to the algebra $\Gamma_0(E)$ of sections of E which vanish at infinity. Further, they prove that two such algebras $A_i = \Gamma_0(E_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism f of X_1 onto X_2 such that $E_1 \cong f^*E_2$ as bundles over X_1. Our goal here is to use elementary homotopy theory to describe the M_N-bundles over tori and hence classify the homogeneous C^*-algebras whose spectra are homeomorphic to tori.

A similar analysis for homogeneous algebras over spheres has been made by Krauss and Lawson [9], and was successful in low dimensions. However, their examples of non-trivial homogeneous C^*-algebras, and others in the literature (see, for example, [12, Section 3]), are constructed only in a homotopy-theoretic fashion. We were attracted to the analogous problem over tori by the abundance of naturally occurring examples: homogeneous C^*-algebras with spectra homeomorphic to $\mathbf{T}^k$ can arise as twisted group algebras of abelian groups, as transformation group C^*-algebras and as more general crossed products or twisted crossed products. Well-known examples are the twisted group algebras of $\mathbf{Z}^k$ by a type I multiplier and the rational rotation algebras, which are the transformation group C^*-algebras of the actions of $\mathbf{Z}$ on $\mathbf{T}$ by rotation through (2π times) a rational angle. It will follow from our results that many of these constructions give non-trivial homogeneous C^*-algebras—that is, not isomorphic to $C(\mathbf{T}^k, M_N)$ —and we shall often be able to determine their isomorphism class explicitly. Since algebras constructed like these carry natural group actions, we shall exhibit concretely a wide variety of C^*-dynamical systems involving non-trivial homogeneous C^*-algebras. We hope these examples may prove useful in that our results facilitate detailed calculations with them.

Our main results concern homogeneous C^*-algebras over $\mathbf{T}^2$ and $\mathbf{T}^3$. Bundles with fibre $M_N = M_N(\mathbf{C})$ over $\mathbf{T}^2$ are described by a residue class $[p]$ in $\mathbf{Z}/N\mathbf{Z} = \pi_1(PU_N)$, and two such bundles with classes $[p_i]$ give isomorphic C^*-algebras if and only if $[p_1] = \pm[p_2]$. An M_N-bundle over $\mathbf{T}^3$ is determined by three residue classes in $\mathbf{Z}/N\mathbf{Z}$, but many bundles can give rise to isomorphic C^*-algebras. It turns out that every homogeneous C^*-algebra with spectrum $\mathbf{T}^3$ is isomorphic to one of the form $B \otimes C(\mathbf{T})$, where B is homogeneous with spectrum $\mathbf{T}^2$, and that it is possible to have $B_1 \otimes C(\mathbf{T}) \cong B_2 \otimes C(\mathbf{T})$ without having $B_1 \cong B_2$. We have been able to realise all these homogeneous C^*-algebras over $\mathbf{T}^2$ and $\mathbf{T}^3$ in various different ways as transformation group C^*-algebras and twisted group C^*-algebras, at least up to tensoring on a matrix algebra $M_m(\mathbf{C})$ (see Propositions 2.8, 2.9 and 3.10). The situation over $\mathbf{T}^4$ is not so clear: we can show that there are infinitely many isomorphism classes of N-homogeneous C^*-algebras over $\mathbf{T}^4$ (for each N), but we do not know if these can be constructed in some natural C^*-algebraic fashion.

Although we were not aware of this when we worked out our classification over $\mathbf{T}^2$, there have been two other determinations of the isomorphism classes of the rational rotation algebras, by Høegh-Krohn and Skjelbred [7] and by Rieffel [16]. The calculations of [7] are in the same spirit as ours; however we feel that, as we have started from the problem of classifying all algebras over $\mathbf{T}^2$, and only specialised to the rotation algebras later, our approach seems more rational. (Sorry.) As a bonus, it follows from our work that the rational rotation algebras are esssentially the only homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ (see Proposition 2.8). We thank Dana Williams for drawing our attention to [7] and [16].

We have been particularly interested in describing the M_N-bundles which arise when calculating specific C^*-algebras using an appropriate version of the Mackey machine, and our classification is geared to this purpose. Another, more sophisticated classification of M_N-bundles up to bundle isomorphism, which works over an arbitrary 4-complex, has been made by Woodward [19]. When the underlying space is $\mathbf{T}^2$ or $\mathbf{T}^3$, however, this part of our problem is quite easy and we have used elementary homotopy-theoretic arguments in preference to calculating Woodward's cohomological invariants for our concretely constructed bundles. On the other hand, his results do give useful information on $\mathbf{T}^4$, where the effect of homeomorphisms on the $H^4(\mathbf{T}^4, \mathbf{Z}) = \mathbf{Z}$ part of his invariant is clear.

Our work is arranged as follows. We begin with an introductory section in which we set up notation and state convenient versions of the results of [5], [17] and [3] describing the representation theory of crossed products and twisted group algebras. Section 2 contains our results on homogeneous C^*-algebras with spectrum $\mathbf{T}^2$. We classify principal PU_N-bundles over $\mathbf{T}^2$, investigate the effect of homeomorphisms and then show how to determine the isomorphism class of algebras constructed in different ways (Proposition 2.7). These enable us to describe a wide variety of transformation group C^*-algebras, twisted group C^*-algebras, crossed products, and so on. In Section 3 we follow a similar programme over $\mathbf{T}^3$. Since the main result here (Theorem 3.9) indicates—rather surprisingly, we feel—that these algebras are likely to be less interesting than those over $\mathbf{T}^2$, we consider only two of the more obvious constructions. We finish with a short section of concluding remarks, including a discussion of our inconclusive results on bundles over $\mathbf{T}^4$.

1. Preliminaries

Let A be a C^*-algebra, G a locally compact group and $\alpha: G \to \operatorname{Aut} A$ a strongly continuous automorphism group. In later sections we shall want to calculate the spectrum of the crossed product $C^*(A, G)$ and represent $C^*(A, G)$ concretely on it: to do this, we use Takesaki's version of Mackey's method for covariant representations. As the groups we consider are all abelian, some simplification of his results is possible, and we state two convenient versions here.

We first set up some notation and conventions. Our C^*-algebras and locally compact groups are all separable. If (π, U) is a covariant representation of (A, G), we denote the corresponding representation of $C^*(A, G)$ by $\pi \times U$. The action of G on A induces an action of G on the spectrum $\hat{A}$ defined by $t \cdot \pi = \pi \circ \alpha_t^{-1}$ (we shall frequently confuse an irreducible representation of A with its class in $\hat{A}$), and we denote the stabiliser of π in this action by G_π. We say that α is smooth if the action of G on $\hat{A}$ is smooth—that is, if the orbit space $\hat{A}/G$ is countably separated. All the actions we consider here are smooth, and in fact the orbit space $\hat{A}/G$ is always Hausdorff, which is a much stronger condition (see [17, Theorem 2.5]). If σ is a multiplier on G as in [10] we denote the set of (equivalence classes of) irreducible σ-representations of G by $(G, \sigma)\hat{}$.

All except the last statement in the next result is contained in Theorem 7.2 of [17]—the simplifications are possible because $G_{t\cdot\pi}$ is G_π rather than a conjugate of it—and the rest is a matter of direct calculation. The corollary is the special case where $A = C_0(\Omega)$ and the crossed product is the transformation group C^*-algebra $C^*(G, \Omega)$.

PROPOSITION 1.1. *Let G be a separable locally compact abelian group and let $\alpha\colon G \to \operatorname{Aut} A$ be a smooth action of G on a separable type* I *C^*-algebra A. For each $\pi \in \hat{A}$ there is a multiplier σ_π on G_π and a σ_π-representation L_π on H_π such that*

$$\pi(\alpha_t(a)) = L_\pi(t)\pi(a)L_\pi(t)^* \quad \text{for } a \in A, t \in G_\pi;$$

σ_π and L_π are unique up to equivalence. Then every irreducible representation of $C^*(A, G)$ is equivalent to one of the form

$$\operatorname{Ind}_{G_\pi}^G[(\pi \otimes 1) \times (L_\pi \otimes M)]$$

for some $\pi \in \hat{A}$ and $M \in (G_\pi, \bar{\sigma}_\pi)\hat{}$. Two such representations

$$\operatorname{Ind}[(\pi_i \otimes 1) \times (L_i \otimes M_i)]$$

are equivalent if and only if $\pi_2 \in G \cdot \pi_1$, so we can take $L_2 = L_1$, and M_2 is equivalent to M_1. In fact if we realise $\operatorname{Ind}(\pi \otimes 1) \times (L \otimes M)$ in

$$H = \left\{ \xi\colon G \to H_\pi \otimes H_M \,\middle|\, \begin{array}{l} \xi \text{ is Borel measurable}, \xi(st) = (L(t)^* \otimes M(t)^*)\xi(s) \\ \text{for all } t \in G_\pi, s \in G, \text{ and } \int \|\xi(sG_\pi)\|^2 d(sG_\pi) < \infty \end{array} \right\},$$

according to

$$[\operatorname{Ind}(\pi \otimes 1) \times (L_\pi \otimes M)(z)\xi](t) = \int_G \big(\pi(\alpha_t^{-1}(z(s))) \otimes 1\big)(\xi(s^{-1}t))\, ds \qquad (z \in L^1(G, A)),$$

and define a unitary W_u on H by $(W_u\xi)(t) = \xi(tu)$, then

$$W_u[\operatorname{Ind}(\pi \otimes 1) \times (L_\pi \otimes M)]W_u^* = \operatorname{Ind}(u \cdot \pi \otimes 1) \times (L_\pi \otimes M).$$

COROLLARY 1 [5]. *Let (G, Ω) be a separable smooth locally compact transformation group with G abelian, and let $\varepsilon_x \in C_0(\Omega)\hat{}$ be evaluation at $x \in \Omega$. Then*

$$C^*(G, \Omega)\hat{} = \left\{ \operatorname{Ind}_{G_x}^G(\varepsilon_x \times \gamma) \,\middle|\, x \in \Omega, \gamma \in \hat{G}_x \right\},$$

and $\operatorname{Ind} \varepsilon_x \times \gamma$ *is equivalent to* $\operatorname{Ind} \varepsilon_y \times \chi$ *if and only if $y \in G \cdot x$ and $\gamma = \chi$; an intertwining operator is as described in the proposition with $L = \gamma$ and M, H_M omitted.*

Let σ be a multiplier on a locally compact group, and let $L^1(G, \sigma)$ be the Banach *-algebra consisting of $L^1(G)$ with multiplication and involution given by

$$f * g(t) = \int_G f(s) g(s^{-1}t) \sigma(s, s^{-1}t)\, dt,$$
$$f^*(s) = \left[\sigma(s, s^{-1}) f(s^{-1})\right]^-.$$

As usual, integrating gives a one-to-one correspondence between σ-representations of G and non-degenerate *-representations of $L^1(G, \sigma)$. The C^*-enveloping algebra $C^*(G, \sigma)$ of $L^1(G, \sigma)$ is called the twisted group C^*-algebra of G by σ; by construction $C^*(G, \sigma)\hat{} = (G, \sigma)\hat{}$, and this gives a topology on the latter set. To calculate twisted group C^*-algebras we need the following basic results on multiplier representations of abelian groups due to Baggett and Kleppner [3]. We observe that although our multipliers are not usually normalised in the sense of [3], each is equivalent to a normalised one, so their results apply. A non-degenerate multiplier σ is one whose normalised version is totally skew—in other words, $S_\sigma = \{e\}$ (see below).

PROPOSITION 1.3. *Let G be a locally compact abelian group with multiplier σ, and let S_σ be the closed subgroup of those $s \in G$ such that*

$$\tilde{\sigma}(s, t) = \sigma(s, t)\overline{\sigma(t, s)} = 1 \quad \textit{for all } t \in G.$$

Then σ is equivalent to a multiplier lifted from a non-degenerate multiplier σ_1 on G/S_σ, and σ is type I *if and only if σ_1 is type* I. *If σ is a type* I *multiplier, then up to equivalence G/S_σ has a unique irreducible σ_1-representation M_1, and if M is the corresponding σ-representation of G, then every irreducible σ-representation of G is equivalent to one of the form $\gamma \cdot M$ for some $\gamma \in \hat{G}$. Two such representations $\gamma M, \chi M$ are equivalent if and only if $\gamma(s) = \chi(s)$ for all $s \in S_\sigma$, and the map $\gamma \to \gamma M$ induces a homeomorphism of $\hat{S}_\sigma = \hat{G}/S_\sigma^\perp$ onto $(G, \sigma)\hat{}$. If $|G\colon S_\sigma| < \infty$, then every $\tau \in S_\sigma^\perp$ has the form $\tilde{\sigma}(s, \cdot)$ for some $s \in G$, and then $M(s)$ intertwines γM and $\tau\gamma M$.*

PROOF. All except the last sentence is contained in [3] (see Theorem 3.1, Theorem 3.3 and its corollary). By definition the homomorphism $s \to \tilde{\sigma}(s, \cdot)$ has kernel S_σ, and if $|G\colon S_\sigma| < \infty$ then it must be an isomorphism of G/S_σ onto $S_\sigma^\perp = (G/S_\sigma)\hat{}$ since they have the same size. It is easy to check that if $g, h \in G/S_\sigma$ then

$$M_1(g) M_1(h)^* = \tilde{\sigma}_1(g, h) M_1(h)^* M_1(g),$$

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and it follows that

$$M(s)\gamma(t)M(t)M(s)^* = \tilde{\sigma}(s,t)\gamma(t)M(t),$$

as required.

