J. Aust. Math. Soc. **108** (2020), 412–430
doi:10.1017/S1446788719000338

CONTINUITY OF A CONDITION SPECTRUM AND ITS LEVEL SETS

D. SUKUMAR and S. VEERAMANI[✉]

(Received 15 March 2018; accepted 10 July 2019; first published online 9 September 2019)

Communicated by L. O. Clark

Abstract

Let $\mathcal{A}$ be a complex unital Banach algebra, let a be an element in it and let $0 < \epsilon < 1$. In this article, we study the upper and lower hemicontinuity and joint continuity of the condition spectrum and its level set maps in appropriate settings. We emphasize that the empty interior of the ϵ-level set of a condition spectrum at a given (ϵ, a) plays a pivotal role in the continuity of the required maps at that point. Further, uniform continuity of the condition spectrum map is obtained in the domain of normal matrices.

2010 *Mathematics subject classification*: primary 46H05; secondary 47A10.

Keywords and phrases: condition spectrum, upper and lower hemicontinuous correspondence.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with unity e and $a \in \mathcal{A}$. We identify $\lambda \cdot e$ as λ for any $\lambda \in \mathbb{C}$. The spectrum of a is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : (a - \lambda) \text{ is not invertible in } \mathcal{A}\}.$$

It is well known that $\sigma(a)$ is a compact subset of $\mathbb{C}$. The complement of $\sigma(a)$ is called the resolvent set of a. Throughout this note, $K(\mathbb{C})$ denotes the set of all compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric. If $\triangleleft$ and $\triangleright$ are two elements in $K(\mathbb{C})$, then the Hausdorff distance between $\triangleleft$ and $\triangleright$ is defined as

$$H(\triangleleft, \triangleright) = \max\Big\{\sup_{s\in\triangleleft} d(s, \triangleright), \sup_{t\in\triangleright} d(t, \triangleleft)\Big\}, \tag{1.1}$$

where $d(s, \triangleright) = \inf\{|s - \mu| : \mu \in \triangleright\}$ and $d(t, \triangleleft) = \inf\{|t - \lambda| : \lambda \in \triangleleft\}$.

Consider the function

$$S : \mathcal{A} \to K(\mathbb{C}) \text{ defined by } S(a) = \sigma(a).$$

The second author thanks the University Grants Commission (UGC), India for the financial support (Ref. no. 23/12/2012(ii)EU-V) provided as a form of Research Fellowship to carry out this research work at IIT Hyderabad.

Investigating the continuity of S at given $a \in \mathcal{A}$ is a long-standing problem in mathematics (see [4]). Newburgh showed that S is continuous at all $a \in \mathcal{A}$ if $\mathcal{A}$ is commutative [9, Corollary of Theorem 4], and in the noncommutative case, if $\sigma(a)$ is totally disconnected, then S is continuous at a [9, Theorem 3]. He proved that S is upper semicontinuous [9, Theorem 1] in any Banach algebra $\mathcal{A}$. In [4], it is observed that if the Banach algebra is a finite-dimensional modulo radical, then S is continuous. Kakutani provided an element T in the Banach algebra $B(\ell_2)$ such that S is not continuous at T (see [10, page 282]). For more details about the continuity of S at a particular element, one can refer to the review article [4].

There are many generalized notions of the spectrum of an element, for example the Ransford spectrum, the ϵ-pseudospectrum (where $\epsilon > 0$) and the ϵ-condition spectrum (where $\epsilon \in (0, 1)$).

For $a \in \mathcal{A}$ and $\epsilon > 0$, the ϵ-pseudospectrum (see [7, Definition 2.1]) is defined as

$$\Lambda_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(a-\lambda)^{-1}\| \geq \frac{1}{\epsilon}\right\}$$

with the convention that $\|(a-\lambda)^{-1}\| = \infty$ if $(a-\lambda)$ is not invertible. By [7, Theorem 2.3], $\Lambda_\epsilon(a)$ is a nonempty compact subset of $\mathbb{C}$. Consider the following three set valued maps. For $a \in \mathcal{A}$,

$$\mathcal{P}_a : \mathbb{R}^+ \to K(\mathbb{C}) \text{ defined by } \mathcal{P}_a(\epsilon) = \Lambda_\epsilon(a).$$

For $\epsilon \in \mathbb{R}^+$,

$$\mathcal{P}_\epsilon : \mathcal{A} \to K(\mathbb{C}) \text{ defined by } \mathcal{P}_\epsilon(a) = \Lambda_\epsilon(a)$$

and

$$\mathcal{P} : \mathbb{R}^+ \times \mathcal{A} \to K(\mathbb{C}) \text{ defined by } \mathcal{P}(\epsilon, a) = \Lambda_\epsilon(a),$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^+ \times \mathcal{A}$ is a metric space with respect to the metric given by

$$d((\epsilon_1, a_1), (\epsilon_2, a_2)) = \|a_1 - a_2\| + |\epsilon_1 - \epsilon_2|. \tag{1.2}$$

It is natural to ask the question: 'Are the above three maps continuous at a given point?'.

For given ϵ and a, the answer to these questions is closely related to the question

'Does the set $\mathcal{LP}_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(a-\lambda)^{-1}\| = \frac{1}{\epsilon}\right\}$ have empty interior?'.

Globevnik in [5] showed that the interior of $\mathcal{LP}_\epsilon(a)$ is empty in the unbounded component of the resolvent set, and if X is a complex uniformly convex Banach space (see [11, Definition 2.4]) and $T \in B(X)$, then the interior of $\mathcal{LP}_\epsilon(T)$ is empty. In [11] (Theorem 3.1), Shargorodsky proved that there exists an invertible bounded operator T acting on a particular Banach space X such that the interior of $\mathcal{LP}_\epsilon(T)$ is nonempty. This example says that pseudospectra functions may have jump discontinuities.

The research article [7] discusses the continuity of the maps $\mathcal{P}_a, \mathcal{P}_\epsilon$ and $\mathcal{P}$ and how this continuity is related to the interior property of $\mathcal{LP}_\epsilon(a)$. For fixed $a \in \mathcal{A}$, Theorem 4.1 in [7] says that it is necessary and sufficient that the interior of $\mathcal{LP}_\epsilon(a)$ is empty for the continuity of $\mathcal{P}_a$. Theorem 4.3 in [7] proves that the maps $\mathcal{P}_\epsilon$ and $\mathcal{P}$ are continuous by assuming the continuity of $\mathcal{P}_a$ for every $a \in \mathcal{A}$.

In this article, we consider the correspondence between two topological spaces. The following is the definition of a correspondence.

Definition 1.1 [1, Definition 17.1]. A correspondence ϕ from a set X to a set Y assigns to each x in X a subset $\phi(x)$ of Y. We write the correspondence ϕ between X and Y as $\phi : X \twoheadrightarrow Y$.

For $\epsilon \in (0, 1)$ and $a \in \mathcal{A}$, the ϵ-condition spectrum is defined by Kulkarni and Sukumar in [8] as the following set.

Definition 1.2 [8, Definition 2.5].

$$\sigma_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(a-\lambda)\|\,\|(a-\lambda)^{-1}\| \geq \frac{1}{\epsilon}\right\}$$

with the convention that $\|(a-\lambda)\|\,\|(a-\lambda)^{-1}\| = \infty$ if $(a-\lambda)$ is not invertible.

Further, for $\epsilon \in (0, 1]$, the ϵ-level set of the condition spectrum of $a \in \mathcal{A}$ is defined in [13] as the following set.

Definition 1.3 [13, