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# Profinite posets

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The class of ordered topological spaces which are projective limits of finite partially ordered sets (equipped with the restriction of the product of the discrete topologies) is shown to coincide with the class of compact totally order-disconnected ordered topological spaces. Hence this is another category of spaces equivalent to the category of distributive lattices with zero and unit.

## 1. Introduction

In her papers [7], [8], Miss Priestley has discussed in detail the equivalence of the category of compact totally order-disconnected ordered topological spaces (with continuous monotone maps) and the category of distributive lattices with zero and unit (with zero and unit preserving lattice homomorphisms). More recently it has been shown [10] that the partially ordered set (= poset) of all prime ideals of such a lattice must be of the form $\varprojlim_{\alpha \in I} X_\alpha$ where each $X_\alpha$ $(\alpha \in I)$ is a finite poset. A synthesis of these two results immediately suggests itself, and we prove the following:

**THEOREM.** *Let* $X$ *be an ordered topological space. Then* $X$ *is compact and totally order-disconnected iff* $X \cong \varprojlim_{\alpha \in I} X_\alpha$ *, where* $\{X_\alpha, f_{\alpha\beta}\}$ *is an inverse system of finite posets each equipped with the discrete topology.*

We prove this theorem in §§3, 4. An ordered topological space which

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is of the form $\varprojlim_{\alpha\in I} X_\alpha$ for an inverse system $\{X_\alpha, f_{\alpha\beta}\}$ of finite discretely topologised posets will be called a *profinite poset* by analogy with the group case. Thus the theorem above is an analogue of the well known characterization of profinite groups; see also [*6*] for other related results.

## 2. Preliminaries

The notation and terminology of [*7*], [*8*] will be adopted without further comment. Let us write $A \nleq B$ for subsets $A, B$ of a poset $(X; \leq)$ iff for all $a \in A$, $b \in B$ we have $a \nleq b$.

LEMMA 1. *Let* $(X, \tau, \leq)$ *be a compact totally order-disconnected space. Then for disjoint closed sets* $A, B$ *we have* $A \nleq B$ *iff there is an order-disconnection* $(U|L)$ *such that* $A \subseteq U$, $B \subseteq L$.

Proof. Assume $A \nleq B$. Then since $X$ is totally order-disconnected, for any $x \in A$, $y \in B$ there is an order-disconnection $(U_{x,y}|L_{x,y})$ such that $x \in U_{x,y}$, $y \in L_{x,y}$. Fix $x$. Then the family $\{L_{x,y} : y \in B\}$ constitutes an open cover of $B$, and so there exists a finite sub-cover $\left\{L_{x,y_j} : j = 1, 2, \ldots, n\right\}$. Put $U_x = \bigcap_{j=1}^{n} U_{x,y_j}$ and $L_x = \bigcup_{j=1}^{n} L_{x,y_j}$ and we observe that $(U_x|L_x)$ is an order-disconnection with $x \in U_x$, $B \subseteq L_x$. Now the family $\{U_x : x \in A\}$ is an open cover of $A$ and so has a finite subcover $\left\{U_{x_i} : i = 1, 2, \ldots, m\right\}$. Put $U = \bigcup_{i=1}^{m} U_{x_i}$ and $L = \bigcap_{i=1}^{n} L_{x_i}$ and we have an order-disconnection $(U|L)$ such that $U \supseteq A$, $L \supseteq B$ as required.

REMARK. This lemma shows that, as one would expect, compact subsets behave in much the same way as points in compact ordered spaces. For further evidence of this see Theorem 4, p. 46 of [*4*]. When the order is trivial, Lemma 1 reduces to a well known result for boolean algebras.