We shall also need some well-known facts about the topology of the complex unitary group U_N and projective unitary group PU_N for $N \geqslant 2$. Both are path-connected compact topological groups. The fundamental group $\pi_1(U_N)$ is isomorphic to $\mathbf{Z}$ under the map which sends a loop ϕ to the winding number deg(det ϕ) of the determinant of ϕ, and $\pi_2(U_N) = 0$; both these can be deduced inductively from the corresponding facts about $U_1 = \mathbf{T}$ using the long exact sequence of homotopy groups for the bundle $U_N \to S^{2N-1}$ with fibre U_{N-1}. The same exact sequence for the bundle Ad: $U_N \to PU_N$ shows that $\pi_1(PU_N) \cong \mathbf{Z}/N\mathbf{Z}$ and that $\pi_2(PU_N) = 0$. The simply connected covering of PU_N is given by Ad: $SU_N \to PU_N$.

Because of the theorem of Fell and Tomiyama-Takesaki discussed in the introduction, we are interested in locally trivial bundles with fibre M_N and structure group PU_N. We shall frequently refer to these merely as M_N-bundles, and we shall move frequently between M_N-bundles and the corresponding (locally trivial) principal PU_N-bundles.

Finally, if L, M are integers we shall write (L, M) for their highest common factor and $[L, M]$ for their lowest common multiple.

2. Homogeneous C^*-algebras with spectrum $\mathbf{T}^2$

To describe the N-homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ we have to classify the locally trivial M_N-bundles over $\mathbf{T}^2$, or, equivalently, the principal PU_N-bundles. We first do this up to the usual notion of bundle isomorphism and then investigate the effect of homeomorphisms of $\mathbf{T}^2$. The following lemma will form the basis for our work on every $\mathbf{T}^k = \mathbf{T}^{k-1} \times \mathbf{T}$: we shall frequently work with a general group G to help isolate the crucial properties of PU_N.

LEMMA 2.1. *Let G be a topological group and X a paracompact space. For each principal G-bundle E over X and automorphism τ of E, we denote by $E \times_\tau \mathbf{T}$ the bundle over $X \times \mathbf{T}$ obtained from $E \times I$ by pasting the subsets $E \times \{1\}$ and $E \times \{0\}$ using τ. Then every principal G-bundle over $X \times \mathbf{T}$ is isomorphic to some $E \times_\tau \mathbf{T}$, and two such bundles $E_i \times_{\tau_i} \mathbf{T}$ are isomorphic if and only if there is an isomorphism σ: $E_1 \to E_2$ such that $\sigma^{-1}\tau_2\sigma$ is homotopic to τ_1.*

PROOF. Suppose first of all that F is a principal G-bundle over $X \times \mathbf{T}$, and let p: $X \times I \to X \times \mathbf{T}$ be the identification map. If $E = F|_{X\times\{0\}}$, then there is an isomorphism ϕ: $E \times I \to p^*F$ [8, page 50]. We denote the restriction of ϕ to $E = E \times \{t\}$ by ϕ_t, and define $\tau \in \operatorname{Aut}(E)$ by $\tau = \phi_1^{-1}\phi_0$. Then ϕ provides an isomorphism of $E \times_\tau \mathbf{T}$ onto F, which proves our first assertion. If E_i, τ_i, σ are as above and ϕ_t: $I \to \operatorname{Aut}(E)$ is a homotopy joining τ_1 and $\sigma^{-1}\tau_2\sigma$, then

$$\Phi(e, t) = (\sigma \circ \phi_t(e), t)$$

defines an isomorphism of $E_1 \times_{\tau_1} \mathbf{T}$ onto $E_2 \times_{\tau_2} \mathbf{T}$. Conversely, such an isomorphism is given by a path of isomorphisms ψ_t: $E_1 \to E_2$ such that $\psi_1 \circ \tau_1 = \tau_2 \circ \psi_0$; then $\phi_t = \tau_1\psi_0^{-1}\psi_t$ is a homotopy joining τ_1 to $\psi_1^{-1}\tau_2\psi_1$. This proves the lemma.

Corollary 2.2. *Let G be a path-connected topological group. Every principal G-bundle over $\mathbf{T}^2$ is isomorphic to the bundle E_τ obtained from $\mathbf{T} \times I \times G$ by pasting $\mathbf{T} \times \{1\} \times G$ and $\mathbf{T} \times \{0\} \times G$ via a continuous map τ: $\mathbf{T} \to G$ satisfying $\tau(1) = e$. The isomorphism class of the bundle depends only on the class of τ in $\pi_1(G)$.*

Proof. An application of Lemma 2.1 to the one point space X shows that when G is path-connected all principal G-bundles over $\mathbf{T}$ are trivial. Hence it also follows from Lemma 2.1 that every principal G-bundle over $\mathbf{T}^2$ is isomorphic to one of the form $(\mathbf{T} \times G) \times_\tau \mathbf{T}$ for some τ: $\mathbf{T} \to G$. Since G is path-connected each such τ is homotopic to one with $\tau(1) = e$, and two such maps are homotopic iff they define the same class in $\pi_1(G)$. The group $\pi_1(G)$ is abelian, so the final assertion also follows from Lemma 2.1.

To describe the effect to the homeomorphisms of $\mathbf{T}^2$, we need to know what they look like, at least up to homotopy. The first lemma will also be useful later.

Lemma 2.3. *Let G be a topological group with $\pi_2(G) = 0$, and let $\tau: I \to G$ be a continuous map satisfying $\tau(0) = \tau(1) = e$. Let $\mathcal{F}$ denote the family of continuous maps ϕ: $I^2 \to G$ which map the corners of I^2 to e and satisfy*

$$\phi(s,0) = \phi(s,1) \quad \text{for all } s \in I, \text{ and}$$
$$\phi(1,t) = \tau(t)^{-1}\phi(0,t)\tau(t) \quad \text{for all } t \in I.$$

Then $\phi, \psi \in \mathcal{F}$ are homotopic in $\mathcal{F}$ if and only if

$$[\phi|_{s=0}] = [\psi|_{s=0}], \quad [\phi|_{t=0}] = [\psi|_{t=0}] \qquad \text{in } \pi_1(G).$$

Proof. The necessity of these conditions is obvious. Suppose that h_u, k_u: $I \to G$ are homotopies relative to $\{0, 1\}$ which join $\phi|_{s=0}$ and $\psi|_{s=0}$, $\psi|_{t=0}$ and $\psi|_{t=0}$ respectively. Define L: $\partial I^3 \to G$ by

$$L(s,t,0) = \phi(s,t), \qquad L(s,t,1) = \psi(s,t),$$

$$L(s,0,u) = L(s,1,u) = h_u(s),$$
$$L(0,t,u) = k_u(t), \qquad L(1,t,u) = \tau(t)^{-1}k_u(t)\tau(t).$$

Since $\pi_2(G)$ is trivial, there is no obstruction to extending L to all of I^3, and such an extension is a homotopy joining ϕ to ψ in $\mathcal{F}$.

LEMMA 2.4. *Define* ϕ: $\mathbf{Z}^2 \to \pi_1(\mathbf{T}^2)$ *by sending the generators* $(1,0)$, $(0,1)$ *to the loops* $\mathbf{T} \to \mathbf{T} \times \{1\}$, $\mathbf{T} \to \{1\} \times \mathbf{T}$ *respectively. Then* ϕ *is an isomorphism, and we can define a map* ψ: $[\mathbf{T}^2, \mathbf{T}^2] \to M_2(\mathbf{Z})$ *by* $\psi(f) = \phi^{-1} \circ f_* \circ \phi$. *If* f, g *are homeomorphisms of* $\mathbf{T}^2$ *such that* $\psi(f) = \psi(g)$, *then* f *is homotopic to* g.

PROOF. The identity map on $\mathbf{T}$ generates $\pi_1(\mathbf{T}) = \mathbf{Z}$, and $\pi_1(X \times Y)$ is always isomorphic to $\pi_1(X) \times \pi_1(Y)$, so ϕ is an isomorphism. The map ψ converts composition into matrix multiplication, so it is enough to show that if $\psi(f) = 1_2$, then f is homotopic to the identity. Without loss of generality, suppose $f(1,1) = (1,1)$. Then the statements

$$\psi(f)(1,0) = (1,0), \qquad \psi(f)(0,1) = (0,1)$$

say that the loops $z \to f(z,1)$ and $w \to f(1,w)$ are homotopic to $z \to (z,1)$ and $w \to (1,w)$ respectively. Since $\pi_2(\mathbf{T}^2) = (\pi_2(\mathbf{T}))^2 = 0$, the result follows by taking $\tau \equiv (1,1)$ in the lemma.

COROLLARY 2.5. *Every homeomorphism of* $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ *is homotopic to one of the form*

$$f_A(x + \mathbf{Z}^2) = Ax + \mathbf{Z}^2 \qquad (x \in \mathbf{R}^2),$$

for some matrix $A \in M_2(\mathbf{Z})$ *of determinant* ± 1.

PROOF. If f is a homeomorphism then f_*: $\pi_1(\mathbf{T}^2) \to \pi_1(\mathbf{T}^2)$ is an isomorphism, and $\psi(f)$ must have determinant ± 1. However, it is straightforward to check that $\psi(f_A) = A$, so it follows from the lemma that f is homotopic to $f_{\psi(f)}$.

The next lemma shows that, if we realise bundles over $\mathbf{T}^2$ appropriately, then it is easy to describe their pull-backs along maps of the form f_A. Notice we do not need to assume $\det A = \pm 1$; this will be convenient later.

LEMMA 2.6. *Let* $V, W \in U_N$ *satisfy* $\operatorname{Ad} VW = \operatorname{Ad} WV$, *and let* $E(V,W)$ *be the* M_N*-bundle over* $\mathbf{T}^2$ *obtained from* $I^2 \times M_N$ *by pasting via* $\operatorname{Ad} V$ *along* $\{0,1\} \times I$ *and via* $\operatorname{Ad} W$ *along* $I \times \{0,1\}$. *Let* $A \in GL_2(\mathbf{R})$ *have integral entries, and let* f_A *be the induced map on* $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. *Then*

$$f_A^* E(V,W) \cong E(V^{a_{11}}W^{a_{21}}, V^{a_{12}}W^{a_{22}}).$$

PROOF. $E(V,W)$ can be regarded as the quotient of $\mathbf{R}^2 \times M_N$ by the equivalence

$$(*) \qquad (s,t,U) \equiv (s+m, t+n, (\operatorname{Ad} V^m W^n)U)$$

for $s, t \in \mathbf{R}$, $m, n \in \mathbf{Z}$ and $U \in M_N$. Let $q\colon \mathbf{R}^2 \to \mathbf{T}^2$ be the quotient map. Then a bundle F on $\mathbf{T}^2$ is isomorphic to $E(V,W)$ if there is an isomorphism ϕ: $\mathbf{R}^2 \times M_N \to q^*F$ compatible with the quotient maps from $\mathbf{R}^2 \times M_N$ to $E(V,W)$ and q^*F to F. This condition is

$$\phi^{-1}_{s+m,t+n} \circ \phi_{s,t} = \operatorname{Ad} V^m W^n.$$

Let $p_{s,t}$ be the restriction to $\{(s,t)\} \times M_N = M_N$ of the quotient map from $\mathbf{R}^2 \times M_N$ to $E(V,W)$. As $f_A \circ q = q \circ A$ we can define an isomorphism ϕ from $\mathbf{R}^2 \times M_n$ to $q^* f_A^* E(V,W)$ by $\phi(s,t,U) = (s,t,p_{A(s,t)}U)$. Now $(*)$ implies that $P_{s+m,t+n} \circ \operatorname{Ad} V^m W^n = p_{s,t}$. Hence

$$P_{A(s+m,t+n)} \operatorname{Ad} V^{a_{11}m + a_{12}n} W^{a_{21}m + a_{22}n} = P_{A(s,t)};$$

therefore

$$\phi^{-1}_{s+m,t+n} \circ \phi_{s,t} = \operatorname{Ad} V^{a_{11}m + a_{12}n} W^{a_{21}m + a_{22}n},$$

which proves that $f_A^* E(V,W) \cong E(V^{a_{11}} W^{a_{21}}, V^{a_{12}} W^{a_{22}})$.