Let $(X; \leq)$ be a poset and $\rho \subseteq X \times X$ an equivalence relation on

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$X$. Then one way of defining a quasi-order on $X/\rho$ is to write $x/\rho \le' y/\rho$ iff there exists $x_1 \equiv x\ (\rho)$, $y_1 \equiv y\ (\rho)$ such that $x_1 \le y_1$. Unfortunately this relation $\le'$ is not always a partial order on $X/\rho$; when it is we say that $\rho$ is *order compatible*. Thus the equivalence $\rho$ on $X$ is order compatible iff for any $x_1, y_1$ in $X$, if $x_1 \equiv x_2\ (\rho)$ and $y_1 \equiv y_2\ (\rho)$ and $x_1 \le y_1$, $x_2 \ge y_2$ then $x_1 \equiv x_2 \equiv y_1 \equiv y_2\ (\rho)$. Equivalently, $\rho$ is order compatible iff for any $x, y \in X$ such that $x \not\equiv y\ (\rho)$, we have either $\{x_1 : x_1 \equiv x\ (\rho)\} \nleq \{y_1 : y_1 \equiv y\ (\rho)\}$ or $\{x_1 : x_1 \equiv x\ (\rho)\} \ngeq \{y_1 : y_1 \equiv y\ (\rho)\}$.

## 3. First proof of the theorem

Suppose $X = \varprojlim_{\alpha \in I} X_\alpha$ where $\{X_\alpha, f_{\alpha\beta}\}$ is an inverse system of finite posets each equipped with the discrete topology, and $I$ is a directed set. Then $X$ is certainly a compact space ([1], Chapter I, §9.6, Proposition 8). For any $\alpha \in I$ and $x'_\alpha \in X_\alpha$ write $U_{x'_\alpha} = \{x \in X : x_\alpha \ge x'_\alpha\}$, $L_{x'_\alpha} = \{x \in X : x_\alpha \le x'_\alpha\}$ and $T_{x'_\alpha} = \{x \in X : x_\alpha = x'_\alpha\}$, where $x = \langle x_\alpha \rangle_{\alpha \in I}$ denotes a typical element of $X$. Then $T_{x'_\alpha}$ is clopen, and (since each $X_\alpha$ is discrete) so are $U_{x'_\alpha}$, $L_{x'_\alpha}$. Further $U_{x'_\alpha}$ is increasing and $L_{x'_\alpha}$ is decreasing. We now prove that $X$ is totally order-disconnected. Suppose $x \nleq y$ in $X$; then for some $\alpha \in I$ we must have $x_\alpha \nleq y_\alpha$. Thus $\left(U_{x_\alpha} \middle| L_{y_\alpha}\right)$ is an order-disconnection and $x \in U_{x_\alpha}$, $y \in L_{y_\alpha}$, and so the result is proved.

For the converse we suppose that $X$ is compact and totally order-disconnected. Let $R$ denote the family of all clopen order compatible equivalences $\rho$ on $X$, that is, all order compatible equivalences of the form $\rho = \bigcup_{i=1}^{m} V_i \times V_i$ for some finite partition $\{V_i\}$ of $X$ into open sets. Then $X_\rho = X/\rho$ is a finite poset, and, when equipped with the discrete topology, is a continuous monotone image of $X$ under the canonical projection $pr_\rho : X \to X/\rho$.

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Now Lemma 1 implies that the equivalence $\rho$ is order compatible iff $V_i \neq V_j$ implies that there exists an order disconnection $(U|L)$ such that $V_i \subseteq U$ , $V_j \subseteq L$ or $V_j \subseteq U$ , $V_i \subseteq L$ . We now prove that the family of all clopen order compatible equivalences is directed, and that $\cap\{\rho : \rho \in R\} = \Delta$ , the diagonal of $X \times X$ . The last remark is easy, for if $x \neq y$ then either $x \nleq y$ or $y \nleq x$ . Suppose $x \nleq y$ ; then there is an order-disconnection $(U|L)$ such that $x \in U$ , $y \in L$ . But it is easily checked that $\{U, L, U^c \cap L^c\}$ is a partition which induces an order compatible equivalence $\rho$ , and hence $x \nleq y\ (\rho)$ .