PROPOSITION 2.7. (1) *Let $U\colon \mathbf{T} \to U_N$ be a continuous map, and let*

$$A(U) = \{a \in C(I \times \mathbf{T}, M_N) \mid a(1,z) = \operatorname{Ad} U(z)(a(0,z)) \text{ for } z \in \mathbf{T}\}.$$

Then every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$ is isomorphic to some $A(U)$, and two such algebras $A(U_i)$ are isomorphic if and only if $\deg(\det U_1) = \pm \deg(\det U_2) \bmod N$, *where* deg *stands for the winding number about* 0.

(2) *Let $U\colon I \to SU_N$ be a continuous map such that* $\operatorname{Ad} U(0) = \operatorname{Ad} U(1)$, *and define $A(U)$ as in* (1). *Every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$ has this form, and two such algebras $A(U_i)$ are isomorphic if and only if the multiples of the identity $U_i(0)^* U_i(1)$ are equal or adjoint.*

(3) *Let $V, W \in U_N$ satisfy* $\operatorname{Ad} VW = \operatorname{Ad} WV$, *and let*

$$B(V,W) = \{b \in C(I^2, M_n) \mid b(1,t) = \operatorname{Ad} V(b(0,t)), \quad b(s,1) = \operatorname{Ad} W(b(s,0)) \text{ for } s,t \in I\}.$$

Every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$ is isomorphic to one of this form, and two such algebras $B(V_i, W_i)$ are isomorphic if and only if the commutators $\{V_i, W_i\} = V_i^* W_i^* V_i W_i$ are equal or adjoint.

(4) *Let $U\colon \mathbf{T} \to U_N$ be continuous, and let $V, W \in M_N$ satisfy* $\operatorname{Ad} VW = \operatorname{Ad} WV$. *Then $A(U) \cong B(V,W)$ if and only if*

$$V^* W^* V W = \exp\{2\pi i (\pm \deg(\det U))/N\} 1_N.$$

PROOF. We begin by showing that (1) and (2) are equivalent statements. Let $U\colon \mathbf{T} \to U_N$, and define $U_1\colon I \to SU_N$ by

$$U_1(t) = [\det U(\exp(2\pi it))]^{-1/N} U(\exp(2\pi it)),$$

where $z^{1/N}$ is the principal branch of the Nth root function: note that $\operatorname{Ad} U_1(1) = \operatorname{Ad} U_1(0)$ since U is a loop. Conversely, if U_1 is as in (2), and $U_1(1) = e^{2\pi ik/N} U_1(0)$, we define a loop U in U_N by

$$U(t) = \exp(-2\pi ikt/N) U_1(t) \quad \text{for } t \in [0,1].$$

Since we have only changed our maps by scalars, we have $\operatorname{Ad} U = \operatorname{Ad} U_1$, and $A(U) = A(U_1)$. The invariant $\deg(\det U)$ is the integer k such that $U_1(1) = e^{2\pi ik/N} U_1(0)$, so the conditions in (1) and (2) agree under this correspondence.

We now show that (2) and (3) are equivalent. Let $V, W \in U_N$ satisfy $\operatorname{Ad} VW = \operatorname{Ad} WV$: note that without changing $B(V, W)$ we may suppose $V, W \in SU_N$. We choose paths V_s, W_t in SU_N joining $V = V_1$, $W = W_1$ to 1_N, and let $U(t) = V^* W_t^* V W_t$. Then a calculation shows that

$$\Phi(a)(s,t) = \operatorname{Ad} V_s^* W_t^* (a(s,t))$$

is an isomorphism of $A(U)$ onto $B(V, W)$. Conversely, given $U\colon I \to SU_N$ with $U(1) = e^{2\pi ik/N} U(0)$, we let $w = e^{2\pi i/N}$ and

$$V = \begin{pmatrix} 0 & 1_{N-k} \\ 1_k & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w^{N-1} \end{pmatrix}.$$

We picked these so that $V^* W^* V W = e^{2\pi ik/N} 1_N$, and hence if W_t joins 1 to W then $V^* W_t^* V W_t U(0)$ has the same endpoints as U. Since SU_N is the simply connected covering space of PU_N, it follows that the two loops $\operatorname{Ad} U$ and $\operatorname{Ad} V^* W_t^* V W_t U(0)$ in PU_N are homotopic. Hence patching with them gives isomorphic M_N-bundles over $\mathbf{T} \times \mathbf{T}$, and their section algebras are isomorphic A's. Thus we have a 1-1 correspondence between the (spectrum-fixing) isomorphism classes of the algebras in (2) and (3): it should be clear that the condition on $U(0)^* U(1)$ is carried into a similar condition on $V^* W^* V W$.

Corollary 2.2 shows that every principal PU_N-bundle E over $\mathbf{T}^2$ is isomorphic to one of the form $(\mathbf{T} \times PU_N) \times_\tau \mathbf{T}$ for some loop $\tau\colon \mathbf{T} \to PU_N$. Lifting this to a path in the simply-connected cover SU_N gives a path $U\colon I \to SU_N$ with $\tau = \operatorname{Ad} U$, and the section algebra of the M_N-bundle corresponding to E is isomorphic to $A(U)$ as in (2). It follows from the theorem of Fell and Tomiyama-Takesaki that every N-homogeneous C^*-algebra is isomorphic to one of the form (2), hence also to ones of the form (1) and (3) by the previous two paragraphs.

To prove that the isomorphism classes are as claimed we use the realisation (3). So let $E(V, W)$ be as in Lemma 2.6; note that our proof of the equivalence of (2) and (3) shows also that the isomorphism class of $E(V, W)$ is determined by the commutator $\{V, W\}$. The homeomorphism $(s, t) \to (s, 1-t)$ converts $E(V, W)$ into $E(V, W^*)$, and a simple calculation using the condition $\operatorname{Ad} VW = \operatorname{Ad} WV$ shows that $\{V, W^*\} = \{W, V\}$, so the alternative in (3) is necessary. Now the isomorphism class of $f^*(E(V, W))$ depends only on the homotopy class of f, so by Corollary 2.5 it will be enough to prove that if $\det A = \pm 1$ and $f_A^* E(V, W) \cong E(V_1, W_1)$ then $\{V_1, W_1\}$ is $\{V, \mathrm{W}\}^{\pm 1}$. However, Lemma 2.6 shows that

$$f_A^* E(V, W) \cong E(V^{a_{11}} W^{a_{21}}, V^{a_{12}} W^{a_{22}}),$$

and simple calculations show that if $VW = zWV$ then

$$\begin{aligned}(V^{a_{11}} W^{a_{21}})(V^{a_{12}} W^{a_{22}}) &= z^{-a_{21}a_{12}} V^{a_{11}} W^{a_{12}} W^{a_{21}} W^{a_{22}} \\ &= z^{-a_{21}a_{12}} z^{a_{11}a_{22}} V^{a_{12}} W^{a_{22}} V^{a_{11}} W^{a_{21}} \\ &= z^{\det A} (V^{a_{12}} W^{a_{22}})(V^{a_{11}} W^{a_{21}}).\end{aligned}$$

Since the isomorphism class of $E(V_1, W_1)$ is determined by the commutator, it follows that

$$\{V_1, W_1\} = \{V^{a_{11}} W^{a_{21}}, V^{a_{12}} W^{a_{22}}\} = z^{\det A} = z^{\pm 1} = \{V, W\}^{\pm 1},$$

as required. The completes the proof of Proposition 2.7.

REMARK. In Corollary 2.2 the principal PU_N-bundles $E \times_\tau \mathbf{T}$ are classified by the class of τ in $\pi_1(PU_N)$; the invariants appearing in (1) and (2) merely describe the class of $\tau = \operatorname{Ad} U$ in two different realisations of $\pi_1(PU_N)$. In (1), we regard $\pi_1(PU_N) = \mathbf{Z}/N\mathbf{Z}$ as the quotient of $\pi_1(U_N) = \mathbf{Z}$ by the kernel of $\operatorname{Ad}_*: \pi_1(U_N) \to \pi_1(PU_N)$. In (2), we view $\pi_1(PU_N)$ as the fibre over the identity in the simply connected cover Ad: $SU_N \to PU_N$, and then the class of τ is determined by the difference between the endpoints of a lift of τ to SU_N.

We now use Proposition 2.7 to study various crossed product C^*-algebras.

Rational rotation algebras.

Let $\theta \in [0, 1)$ and define an action of $\mathbf{Z}$ on $\mathbf{T} = I/\{0, 1\}$ by $n \cdot t + n\theta \pmod 1$. According as θ is rational or not, the transformation group C^*-algebra $A_\theta = C^*(\mathbf{Z}, \mathbf{T})$ is called a rational or irrational rotation C^*-algebra. If θ is rational, the action is smooth and we can compute $\hat{A}_\theta$ using Corollary 1.2.

Let $\theta = p/N$ with $N \in \mathbf{N}$, $p \in \{0, 1, \ldots, N-1\}$ and $(p, N) = 1$. Then the stabilizer of each $z \in \mathbf{T}$ is $N\mathbf{Z}$, a complete set of representatives for the orbits is $[0, 1/N)$, and so the spectrum of A_θ consists of

$$(1) \qquad \{\operatorname{Ind}_{N\mathbf{Z}}^{\mathbf{Z}}(\epsilon_t \times \gamma_s) : t \in [0, 1/N), s \in [0, 1/N)\},$$

where for $s \in [0, 1/N)$, $\gamma_s \in (N\mathbf{Z})\hat{}$ is given by $\gamma_s(Nn) = \exp(2\pi i s N n)$. Note in particular that $\gamma_{1/N} = \gamma_0$. Let H_s denote the Hilbert space

$$H_s = \{\xi : \mathbf{Z} \to \mathbf{C} : \xi(m + Nn) = \gamma_s(Nn)\xi(m)\}$$

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as in Proposition 1.1, let H be the space of functions from $\{0, 1, \ldots, N-1\}$ to $\mathbf{C}$, define unitaries $V_s\colon H_s \to H$ by $(V_s\xi)(m) = \xi(m)$, and write

$$\rho(t, s) = V_s\big(\mathrm{Ind}_{N\mathbf{Z}}^{\mathbf{Z}}(\epsilon_t \times \gamma_s)\big)V_s^*.$$

If we choose q such that $0 \leqslant q < N$ and $qp = 1 \pmod N$, then

$$\rho(1/N, s) = \rho(q \cdot 0, s) = V_s W_q^* V_s^* \rho(0, s) V_s W_q V_s^* \tag{2}$$

where $(W_q\xi)(m) = \xi(m+q)$ as in Proposition 1.1. Direct calculations show that

$$V_s W_q^* V_s^* \eta(m)$$
$$= \exp\{-2\pi i s(m - q - \mathrm{Res}(m-q, \mathrm{mod}\, N))\}\eta(\mathrm{Res}(m-q, \mathrm{mod}\, N)),$$

so that if we identify H and $\mathbf{C}^N$ in the obvious way, $V_s W_q^* V_s^*$ has matrix

$$U(s) = \begin{pmatrix} 0 & e^{2\pi i s}1_q \\ 1_{N-q} & 0 \end{pmatrix}.$$

We define a homomorphism Φ of A_θ into $A(U_1)$ by

$$\Phi(x)(t, s) = \rho(t/n, s/N)(z), \quad U_1(s) = U(s/N) \qquad \text{for } z \in C_c(\mathbf{Z}, C(\mathbf{T}));$$

that $\Phi(z)$ patches correctly on the edges of I^2 follows from (2) and the fact that $\rho(t, 1/N) = \rho(t, 0)$, and the continuity of $\Phi(z)$, from a simple direct calculation. Composing the irreducible representations of $A(U_1)$ with Φ gives the representations (1) of A_θ, so Φ is injective and its range is a rich subalgebra of $A(U_1)$. Thus Φ is an isomorphism onto $A(U_1)$ by [4, 11.1.4].

By Proposition 2.7(1) the isomorphism class of A_θ is therefore determined by $\deg(\det U_1)$. Now $\det U(s) = \pm(\exp(2\pi i s N))^q = \pm\exp(2\pi i s N q)$, which is a map from $\mathbf{T} = [0, 1/N]/\{0, 1/N\}$ has winding number q about 0. However, since $(p, N) = 1$ its inverse $q \bmod N$ is determined uniquely, and since $(-p)$ has inverse $(-q)$, the isomorphism class of A_θ depends only on $\pm p \pmod N$.

This argument also shows that every N-homogeneous algebra $A(U)$ with $(\deg(\det U), N) = 1$ is isomorphic to some A_θ. If $m = (\deg(\det U), N)$ is not 1, let $\theta = \deg(\det U)/N$. According to the theory above, A_θ is N/m-homogeneous, and is isomorphic to $A(U_1)$ where $\deg(\det U_1)$ is $\deg(\det U)/m$. The N-homogeneous C^*-algebra $A(U_1) \otimes M_m$ is isomorphic to $A(U_1 \otimes 1_m)$, and because

$$\deg(\det(U_1 \otimes 1_m)) = m \deg(\det U_1) = \deg(\det U)$$

we have

$$A(U) \cong A(U_1 \otimes 1_m) \cong A(U_1) \otimes M_m(\mathbf{C}) \cong A_\theta \otimes M_m(\mathbf{C}).$$

We have therefore proved:

PROPOSITION 2.8. *Two rational rotation algebras A_θ, A_ϕ are isomorphic if and only if $\theta = \pm\phi \pmod 1$. Every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$ is isomorphic to one of the form $A_\theta \otimes M_m(\mathbf{C})$.*

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The first part of this result has also been proved by Høegh-Krohn and Skjelbred [7] and by Rieffel [16, Section 3].

Twisted group algebras.

Let $\theta \in \mathbf{R}$ and define a multiplier ω_θ on $\mathbf{Z}^2$ by

$$\omega_\theta((m_1, n_1), (m_2, n_2)) = \exp 2\pi i(-m_1 n_2 \theta).$$

According to [1, Theorem 3.2], every multiplier on $\mathbf{Z}^2$ is equivalent to one of this form. It is well-known that the twisted group C^*-algebra $C^*(\mathbf{Z}^2, \omega_\theta)$ is isomorphic to the rotation algebra A_θ. To see this, let $\mathcal{F}$: $C^*(\mathbf{Z}) \to C(\mathbf{T})$ be the Fourier transform given by

$$(\mathcal{F}a)(e^{2\pi it}) = \sum_{n=-\infty}^{\infty} a(n) e^{-2\pi int} \quad \text{for } a \in l^1(\mathbf{Z}).$$

Then $\mathcal{F}$ converts the action of rotation through $2\pi\theta$ on $C(\mathbf{T})$ into the action β: $\mathbf{Z} \to \operatorname{Aut} C^*(\mathbf{Z})$ given by

$$\beta_m(a)(n) = \exp(-2\pi imn\theta)a(n) \quad \text{for } a \in l^1(\mathbf{Z}),\ m, n \in \mathbf{Z},$$

and A_θ is isomorphic to $C^*(C^*(\mathbf{Z}), \mathbf{Z})$. Straightforward calculations show that the multiplications and involutions on $C_c(\mathbf{Z} \times \mathbf{Z})$ regarded as subalgebras of $C^*(C^*(\mathbf{Z}), \mathbf{Z})$ and $C^*(\mathbf{Z}^2, \omega_\theta)$ coincide, so the two enveloping algebras are isomorphic. We could therefore deduce at once from Proposition 2.8 how the isomorphism class of $C^*(\mathbf{Z}^2, \omega_\theta)$ depends on $\theta \in \mathbf{Q}$. However, it is easy enough to do this directly using Proposition 1.3, and the method we use will work in other situations.

Let $\theta = p/N$ with $(p, N) = 1$, and write $\omega = \omega_\theta$. Then $S_\theta = N\mathbf{Z} \times N\mathbf{Z}$; thus if M is a fixed irreducible ω-representation of $\mathbf{Z} \times \mathbf{Z}$ and

$$\gamma_{s,t}(m, n) = \exp 2\pi i(ms + nt)$$

then the spectrum $(G, \omega)\hat{}$ can be identified with

$$\{\gamma_{s,t}M \mid s, t \in [0, 1/N)\}.$$

Note that if $qp = 1 \pmod N$, then

$$\tilde{\omega}((0, q), (m, n)) = \exp 2\pi iqm\theta = \gamma_{1/N,0}(m, n),$$
$$\tilde{\omega}((-q, 0), (m, n)) = \exp 2\pi iqn\theta = \gamma_{0,1/N}(m, n).$$

If follows from Proposition 1.3 that the map Φ of $C_c(\mathbf{Z}^2)$ into $C(I^2, M_N)$ defined by

$$\Phi(f)(s, t) = (\gamma_{s/N,t/N}M)(f)$$

extends to an isomorphism of $C^*(\mathbf{Z}^2, \omega_\theta)$ onto $B(M(0, q), M(-q, 0))$. A simple calculation shows that

$$\begin{aligned} M(0,q)^*M(-q,0)^*M(0,q)M(-q,0) &= \tilde{\omega}((0,q),(-q,0))M(-q,q)^*M(-q,q) \\ &= \exp(2\pi i q^2\theta)1_N \\ &= \exp(2\pi i q/N)1_N, \end{aligned}$$

and we can apply Proposition 2.7(3) to determine the isomorphism class of $C^*(\mathbf{Z}^2, \omega_\theta)$.

PROPOSITION 2.9. *Let θ, ϕ be two rational numbers. Then*

(1) $C^*(\mathbf{Z}^2, \omega_\theta) \cong C^*(\mathbf{Z}^2, \omega_\phi)$ *if and only if* $\theta = \pm\phi \pmod 1$.

(2) *there is an isomorphism of $C^*(\mathbf{Z}^2, \omega_\theta)$ onto $C^*(\mathbf{Z}^2, \omega_\phi)$ which preserves the dual action of $\mathbf{T}^2$ if and only if* $\theta = \phi \pmod 1$.

PROOF. Part (1) follows from the preceding discussion. Suppose that Φ is an isomorphism as in (2); note that by part (1) we can write $\theta = p_1/N$, $\phi = p_2/N$ with $0 \leqslant p_i < N$ and $(p_i, N) = 1$ (and in fact we also know $p_1 = p_2$ or $N - p_2$). Let q_i be the inverses modulo N of p_i, and let M be any irreducible ω_θ-representation of $\mathbf{Z}^2$. Then according to the construction above

$$\Psi(a)(s,t) = \left(\gamma_{s/N,t/N}M\right)(a), \qquad \Theta(a)(s,t) = \left(\gamma_{s/N,t/N}\left(M \circ \Phi^{-1}\right)\right)(a),$$

define isomorphisms Ψ, Θ of $C^*(\mathbf{Z}^2, \omega_\phi)$, $C^*(\mathbf{Z}^2, \omega_\phi)$ onto $B(V_1, V_2)$, $B(W_1, W_2)$ respectively, where the V_i, W_i satisfy

$$\{V_1, V_2\} = \exp(2\pi i q_1/N)1_N, \qquad \{W_1, W_2\} = \exp(2\pi i q_2 N)1_N.$$

Since Φ preserves the dual action, we have

$$\left[\gamma\left(M \circ \Phi^{-1}\right)\right](a) = M \circ \Phi^{-1}\left(\gamma^{-1} \cdot a\right) = M\left(\gamma^{-1}\left(\Phi^{-1}(a)\right)\right) = \gamma M\left(\Phi^{-1}(a)\right),$$

for any $\gamma \in (\mathbf{Z}^2)\hat{}$, $a \in C^*(\mathbf{Z}^2, \omega_\phi)$. It follows that the isomorphism $\Theta \circ \Phi \circ \Psi^{-1}$ of $B(V_1, V_2)$ onto $B(W_1, W_2)$ induces the identity map on spectra, so that the underlying bundles $E(V_1, V_2)$ and $E(W_1, W_2)$ are isomorphic. But this implies that $\{V_1, V_2\} = \{W_1, W_2\}$ (see the proof of Proposition 2.7), so that $q_1 = q_2 \pmod N$ and $p_1 = p_2$. Thus $\theta = \phi$ and the proposition is proved.

There are other multipliers on abelian groups whose twisted group algebras are homogeneous with spectrum $\mathbf{T}^2$. For example, let $L \in \mathbf{N}$, $\theta \in \mathbf{Q}$, $G = \mathbf{Z}/L\mathbf{Z} \times \mathbf{Z}^2$ and define

$$\omega(([k], m, n), ([k_1], m_1, n_1)) = \exp 2\pi i(-mn_1\theta + km_1/L).$$

A little work shows that if $\theta = p/M$ and $(p, M) = 1$, then

$$S_\omega = \{([-jp], k[L, M], jM/(L, M)) \colon k, j \in \mathbf{Z}\},$$

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which has index $[L, M]^2$, so that $C^*(G, \omega)$ is $[L, M]$-homogeneous by Proposition 1.3 and [3, Lemma 3.1]. For $s, t \in I$ we set

$$\gamma_{s,t}([-jp], k[L, M], jM/(L, M)) = \exp 2\pi i(sk[L, M] + tjM/(L, M)),$$

and we identify $\hat{S}_\omega$ with $[0, 1/[L, M]) \times [0, (L, M)/M)$. If we pick l, m, n so that

$$l/L + m\theta = 0, \quad -m\theta = (L, M)/M, \quad -m/L = 0 \qquad (\text{mod } 1),$$

then

$$\tilde{\omega}((l, 0, n), \cdot) = \gamma_{1/[L,M],0} \quad \text{and} \quad \tilde{\omega}((0, m, 0), \cdot) = \gamma_{0,(L,M)/M}.$$

An argument just like the one we used for $\mathbf{Z}^2$ shows that $C^*(G, \omega)$ is isomorphic to $B(V, W)$ where

$$V^*W^*VW = \tilde{\omega}((l, 0, n), (0, m, 0)) = \exp 2\pi i(-n(L, M)/M).$$

It is interesting to note that the invariant $\{V, W\}$, when put in the form $\exp 2\pi iq/[L, M]$, does not satisfy $(q, [L, M]) = 1$, as is always the case for the algebras $C^*(\mathbf{Z}^2, \omega_\theta)$ considered above. In particular, if $M = (L, M)$ we have $C^*(G, \omega) \cong C(\mathbf{T}^2, M_L)$. It would be interesting to know if every root of unity was attained by some twisted group algebra (although we can do this other ways, as we show later).

Note added in proof. Alex Kumjian has pointed out to us that this is a consequence of Proposition 2.8. For $M_m(\mathbf{C})$ is a twisted group algebra of $\mathbf{Z}_m \times \mathbf{Z}_m$, and the tensor product of two twisted group algebras is another twisted group algebra.

Transformation group C^*-algebras of covering spaces.

In general, if $p: E \to X$ is a finite covering, then the group G of deck transformations acts freely on E, and in fact E is a principal G-bundle. The transformation group C^*-algebra $C^*(G, E)$ will then be a $|G|$-homogeneous C^*-algebra with spectrum X. (This follows easily from the version of Corollary 1.2 for non-abelian groups [5], [17].) In particular, N-sheeted coverings of $\mathbf{T}^2$ will give N-homogeneous C^*-algebras over $\mathbf{T}^2$.

The covering spaces of $\mathbf{T}^2$ are in one-to-one correspondence with the subgroups of $\pi_1(\mathbf{T}^2) \cong \mathbf{Z}^2$, and the appropriate quotient of $\pi_1(\mathbf{T}^2)$ acts as deck transformations. In particular, every finitely sheeted covering of $\mathbf{T}^2$ carries an action of a finite quotient of $\mathbf{Z}^2$. Now it is standard group theory that every subgroup H of $\mathbf{Z}^2$ of finite index has the form $\mathbf{Z}(m_1, n_1) + \mathbf{Z}(m_2, n_2)$ for some pair $(m_i, n_i) \in \mathbf{Z}^2$ with $m_1 n_2 - n_1 m_2 \neq 0$, and that the index $|\mathbf{Z}^2/H|$ is precisely $\Delta = |m_1 n_2 - n_1 m_2|$. It is routine to check that the corresponding covering of $\mathbf{T}^2$ is given by p: $\mathbf{T}^2 \to \mathbf{T}^2$ where

$$p(z, w) = (z^{m_1} w^{m_2}, z^{n_1} w^{n_2}),$$

and the action of $\pi_1(\mathbf{T}^2)/H = \mathbf{Z}^2/H$ on $\mathbf{T}^2$ is given by

$$((m,n)+H)\cdot(z,w)$$
$$= \left(z\exp\{2\pi i(mn_2 - nm_2)/\Delta\}, w\exp\{2\pi i(m_1 n - n_1 m)/\Delta\}\right).$$

The corresponding transformation group C^*-algebra is Δ-homogeneous. We analyse it using Corollary 1.2.