Suppose $\rho$ and $\rho'$ are two clopen order compatible equivalences induced by the partitions $\{V_i : i = 1, 2, \ldots, m\}$ and $\{V'_j : j = 1, 2, \ldots, n\}$ respectively. Then the partition

$$\{V_i \cap V'_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, V_i \cap V'_j \neq \emptyset\}$$

induces an order compatible equivalence $\rho \vee \rho'$ . For if $V_i \cap V'_j \neq V_{i_1} \cap V'_{j_1}$ , then either $V_i \neq V_{i_1}$ or $V'_j \neq V'_{j_1}$ , say the former. Then either $V_i \nleq V_{i_1}$ or $V_{i_1} \nleq V_i$ , again suppose the former. By Lemma 1 there is an order-disconnection $(U|L)$ such that $V_i \subseteq U$ and $V_{i_1} \subseteq L$ . But now $V_i \cap V'_j \subseteq U$ and $V_{i_1} \cap V'_{j_1} \subseteq L$ which proves that $V_i \cap V'_j \nleq V_{i_1} \cap V'_{j_1}$ and so $\rho \vee \rho'$ is order compatible.

We now collect the foregoing results: the system $\{X_\rho : \rho \in R\}$ where for $\rho \subseteq \rho'$ the canonical map $f_{\rho\rho'} : X_{\rho'} \to X_\rho$ is continuous and monotone, and $R$ is directed, becomes an inverse system $\{X_\rho, f_{\rho\rho'}\}$ . The map $\phi : X \to \varprojlim_{\rho \in R} X_\rho$ given by $\phi(x) = \left\langle pr_\rho(x) \right\rangle_{\rho \in R}$ is continuous, bijective, and an order isomorphism, and so $X$ and $\varprojlim_{\rho \in R} X_\rho$ are homeomorphic as required.

## 4. Second proof of the theorem

We quickly sketch an alternative, shorter, proof of the theorem. It

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does however, have the disadvantage of using results from [2], [5], [10] of a non-topological nature, but is the way the theorem was originally deduced.

Suppose $X = \varprojlim_{\alpha \in I} X_\alpha$ is a projective limit of finite, discretely topologised posets. Then $X_\alpha = \text{Patch} A_\alpha$ for a unique distributive lattice $A_\alpha$. Thus $X = \varprojlim X_\alpha \cong \varprojlim \text{Patch} A_\alpha \cong \text{Patch}\left(\varinjlim A_\alpha\right) = \text{Patch}\ A$ where $A = \varinjlim A_\alpha$ is the direct limit of the direct system $\{A_\alpha, f^*_{\alpha\beta}\}$, and where $f^*_{\alpha\beta} : A_\alpha \to A_\beta$ is the dual map to $f_{\alpha\beta} : X_\beta \to X_\alpha$ for $\alpha \le \beta$. By the main result of [7] and some remarks of [2], $X = \text{Patch}\ A$ is compact and totally order-disconnected.

Conversely, suppose $X$ is compact and totally order-disconnected. By the main result of [7], $X = \text{Patch}\ A$ for a unique distributive lattice $A$. Write $A = \varinjlim A_\alpha$ as a direct limit of its finitely generated (finite) sublattices $A_\alpha$. Then

$$\text{Patch}\ A \cong \text{Patch}\left(\varinjlim A_\alpha\right) \cong \varprojlim \text{Patch}\ A_\alpha = \varprojlim X_\alpha$$

where $\{X_\alpha\}$ is a family of finite posets equipped with discrete topologies. The details of this proof can be reconstructed from [2], [5].

In a notice which appeared after this note was written, Joyal [3] states a theorem closely related to our main result. His proof is probably more like the one sketched above.

## 5. Final remarks

The theorem of this note and other results show that the following categories are equivalent:

(i) distributive lattices with zero and unit (with zero and unit preserving homomorphisms);

(ii) spectral spaces (with spectral maps);

(iii) compact totally order-disconnected spaces (with continuous monotone maps);

(iv) profinite posets (with continuous monotone maps).

The study of the relations between (i) and (ii) was begun by Stone in [*11*]; some further details are in [*9*] and the forthcoming part II, while much useful information is in [*2*]. The relation (i) ↔ (iii) is the object of [*7*], [*8*], and the connections between (i), (ii) and (iii) are being studied at the moment.

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