We can always find m, n such that $mn_2 - nm_2 = (m_2, n_2)$, and this is obviously the smallest positive integer we can get this way, so a complete set of orbit representatives is

$$\exp 2\pi i[0,(m_2,n_2)/\Delta) \times \exp 2\pi i[0,(m_1,n_1)/\Delta).$$

The action of $G = \mathbf{Z}^2/H$ is free, and hence Corollary 1.2 implies that a complete set of irreducible representations of $C^*(G, \mathbf{T}^2)$ is

$$\left\{\operatorname{Ind} G_{\{e\}}\epsilon_{s,t}: s\in[0,(m_2,n_2)/\Delta), t\in[0,(m_1,n_1)/\Delta)\right\},$$

where $\epsilon_{s,t}$ is evaluation at $(\exp 2\pi is, \exp 2\pi it) \in \mathbf{T}^2$. Note that these representations all act in the same space $L^2(G)$. If we fix $(m,n), (m',n')$ satisfying

$$mn_2 - nm_2 = (m_2,n_2), \qquad m'n_1 - n'm_1 = (m_1,n_1),$$

then $W_1 = W_{(m,n)+H}$, $W_2 = W_{(m',n')+H}$ respectively intertwine

$$\operatorname{Ind}\epsilon_{0,t} \text{ and } \operatorname{Ind}\epsilon_{(m_2,n_2)/\Delta,t}, \qquad \operatorname{Ind}\epsilon_{s,0} \text{ and } \operatorname{Ind}\epsilon_{s,(m_1,n_1)/\Delta}.$$

Thus the map Ψ: $C^*(G,\mathbf{T}^2) \to C(I^2, M_\Delta)$ defined on $C_c(G, C(\mathbf{T}^2))$ by

$$\Psi(z)(s,t) = \operatorname{Ind}_{\{e\}}^{G}\epsilon_{s(m_2,n_2)/\Delta,t(m_1,n_1)/\Delta}(z)$$

gives an isomorphism of $C^*(G,\mathbf{T}^2)$ onto $B(W_1, W_2)$. However, it is easy to see that the W's commute, so $C^*(G,\mathbf{T}^2)$is isomorphic to $C(\mathbf{T}^2, M_\Delta)$ by Proposition 2.7(3).

We observe that this last result does not appear to be predictable on general grounds—there are finite covering spaces p: $E \to X$ whose associated transformation group C^*-algebra $C^*(G, E)$ is not trivial. For example, if p: $S^2 \to \mathbf{R}P^2$ is the canonical 2-sheeted covering of real projective space, then $C^*(\mathbf{Z}_2, S^2)$ is not isomorphic to $C(\mathbf{R}P^2, M_2)$, as is pointed out in [15, page 301]. In fact, another application of Corollary 1.2 shows that

$$C^*(\mathbf{Z}_2, S^2) \cong \left\{f\in C(S^2, M_2) \middle| f(x) = Wf(-x)W^* \text{ for } x\in S^2\right\},$$
$$\text{where } W = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

An isomorphism of the corresponding M_2-bundle with $\mathbf{R}P^2 \times M_2$ would give a continuous map Ψ: $S^2 \to PU_2$ satisfying $\psi(x) = \operatorname{Ad} W \circ \psi(-x)$ for all x. The restriction of ψ to the equator is a loop which is homotopically trivial and hence has the form $\operatorname{Ad} V(x)$ for some loop V: $S^1 \to SU_2$. The condition on Ψ implies

$V(x) = \lambda(x)WV(-x)$ for some map $\lambda: S^1 \to S^1$, and taking determinants shows λ is constant with $\lambda^2 = -1$. But this is impossible, since then

$$V(1) = \lambda WV(-1) = \lambda W(\lambda WV(1)) = \lambda^2 V(1).$$

Thus no such isomorphism can exist and $C^*(\mathbf{Z}_2, S^2)$ is not trivial as claimed. The same argument works for the canonical coverings of the other projective spaces $\mathbf{R}P^n$ $(n > 2)$.

It is possible to build non-trivial algebras from coverings of $\mathbf{T}^2$ by adding a multiplier σ of the group G of deck transformations, and forming the twisted crossed product $C^*(G, \mathbf{T}^2, \sigma)$ as in [20, Section 2.4]. We could calculate this C^*-algebra using [20], but we can also reduce to the ordinary theory of crossed products. In fact, if $V: G \to M_N$ is a $\bar{\sigma}$-representation and $\alpha: G \to \operatorname{Aut} A$, then the map

$$\Phi: C(G, A) \otimes M_n \to C(G, A \otimes M_N)$$

defined by

$$\Phi(z \otimes T)(s) = z(s) \otimes TV(s)^*$$

extends to an isomorphism of $C^*(A, G, \sigma) \otimes M_N$ onto the crossed product $C^*(A \otimes M_N, G)$ for the action $\alpha \otimes \operatorname{Ad} V$. If (m_i, n_i), H, ρ, Δ are as before, then Proposition 1.1 shows that a complete set of irreducible representations for $C^*(C(\mathbf{T}^2, M_N), G)$ is

$$\left\{\operatorname{Ind}_{\{e\}}^G \epsilon_{s,t} : s \in \left[0, (m_2, n_2)/\Delta\right), t \in \left[0, (m_1, n_1)/\Delta\right)\right\}.$$

We still have W_u intertwining $\operatorname{Ind} \epsilon_x$ and $\operatorname{Ind}(u \cdot \epsilon_x)$, but now $u \cdot \epsilon_x$ is $\operatorname{Ad} V(u)^* \cdot \epsilon_{u \cdot x}$, so we have

$$(1 \otimes V(u))W_u[\operatorname{Ind} \epsilon_x]W_u^*(1 \otimes V(u))^* = \operatorname{Ind} \epsilon_{u \cdot x}.$$

Thus if $(m, n), (m', n')$ are as before and $u_1 = (m, n) + H$, $u_2 = (m', n') + H$ then $C^*(G, \Pi^2, \sigma) \otimes M_N$ is isomorphic to $B(Y_1, Y_2)$, where $Y_i = (1 \otimes V(u_i))W_{u_i}$. However, since V is a $\bar{\sigma}$-representation we have

$$\{Y_1, Y_2\} = \tilde{\sigma}\left((m, n) + H, (m', n') + H\right)^{-} 1_{|G|N},$$

and it is easy to construct examples where this is not 1.

Pull-backs.

Another way to construct homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ is to pull back a given algebra along a covering $p: \mathbf{T}^2 \to \mathbf{T}^2$. In general, if $F: X \to Y$ is a continuous map and $A = \Gamma(E)$ is a homogeneous C^*-algebra with spectrum Y, then the pull-back f^*A is by definition the C^*-algebra $\Gamma(f^*E)$ with spectrum X. Equivalently, f^*A is the C^*-algebraic tensor product $C(X) \otimes_{C(Y)} A$, where $C(Y)$ acts naturally on A and on $C(X)$ via the map f (see [14, Proposition 1.3]).

Let $A = B(V, W)$ have spectrum $\mathbf{T}^2$, and let $p\colon \mathbf{T}^2 \to \mathbf{T}^2$ be the finite covering given by $p(z, w) = (z^{m_1}w^{m_2}, z^{n_1}w^{n_2})$. Then according to Lemma 2.6

$$p^*A = p^*B(V, W) \cong B(V^{m_1}W^{n_1}, V^{m_2}W^{n_2}).$$

As in the proof of Proposition 2.7, if $\{V, W\} = z1_N$, then

$$\{V^{m_1}W^{n_1}, V^{m_2}W^{n_2}\} = z^{m_1n_2 - n_1m_2}1_N.$$

Recall that $\Delta = |m_1n_2 - n_1m_2|$ is the number of sheets in the covering, so that if $z = \exp(2\pi iq/N)$ with $(q, N) = 1$, then we can obtain all the N-homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ by pulling back A along different coverings. In particular, any N-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$ has the form p^*A_θ, or $p^*C^*(\mathbf{Z}^2, \omega_\theta)$, for some $\theta \in \mathbf{Q}$ and some covering map $p\colon \mathbf{T}^2 \to \mathbf{T}^2$.

Crossed products of homogeneous C^*-algebras.

The various homogeneous C^*-algebras we have constructed often carry natural group actions, and taking crossed products by these actions can give more homogeneous C^*-algebras. For example, the dual group $\hat{G}$ always acts on the twisted group algebra $C^*(G, \omega)$ of an abelian group according to the formula

$$(\gamma a)(s) = \gamma(s)a(s) \qquad \big(a \in C_c(G), \gamma \in \hat{G}, s \in G\big).$$

A simple computation shows that if ω is type I and we realise $(G, \omega)\hat{}$ as in Proposition 1.3, then the corresponding action of $\hat{G}$ on $(G, \omega)\hat{}$ is given by $\gamma \cdot (\chi M) = \gamma\chi M$, so that characters in $S_\omega^\perp$ act trivially and others act freely. In particular, let $(G, \omega) = (\mathbf{Z}^2, \omega_\theta)$ for some $\theta \in \mathbf{Q}$, and write $\theta = p/N$ with $(p, N) = 1$, so that $S_\omega = N\mathbf{Z} \times N\mathbf{Z}$. If H is a finite subgroup of $\mathbf{T}^2$ such that $H \cap S_\omega^\perp = \phi$, then H acts freely on $(G, \omega)\hat{} = \mathbf{T}^2$, and the orbit space $(G, \omega)\hat{}/H$ is another torus. By Proposition 1.1 the irreducible representations of $C^*(C^*(\mathbf{Z}^2, \omega_\theta), H)$ are all induced from irreducible representations of $C^*(\mathbf{Z}^2, \omega_\theta)$, and $C^*(C^*(\mathbf{Z}^2, \omega_\theta), H)\hat{}$ can be identified with the orbit space $(G, \omega)\hat{}/H$. (Actually, Proposition 1.1 just gives us a setwise identification, but it is not hard to see it must be a homeomorphism.) Thus, $C^*(C^*(\mathbf{Z}^2, \omega_\theta), H)$ is an $N|H|$-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$.

There are also natural actions on the pull-backs along covering maps $p\colon E \to X$. For the group G of deck transformations acts freely on E, and dualising gives an automorphism group $\delta\colon G \to \operatorname{Aut} C(E)$. If A is N-homogeneous with spectrum X, tensoring with the identity gives an automorphism group $\delta \otimes \mathrm{id}$ of $C(E) \otimes A$, and this in turns defines an automorphism group $p^*\mathrm{id}$ acting on the quotient $p^*A = C(E) \otimes_{C(X)} A$. Since this action is free on the spectrum $(p^*A)\hat{} = E$, Takesaki's results (and a little bit of work) show that $C^*(p^*A, G, p^*\mathrm{id})$ is $N|G|$-homogeneous with spectrum X. More generally, if $\beta\colon G \to \operatorname{Aut} A$ is any group of automorphisms which preserve the $C(X)$-action on A, then $\delta \otimes \beta$

defines a group $p^*\beta$: $G \to \operatorname{Aut} p^*A$, and $C^*(p^*A, G)$ has similar properties: crossed products like these are studied in [14]. Specializing to a covering p: $\mathbf{T}^2 \to \mathbf{T}^2$ will give more homogeneous algebras with spectrum $\mathbf{T}^2$, and it should be possible to calculate the isomorphism class in any given situation using Propositions 1.1 and 2.7. In fact, we have done this above in the case of $p^*(\operatorname{Ad} V)$, where p: $\mathbf{T}^2 \to \mathbf{T}^2$ is a finite cover, $A = p^*A = C(\mathbf{T}^2, M_N)$ and V: $G \to U_N$ is a $\bar{\sigma}$-representation of the group G of deck transformations.

Of course, all the constructions in the preceding paragraphs give algebras which carry natural group actions, and we could continue this process indefinitely.

3. Homogeneous C^*-algebras with spectrum $\mathbf{T}^3$

We tackle the classification of PU_N-bundles over $\mathbf{T}^3$ much as we did over $\mathbf{T}^2$. We first describe them up the usual notion of bundle isomorphism, by applying Lemma 2.1 to $\mathbf{T}^3 = \mathbf{T}^2 \times \mathbf{T}$; this is harder now because we need to know about automorphisms of non-trivial PU_N-bundles over $\mathbf{T}^2$. The second part—namely, the effect of homeomorphisms of $\mathbf{T}^3$—proceeds in exactly the same way. As before, we begin with some results on more general principal G-bundles, and specialise to PU_N later. For the next three lemmas, G is a path-connected topological group such that $\pi_2(G) = 0$.

LEMMA 3.1. *Let F be a principal G-bundle over $\mathbf{T}^2$, and suppose that ρ, σ are two automorphisms of F. Then $\rho \circ \sigma$ is homotopic to $\sigma \circ \rho$.*

PROOF. By Corollary 2.2 we can assume that $F = E_\tau$ for some τ: $\mathbf{T} \to G$, and, since G is path-connected, that $\tau(1) = e$. Then the automorphisms ρ, σ are defined by continuous maps ϕ, ψ: $I^2 \to G$ such that

$$(1) \qquad \begin{aligned} \phi(s,0) &= \phi(s,1) && \text{for all } s \in I, \\ \phi(1,t) &= \tau(t)^{-1}\phi(0,t)\tau(t) && \text{for all } t \in I, \end{aligned}$$

and similarly for ψ. The assumption $\tau(0) = \tau(1) = e$ implies that ϕ maps all the corners of I^2 to the same element g_1 of G: we first show we may take $g_1 = e$. Fix a path $g(u)$ joining g_1 to e, and define a function f: $I^3 \to G$ as follows:

$$f(s,t,0) = \phi(s,t),$$
$$f(0,1,u) = f(1,1,u) = f(1,0,u) = f(0,0,u) = g(u);$$

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define f on the faces $t = 0$, $s = 0$ to be arbitrary continuous extensions of f from the 3 edges of those faces, and finally let

$$(2) \qquad \begin{aligned} f(s,1,u) &= f(s,0,u), \\ f(1,t,u) &= \tau(t)^{-1} f(0,t,u)\tau(t). \end{aligned}$$

Now extend f arbitrarily to all of I^3: it gives a homotopy joining ϕ to an automorphism $f|_{u=1}$ which maps the corners of I^2 to e. We may therefore suppose ϕ, ψ map the corners of I^2 to e.

We now define another function $f\colon \partial I^3 \to G$ by

$$f(s,t,0) = \phi(s,t)\psi(s,t), \quad f(s,t,1) = \psi(s,t)\phi(s,t) \qquad \text{for } s, t \in I,$$
$$f(s,t,u) = e \quad \text{for } s, t \in \{0,1\} \text{ and } u \in I;$$

the loops $s \to \phi(s,0)\psi(s,0)$ and $s \to \psi(s,0)\phi(s,0)$ are homotopic since $\pi_1(G)$ is abelian, so we can extend f to the face $t = 0$, and by a similar argument to the face $t = 1$; finally, we extend f to the remaining faces using (2). Since $\pi_2(G) = 0$, we can extend f to all of I^3, and this gives us a homotopy from $\phi\psi$ to $\psi\phi$ through maps of I^2 into G satisfying (1). It follows that the automorphisms $\rho \circ \sigma$ and $\sigma \circ \rho$ are homotopic.

COROLLARY 3.2. *Every principal G-bundle over* $\mathbf{T}^3$ *is isomorphic to one of the form* $E_\tau \times_\sigma \mathbf{T}$ *(see Corollary 2.2) for some* $\tau\colon \mathbf{T} \to G$ *and some automorphism* σ *of* E_τ. *Such an automorphism is homotopic to one given by a continuous map* $\phi\colon I^2 \to G$ *which maps the corners of* I^2 *to e and satisfies* (1). *The isomorphism class of* $E_\tau \times_\sigma \mathbf{T}$ *is then determined by the classes of the three loops* τ, $\phi|_{s=0}$, $\phi|_{t=0}$ *in* $\pi_1(G)$.

PROOF. Lemma 2.1 shows that each principal G-bundle over $\mathbf{T}^3$ is isomorphic to $F \times_\sigma \mathbf{T}$ for some G-bundle F over $\mathbf{T}^2$ and automorphism σ of F, and Corollary 2.2 shows that F has the form $E_\tau = (\mathbf{T} \times G) \times_\tau \mathbf{T}$ for some $\tau\colon \mathbf{T} \to G$. Every automorphism of E_τ comes from a map $\phi\colon I^2 \to G$ satisfying (1), and the first paragraph of the proof of Lemma 3.1 shows that we may suppose ϕ maps the corners to e. Lemma 2.1 and Lemma 3.1 together show that the isomorphism class of $F \times_\sigma \mathbf{T}$ depends only on the isomorphism class of F and on the homotopy class of σ. However, Corollary 2.2 also shows that E_τ is determined up to isomorphism by $[\tau] \in \pi_1(G)$, and according to Lemma 2.3 the class of σ is determined by $[\phi|_{s=0}]$ and $[\phi|_{t=0}] \in \pi_1(G)$.

LEMMA 3.3. *Let* $p\colon \tilde{G} \to G$ *be the simply-connected covering of G. For each commuting triple* $\{g_i\colon i = 1,2,3\}$ *in G let* $E\{g_i\}$ *be the G-bundle over* $\mathbf{T}^3$ *obtained from* $I^3 \times G$ *by identifying* $(1,t,u,g)$ *with* $(0,t,u,g_1 g)$, $(s,1,u,g)$ *with* $(s,0,u,g_2 g)$ *and* $(s,t,1,g)$ *with* $(s,t,0,g_3 g)$.

(1) *Let* $\{\tilde{g}_i\}$, $\{h_i\}$ *be two commuting triples in* G, *let* $\{\tilde{g}_i\}$, $\{\tilde{h}_i\}$ *be liftings of these triples in* $\tilde{G}$, *and let* $c_{ij} = \{\tilde{g}_i, \tilde{g}_j\}$, $d_{ij} = \{\tilde{h}_i, \tilde{h}_j\}$. *Then* $E\{g_i\} \cong E\{h_i\}$ *if and only if* $c_{ij} = d_{ij}$ *for all* i, j.

(2) *If every triple in the kernel of* p *can be realised as the commutators of a triple* $\{\tilde{g}_i\}$ *in* $\tilde{G}$, *then every principal* G-*bundle over* $\mathbf{T}^3$ *is isomorphic to some* $E\{g_i\}$.

PROOF. Let $\{g_i\}$ be as given, and let α, β, γ be paths in G joining e to g_1, g_2, g_3 respectively. Let τ: $I \to G$ be the loop defined by

$$\tau(t) = \beta(t)g_1\beta(t)^{-1}g_1^{-1}.$$

If $\theta(s, t) = \alpha(s)\beta(t)$ then

$$\theta(0, t)g_1 = \tau(t)\theta(1, t), \qquad \theta(s, 0)g_2 = \theta(s, 1)$$

and it follows that θ defines an isomorphism from $E(g_1, g_2)$ to E_τ. We define a function ϕ: $I^3 \to G$ by

$$\phi(s, t, u) = \gamma(u)\alpha(s)\beta(t) \quad \text{if } s = 0, t = 0 \text{ or } u = 0,$$

$$\phi(s, 1, u) = \phi(s, 0, u)g_2, \tag{3}$$

$$\phi(1, t, u) = \tau(t)^{-1}\phi(0, t, u)g_1, \tag{4}$$

and extending arbitrarily to the rest of I^3. Then simple calculations show that

$$\psi(s, t) = \phi(s, t, 0)g_3\phi(s, t, 1)^{-1} \tag{5}$$

defines an automorphism σ of E_τ, and then (3–5) show that ϕ is an isomorphism of $E\{g_i\}$ onto $E_\tau \times_\sigma \mathbf{T}$. Corollary 3.2 therefore implies that the isomorphism class of $E\{g_i\}$ depends on the classes of τ, $\psi(\cdot, 0)$, $\psi(0, \cdot)$ in $\pi_1(G)$. Note that

$$\psi(s, 0) = \alpha(s)g_3\alpha(s)^{-1}g_3^{-1}, \qquad \psi(0, t) = \beta(t)g_3\beta(t)^{-1}g_3^{-1}.$$

Thus if $\tilde{\alpha}$ and $\tilde{\beta}$ are the based lifts of α and β to $\tilde{G}$, $\tilde{g}_1 = \tilde{\alpha}(1)$, $\tilde{g}_2 = \tilde{\beta}(1)$ and $p(\tilde{g}_3) = g_3$, then $\{\tilde{\beta}(t), \tilde{g}_1\}$, $\{\tilde{\alpha}(s), \tilde{g}_3\}$, $\{\tilde{\beta}(t), \tilde{g}_3\}$ are the based lifts of τ, $\psi(\cdot, 0)$, $\psi(0, \cdot)$. The class of a loop in $\pi_1(G)$ is determined by the end-point of its based lift to $\tilde{G}$, so the isomorphism class of $E\{g_i\}$ depends only on the commutators $\{\tilde{g}_i, \tilde{g}_j\}$ as claimed. (Observe that, since $p^{-1}(e)$ is a discrete normal subgroup of $\tilde{G}$ and hence contained in the centre of $\tilde{G}$, the commutators are independent of the choice of liftings for g_i, and of the paths α, β.) This proves (1).

By Corollary 3.2 every G-bundle is isomorphic to one of the form $E_\tau \times_\sigma \mathbf{T}$. Let ϕ: $I^2 \to G$ define σ as in 3.2, and let c_1, c_2, c_3 be the endpoints of the based liftings of τ, $\phi|_{t=0}$, $\phi|_{s=0}$ to $\tilde{G}$. According to our hypothesis on G, we can pick g_i such that their lifts to $\tilde{G}$ satisfy

$$c_1 = \{\tilde{g}_2, \tilde{g}_1\}, \qquad c_2 = \{\tilde{g}_1, \tilde{g}_3\}, \qquad c_3 = \{\tilde{g}_2, \tilde{g}_3\}.$$

The construction of the previous paragraph and the realisation of $\pi_1(G)$ as the end-points of liftings shows that $E\{g_i\}$ is isomorphic to $E_\tau \times_\sigma \mathbf{T}$. This completes the proof of Lemma 3.3.

In order to apply Lemma 3.3 to the group $G = PU_N$ and its simply-connected cover SU_N, we need to check that PU_N satisfies the hypothesis in (2).

LEMMA 3.4. *Let ω be a primitive Nth root of unity. Then for any $p, q, r \in \mathbf{Z}$ there are $U, V, W \in SU_N$ such that*

$$\{U, V\} = \omega^p 1_N, \qquad \{U, W\} = \omega^q 1_N, \qquad \{V, W\} = \omega^r 1_N.$$

PROOF. We first handle the special case where N is a power of a prime. If $d = (N, p, q, r)$ and $U_1, V_1, W_1 \in SU_{N/d}$ have commutators $\omega^p 1_{N/d}$, etc., then tensoring with 1_d gives a solution to our problem. So we may assume one of the p, q, r—say p—is coprime to N. Let U be the unitary diagonal matrix with entries $1, \omega, \omega^2, \dots, \omega^{N-1}$. Then for any matrix $T = (t_{ij})$, $UTU^* = (\omega^{i-j} t_{ij})$, so

$$\{U, T\} = \omega^s 1_N \Leftrightarrow t_{ij} = 0 \quad \text{for } i - j \neq s \ (\mathrm{mod}\ N).$$

We define V to be the matrix with entries

$$v_{ij} = \begin{cases} 1 & \text{if } i - j = 0 \ (\mathrm{mod}\ N), \\ 0 & \text{otherwise.} \end{cases}$$

Then V is unitary and $\{U, V\} = \omega^p 1_N$. As $(p, N) = 1$ the map $n \to -pn$ is an automorphism of $\mathbf{Z}/N\mathbf{Z}$: let σ denote its inverse, and let W be the matrix with entries

$$w_{ij} = \begin{cases} \omega^{\sigma(i)r} & \text{if } i - j = q \ (\mathrm{mod}\ N), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{V, W\} = \omega^q 1_N$, and $\{V, W\}$ is the diagonal matrix with ith entry

$$\omega^{(\sigma(i-p) - \sigma(i))r} = \omega^r,$$

so U, V, W are unitary matrices with the right commutators. Dividing by scalars allows us to replace them with unitaries of determinant 1.

The general case can now be proved by induction on the number of distinct prime factors of N. For if $N = N_1 N_2$ with $(N_1, N_2) = 1$, then there are $p_i, q_i, r_i \in \mathbf{Z}$ such that

$$N_1 p_1 + N_2 p_2 = p, \qquad N_1 q_1 + N_2 q_2 = q, \qquad N_1 r_1 + N_2 r_2 = r.$$

By the inductive hypothesis we can find U_i, V_i, W_i such that

$$\{U_i, V_i\} = \omega^{p_i N_i} 1_{N_i}, \qquad \{U_i, W_i\} = \omega^{q_i N_i} 1_{N_i}, \qquad \{V_i, W_i\} = \omega^{r_i N_i} 1_{N_i};$$

then $U = U_1 \otimes U_2$, $V = V_1 \otimes V_2$, $W = W_1 \otimes W_2$ are solutions to our problem. This proves the lemma.

We can now describe the principal PU_N-bundles over $\mathbf{T}^3$, and have to consider the effect of homeomorphisms on them. As in the $\mathbf{T}^2$ case, these are effectively all induced from integer matrices acting on $\mathbf{R}^3$, and as the proof is similar we just give the main steps. This part of our argument will work for any torus $\mathbf{T}^k$.

LEMMA 3.5. *Let G be any topological group such that $\pi_2(G) = \pi_3(G) = 0$, and let ϕ, ψ: $\mathbf{T}^3 \to G$ satisfy $\phi(1,1,1) = \psi(1,1,1) = e$. Then ϕ, ψ are homotopic relative to the base point $(1,1,1)$ if and only if the restrictions of ϕ, ψ to the circles $\mathbf{T} \times \{(1,1)\}$, $\{1\} \times \mathbf{T} \times \{1\}$, $\{(1,1)\} \times \mathbf{T}$ are pairwise homotopic relative to $(1,1,1)$.*

PROOF. Suppose the restrictions of ϕ, ψ are homotopic as claimed. Three applications of Lemma 2.3 (with $\tau \equiv e$) show that there are homotopies g_t, h_t, k_t: $\mathbf{T}^2 \to G$ such that

$$g_0 = \phi|_{\mathbf{T}^2\times\{1\}}, \quad g_1 = \psi|_{\mathbf{T}^2\times\{1\}}, \quad h_0 = \phi|_{\mathbf{T}\times\{1\}\times\mathbf{T}}, \quad h_1 = \psi|_{\mathbf{T}\times\{1\}\times\mathbf{T}},$$
$$k_0 = \phi|_{\{1\}\times\mathbf{T}^2}, \quad k_1 = \psi|_{\{1\}\times\mathbf{T}^2}, \quad g_t(1,1) = h_t(1,1) = k_t(1,1) = e.$$

We can regard ϕ, ψ as functions on I^3, g_t, h_t, k_t as functions on $I \times I^2$ and use g, h, k to define a continuous function l on ∂I^4 such that $l|_{\{0\}\times I^3} = \phi$, $l|_{\{1\}\times I^3} = \psi$. Then the vanishing of $\pi_3(G)$ allows us to extend l to all of I^4, and this gives a homotopy joining ϕ to ψ. The converse is clear, so this establishes the lemma.

LEMMA 3.6. *Every homeomorphism of $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ is homotopic to one of the form*

$$f_T(x + \mathbf{Z}^3) = (Tx) + \mathbf{Z}^3 \qquad (x \in \mathbf{R}^3),$$

for some matrix $T \in M_3(\mathbf{Z})$ of determinant ± 1.

PROOF. As in Lemma 2.4, regarding the loops $\mathbf{T} \times \{(1,1)\}$ etc. as generators gives an isomorphism ϕ: $\mathbf{Z}^3 \to \pi_1(\mathbf{T}^3)$, and it follows from Lemma 3.5 that two homeomorphisms f_i of $\mathbf{T}^3$ are homotopic if and only if the automorphisms $\psi(f_i) = \phi^{-1} \circ (f_i)_* \circ \phi$ of $\mathbf{Z}^3$ are equal. It is easy to check that $\psi(f_T) = T$, and every automorphism of $\mathbf{Z}^3$ is given by a matrix $T \in M_3(\mathbf{Z})$ of determinant ± 1, so the result follows.

PROPOSITION 3.7. *Let $\{V_i: i = 1,2,3\} \subset U_N(\mathbf{C})$ be such that $\{\operatorname{Ad} V_i\}$ is a commuting triple in PU_N, and let $B\{V_i\} =$*

$$\left\{ b \in C(I^3, M_N(\mathbf{C})) \,\middle|\, \begin{matrix} b(1,t,u) = \operatorname{Ad} V_1 b(0,t,u),\ b(s,1,u) = \operatorname{Ad} V_2 b(s,0,u) \\ \text{and } b(s,t,1) = \operatorname{Ad} V_3 b(s,t,0) \text{ for all } s,t,u \in I \end{matrix} \right\}.$$

Every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^3$ is isomorphic to one of this form. Let $\{W_i\}$ be another triple, and set

$$c_1 = \{V_2, V_3\}, \quad c_2 = \{V_3, V_1\}, \quad c_3 = \{V_1, V_2\},$$
$$d_1 = \{W_2, W_3\}, \quad d_2 = \{W_3, W_1\}, \quad d_3 = \{W_1, W_2\}.$$

Then $B\{V_i\} \cong B\{W_i\}$ if and only if there is a matrix $T = (t_{ij}) \in M_3(\mathbf{Z})$ with $\det T = \pm 1$ and $c_k = \prod_{i=1}^3 d_i^{t_{ki}}$.

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PROOF. Let $F\{V_i\}$ be the M_N-bundle over $\mathbf{T}^3$ obtained from $I^3 \times M_N$ by pasting along the edges I^3 using the Ad V_i, so that $B\{V_i\} = \Gamma(F\{V_i\})$; part (2) of Lemma 3.3 and Lemma 3.4 imply that every N-homogeneous C^*-algebra with spectrum $\mathbf{T}^3$ is isomorphic to some $B\{V_i\}$. The isomorphism class of f^*F depends only on the homotopy class of f, and hence Lemma 3.6 shows that $B\{V_i\} \cong B\{W_i\}$ if and only there is an invertible matrix $S \in M_3(\mathbf{Z})$ with $\det S = \pm 1$ and $f_S^* F\{V_i\} \cong F\{W_i\}$. The obvious extension of Lemma 2.6 says that

$$f_S^* F\{V_i\} \cong F\{V_1^{s_{1i}}, V_2^{s_{2i}}, V_3^{s_{3i}}\} = F\{W_i\}, \quad \text{say.}$$

Then some messy but straightforward calculations show that if $V_iV_j = z_{ij}V_jV_i$, we have

$$W_iW_j = \left(\prod_{k,l=1}^{3} z_{kl}^{s_{ki}s_{lj}}\right) W_jW_i.$$

If we set $T = S^{-1}$, so that t_{ij} is $(\pm)$ the cofactor of s_{ji}, then this implies that

$$d_k = \left(\prod_{i=1}^{3} c_i^{tdk_i}\right)^{\pm 1}.$$

This proves the necessity of the condition. On the other hand, we can reverse this argument to deduce that if the commutators are related like this, then there is a homeomorphism f_S carrying $B\{V_i\}$ into $B\{W_i\}$, and the proposition is proved.

REMARK. The invariants $\{V_i, V_j\}$ represent three elements of $\pi_1(PU_N)$, realised as the fibre over the identity in the simply-connected covering Ad: $SU_N \to PU_N$. One copy of π_1 describes the class of a bundle E over $\mathbf{T}^2$, and the other two determine the class of an automorphism σ of E which is used to construct a bundle $E \times_\sigma \mathbf{T}$ over $\mathbf{T}^2$ (see Corollary 3.2). As in Proposition 2.7, we could have described the algebras over $\mathbf{T}^3$ in different ways using different realisations of $\pi_1(PU_N)$, but this is not necessary for our present applications, so we have refrained.

LEMMA 3.8. *Let $\{k_i\colon i = 1,2,3\} \subset \mathbf{Z}$ satisfy $(k_1, k_2, k_3) = 1$. Then there is a matrix $T = (t_{ij}) \in SL_3(\mathbf{Z})$ such that $t_{i3} = k_i$ for all i.*

PROOF. Since the k_i are coprime there are integers K_i such that

$$k_1K_1 + k_2K_2 + k_3K_3 = 1;$$

then it is enough to choose t_{ij} $(j = 1,2)$ satisfying

$$t_{21}t_{32} - t_{31}t_{22} = K_1,$$
$$t_{12}t_{31} - t_{11}t_{32} = K_2,$$
$$t_{11}t_{22} - t_{21}t_{12} = K_3.$$

We take

$$t_{12} = 0, \qquad t_{11} = (K_2, K_3), \qquad t_{32} = -K_2/t_{11}, \qquad t_{22} = K_3/t_{11}.$$

Then $(t_{32}, t_{22}) = 1$ and it possible to find t_{21}, t_{31} as required.

THEOREM 3.9. *Every N-homogeneous C^*-algebra A with spectrum $\mathbf{T}^3$ is isomorphic to one to the form $B \otimes C(\mathbf{T})$, where B is N-homogeneous with spectrum $\mathbf{T}^2$. In particular, by Proposition* 2.7(3) *we can suppose $B = B(V, W)$ for some pair $V, W \in SU_N(\mathbf{C})$ satisfying*

$$VW = \exp(2\pi ik/N)WV;$$

then the isomorphism class of A is determined by the integer (k, N).

PROOF. By Proposition 3.7 we may suppose $A = B\{V_i\}$; then A will have the form $B \otimes C(\mathbf{T})$ if $V_3 = 1_N$. Let c_i be the commutators of the V_i as in Proposition 3.7, so that $c_i = \exp(2\pi il_i/N)$ for some $l_i \in \mathbf{Z}$. Let $k_i = l_i/(l_1, l_2, l_3)$, and choose T as in Lemma 3.8. Then

$$T\begin{pmatrix} 0 \\ 0 \\ (l_1, l_2, l_3) \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

so that, if $d_3 = \exp(2\pi i(l_1, l_2, l_3)/N)1_N$ and $d_1 = d_2 = 1_N$, then

$$c_i = d_3^{t_{i3}} = \prod_{j=1}^{3} d_j^{t_{ij}}.$$

It follows from Proposition 3.7 that if $W_3 = 1_N$ and $\{W_1, W_2\} = d_3$ then $B\{V_i\} \cong B\{W_i\} \cong B(W_1, W_2) \otimes C(\mathbf{T})$. Proposition 3.7 also shows that algebras $B\{V_i\}$ corresponding to the commutators $(d, 1, 1)$ and $(c, 1, 1)$ are isomorphic exactly when there is an integer matrix $T = (t_{ij})$ of determinant 1 such that

$$c = d^{t_{11}}, \qquad 1_N = d^{t_{21}}, \qquad 1_N = d^{t_{31}}.$$

Let $c = \exp(2\pi ik/N)1_N$ and $d = \exp(2\pi il/N)1_N$. Then the algebras are isomorphic iff there is an integer $a = t_{11}$ with $k = a\,l \pmod N$ and which can appear as the top left-hand entry in a matrix $(t_{ij}) \in SL_3(\mathbf{Z})$ where t_{21}, t_{31} are multiples of $N/(l, N)$. It follows from Lemma 3.8 (applied to the first column rather than the third) that a can appear in such a column if $(a, N/(l, N)) = 1$ and the converse is easy to prove. If $(a, N/(l, N)) = 1$ then $(al, N) = (l, N)$ and it follows easily that the k's which can arise this way from a fixed l are precisely those with $(k, N) = (l, N)$.

Rational rotation algebras.

There are two obvious generalisations of the rational rotation algebras A_θ which have spectrum $\mathbf{T}^3$. Given $\theta, \phi \in \mathbf{Z}$, we define actions of $\mathbf{Z}$ on $\mathbf{T}^2$ and $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{T}$ by

$$n \cdot (z, w) = (e^{2\pi i n\theta}z, e^{2\pi i n\phi}w),$$
$$(m, n) \cdot z = e^{2\pi i(m\theta + n\phi)}z.$$

We could calculate the corresponding transformation group C^*-algebras $C^*(\mathbf{Z}, \mathbf{T}^2, (\theta, \phi))$ and $C^*(\mathbf{Z}^2, \mathbf{T}, (\theta, \phi))$ using Corollary 1.2, as we did for A_θ. However, Theorem 3.9 shows that they must have the form $B \otimes C(\mathbf{T})$, and it is easier to exhibit this decomposition directly.

For $\theta = p/M$, $\phi = q/N$ let

$$r = \left(\frac{mQ}{(M,N)}, \frac{Np}{(M,N)}\right), \quad a = \frac{Np}{(M,N)r}, \quad c = \frac{Mq}{(M,N)r}, \quad \psi = \frac{r}{[M,N]},$$

and choose $b, d \in \mathbf{Z}$ such that $ad - bc = 1$. Routine calculations show that the map Φ of $C_c(\mathbf{Z} \times \mathbf{T}^2)$ into itself defined by

$$(\Phi f)(m, e^{2\pi i s}, e^{2\pi i t}) = f(m, e^{2\pi i(as+bt)}, e^{2\pi i(cs+dt)}),$$

extends to an isomorphism of $C^*(\mathbf{Z}, \mathbf{T}^2, (\theta, \phi))$ onto $C^*(\mathbf{Z}, \mathbf{T}^2, (\psi, 0))$. Similarly, the map Ψ on $C_c(\mathbf{Z}^2 \times \mathbf{T})$ defined by

$$(\Psi f)(m, n, z) = f(am + cn, bm + dn, z),$$

gives an isomorphism of $C^*(\mathbf{Z}^2, \mathbf{T}, (\theta, \phi))$ onto $C^*(\mathbf{Z}^2, \mathbf{T}, (\psi, 0))$. Now we define $\mathcal{F}\colon C_c(\mathbf{Z}^2 \times \mathbf{T}) \to C_c(\mathbf{Z} \times \mathbf{T}^2)$ by

$$(\mathcal{F} f)(m, z, w) = \sum_{n=-\infty}^{\infty} f(m, n, z)w^{-n};$$

this induces an isomorphism of $C^*(\mathbf{Z}^2, \mathbf{T}, (\psi, 0))$ onto $C^*(\mathbf{Z}, \mathbf{T}^2, (\psi, 0))$. Since this latter algebra is isomorphic to $A_\psi \otimes C(\mathbf{T})$ in an obvious way, we can describe its isomorphism class as follows.

PROPOSITION 3.10. (1) *Let $\theta, \phi \in \mathbf{Q}$, suppose that $\theta = p/M$, $\phi = q/N$ with $(p, M) = (q, N) = 1$, and let $r = (Mq/(M, N), Np/(M, N))$. Then $C^*(\mathbf{Z}^2, \mathbf{T}, (\theta, \phi))$ is an $[M, N]$-homogeneous C^*-algebra, and two such algebras $C^*(\mathbf{Z}^2, \mathbf{T}, (\theta_i, \phi_i))$ are isomorphic if and only if $[M_1, N_1] = [M_2, N_2]$. Every homogeneous C^*-algebra with spectrum $\mathbf{T}^3$ is isomorphic to one of the form $C^*(\mathbf{Z}^2, \mathbf{T}) \otimes M_m(\mathbf{C})$.*

(2) *A similar statement holds for $C^*(\mathbf{Z}, \mathbf{T}^2, (\theta, \phi))$.*

PROOF. The conditions $(p, M) = (q, N) = 1$ imply that r and $[M, N]$ are coprime, so $A_{r/[M,N]}$ is $[M, N]$-homogeneous as shown in the proof of Proposition 2.8. By Theorem 3.9 the isomorphism class of $A_{r/[M,N]} \otimes C(\mathbf{T})$ is uniquely

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determined by the condition $(r, [M, N]) = 1$, and the result follows from the preceding paragraph.

Twisted group algebras.

Let $\theta, \phi, \psi \in \mathbf{R}$, and define a multiplier ω on $\mathbf{Z}^3$ by

$$\omega((m_1, m_2, m_3), (n_1, n_2, n_3)) = \exp 2\pi i \left\{ (m_1\, m_2\, m_3) \begin{pmatrix} 0 & \psi & \theta \\ 0 & 0 & \phi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right\}$$
$$= \exp 2\pi i \{\psi m_1 n_2 + \theta m_1 n_3 + \phi m_2 n_3\}.$$

According to [1, Theorem 3.2], every multiplier on $\mathbf{Z}^3$ is equivalent to one of this form. By [2, Lemma 3.1] and Proposition 1.3 the multiplier ω is type I if and only if the group

$$S_\omega = \{m \in \mathbf{Z}^3 | \omega(m, n) = \omega(n, m) \text{ for all } n \in \mathbf{Z}^3\}$$

has finite index in $\mathbf{Z}^3$, and then $C^*(\mathbf{Z}^3, \omega)$ is $\sqrt{|G: S_\omega|}$-homogeneous. It is not hard to see that this happens exactly when θ, ϕ, ψ are rational.

Therefore when $\theta, \phi, \psi \in \mathbf{Q}$, $C^*(\mathbf{Z}^3, \omega)$ is a homogeneous C^*-algebra, and by Proposition 1.3 its spectrum is $(\mathbf{Z}^3)\hat{}/S_\omega^\perp = \mathbf{T}^3/S_\omega^\perp$. The subgroup S_ω is the lattice in $\mathbf{Z}^3$ generated by a parallelepiped of finite volume $= |\mathbf{Z}^3: S_\omega|$ so $S_\omega^\perp$ is also a lattice in $\mathbf{T}^3$, and $\mathbf{T}^3/S_\omega^\perp$ is homeomorphic to $\mathbf{T}^3$. It is tempting to try to calculate the isomorphism class of $C^*(\mathbf{Z}^3, \omega)$ by Fourier transforming it into a rational rotation algebra, as worked in the $\mathbf{T}^2$-case. However, if we define $\mathcal{F}: C_c(\mathbf{Z}^3) \to C_c(\mathbf{Z}^2 \times \mathbf{T})$ by

$$(\mathcal{F}a)(m, n, z) = \sum_{k=-\infty}^{\infty} a(m, n, k) z^{-k}$$

and a multiplier σ on $\mathbf{Z}^2$ by

$$\sigma((m_1, m_2), (n_1 n_2)) = \exp 2\pi i \psi m_1 n_2,$$

then the usual lengthy calculations show that $\mathcal{F}$ extends to an isomorphism of $C^*(\mathbf{Z}^3, \omega)$ onto the twisted crossed product $C^*(\mathbf{Z}^2, \mathbf{T}, (\theta, \phi), \sigma)$. So to make this approach work we would need to calculate the twisted crossed product using [20]; we think it is probably easier to use Proposition 1.3 and Proposition 3.7. We illustrate by considering the case where $\phi = r_1/N$, $\theta = r_2/N$, $\psi = r_3/N$ and one of the r_i—say r_1—is coprime to N.

The subgroup S_ω is given by

$$\begin{aligned} S_\omega &= \{m: \omega(m, n) = \omega(n, m) \text{ for all } n\} \\ &= \{(m_1, m_2, m_3): -r_3 m_2 - r_2 m_3 = 0, \\ &\qquad r_3 m_1 - r_1 m_3 = 0, r_2 m_1 + r_1 m_2 = 0 \pmod N\}. \end{aligned}$$

Since $(r_1, N) = 1$, we have

$$-r_3m_2 - r_2m_3 \in N\mathbf{Z} \Leftrightarrow r_1[-r_3m_2 - r_2m_3] \in N\mathbf{Z}$$
$$\Leftrightarrow -r_3[r_1m_2 + r_2m_1] + r_2[r_3m_1 - r_1m_3] \in N\mathbf{Z},$$

so

$$S_\omega = \{(m_1, m_2, m_3): r_1m_2 + r_2m_1 \in N\mathbf{Z}, r_3m_1 - r_1m_3 \in N\mathbf{Z}\}.$$

Let q satisfy $qr_1 = 1 \pmod N$ and $0 \leqslant q \leqslant N - 1$. Then

$$S_\omega = \{(m_1, -qr_2m_1 + Nm_2, qr_3m_1 + Nm_3): m_1, m_2, m_3 \in \mathbf{Z}\}:$$

note that $|\mathbf{Z}^3: S_\omega| = N^2$. For $s \in I^3$ we write

$$\gamma_s(n_1, n_2, n_3) = \exp 2\pi i(s_1n_1 + s_2n_2 + s_3n_3).$$

Then $S_\omega^\perp$ can be realised as

$$\{\gamma_s: \exp 2\pi i[s_1m_1 - s_2qr_2m_1 + s_2Nm_2 + s_3qr_3m_1 + s_3Nm_3] = 1 \quad \text{for all } m \in \mathbf{Z}^3\}$$
$$= \{\gamma_s: s_2N = 0, s_3N = 0 \text{ and } s_1 = qr_2s_2 - qr_3s_3 \pmod 1)\}.$$

A basis for the lattice $S_\omega^\perp$ is given by

$$\gamma_1 = \gamma_s \quad \text{for } s = (qr_2/N, 1/N, 0), \qquad \gamma_2 = \gamma_s \quad \text{for } s = (-qr_3/N, 0, 1/N)$$

and

$$\gamma_3 = \gamma_s \quad \text{for } s = (1, 0, 0).$$

(To see that this is a basis, one only has to verify that the volume of the parallelepiped they generate is $1/N^2$, equal to $1/|\mathbf{Z}^3: S_\omega|$.) According to Proposition 1.3, $(\mathbf{Z}^3, \omega)\hat{}$ is obtained from $\mathbf{T}^3$ by pasting along opposite faces of the basic parallelepiped of $S_\omega^\perp$. If M is an irreducible ω-representation of $\mathbf{Z}^3$, then every such representation of $\mathbf{Z}^3$ has the form γM, and

$$M(m)\gamma(n)M(n)M(m) = \tilde{\omega}(m, n)\gamma(n)M(n) \quad \text{for } m, n \in \mathbf{Z}^3.$$

Thus if $m^i = (m_1^i, m_2^i, m_3^i) \in \mathbf{Z}^3$ satisfy $\tilde{\omega}(m^i, \cdot) = \gamma_i$, then $M(m^i)$ will interwine $\gamma_s M$ and $\gamma_i\gamma_s M$ for all $s \in I^3$. In particular, flattening out $S_\omega^\perp$ shows that

$$C^*(\mathbf{Z}^3, \omega) \cong B\{M(m^i)\}.$$

Since $\gamma_3 = 1$, we can take $m^3 = 0$, so $M(m^3) = 1$ and we have

$$C^*(\mathbf{Z}^3, \omega) \cong B(M(m^1), M(m^2)) \otimes C(\mathbf{T}).$$

A simple calculation shows that

$$\{M(m^1), M(m^2)\} = \tilde{\omega}(m^1, m^2)1_N = \gamma_1(m^2)1_N$$
$$= \exp[2\pi i(qr_2m_1^2 + m_2^2)/N]1_N$$
$$= \exp[2\pi iq(r_2m_1^2 + r_1m_2^2)/N]1_N.$$

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However, since $\tilde{\omega}(m^2, \cdot) = \gamma_2$, it follows that $r_2 m_1^2 + r_1 m_2^2 = 1 \pmod N$, so we have

$$\{M(m^1), M(m^2)\} = \exp(2\pi i q/N) 1_N.$$

By definition q is the inverse of r_1 mod N, so $(q, N) = 1$ and by Theorem 3.9 this determines the class $C^*(\mathbf{Z}^3, \omega)$ uniquely.

In general, if ω is given by three rational numbers r_i/N_i, where $(r_i, N_i) = 1$, then $C^*(\mathbf{Z}^3, \omega)$ is $[N_1, N_2, N_3]$-homogeneous (see [2, page 220]). The argument above shows that if $[N_1, N_2, N_3] = N_i$ for some i, then the isomorphism class of $C^*(\mathbf{Z}^3, \omega)$ is that of $A_{1/N_i} \otimes C(\mathbf{T})$. It would be interesting to carry through a similar analysis for the general case, but the calculations do get much more complicated—it is not so easy to write down S_ω, for example. However, we imagine that it should be possible to do the necessary computations, at least for a specific multiplier ω.

4. Concluding remarks.

4.1. As we mentioned in the introduction, the classification of M_N-bundles over $\mathbf{T}^4$ differs substantially from that over $\mathbf{T}^2$ or $\mathbf{T}^3$. In particular, for each N there are infinitely many non-isomorphic N-homogeneous C^*-algebras whose spectra are homeomorphic to $\mathbf{T}^4$; as we shall see, this is a direct consequence of Woodward's analysis [19] of principal PU_N-bundles.

The main result of [19] identifies naturally the set of isomorphism classes of PU_N-bundles over a 4-complex with a subgroup H of $H^2(X, \mathbf{Z}_N) \oplus H^4(X, \mathbf{Z})$. We have $H^2(\mathbf{T}^4, \mathbf{Z}_N) \cong \mathbf{Z}_N^6$ and $H^4(\mathbf{T}^4, \mathbf{Z}) \cong \mathbf{Z}$, and it follows from part (i) of the classification theorem in [19] that the subgroup H contains $\{0\} \oplus 2N\mathbf{Z}$, thus giving us an infinite class of PU_N-bundles over $\mathbf{T}^4$. The group $\mathbf{Z} \cong 2N\mathbf{Z}$ has only two automorphisms, so after allowing for the effect of homeomorphisms we still have infinitely many bundles, and hence infinitely many mutually non-isomorphic N-homogeneous C^*-algebras.

We have not worked out a detailed description of the M_N-bundles over $\mathbf{T}^4$, or even written down representatives of an infinite class of examples, although in principle it appears to be possible. However, it seems likely that the crossed product constructions we have used earlier can only give finitely many distinct N-homogeneous algebras for each N, and it could be very interesting to find a natural C^*-algebraic construction of an infinite family of such algebras.

4.2. We have shown that non-trivial homogeneous C^*-algebras can arise in surprisingly many different ways. There are at least two more, however, which we have not investigated. In operator theory C^*-algebras generated by n-normal operators, or those generated by essentially n-normal operators modulo the compacts, have all their irreducible representations of dimension $\leqslant n$ and they can be homogeneous (see, for example, [11, Section 3, particularly Remark 3.10(b)]). Secondly, although group C^*-algebras cannot be homogeneous without being abelian—there is always the trivial one-dimensional representation—they can be direct sums of homogeneous algebras. (See, for example, [13, Proposition 4], which is probably true in more generality.) In fact, it was an example which arose in this context [13, Proposition 7] which first stimulated our interest in algebras with spectrum $\mathbf{T}^2$. It is not hard to see that the 2-homogeneous algebra B in [13, Proposition 7] is the twisted group algebra $C^*(\mathbf{Z}^2, \omega_{1/2})$.

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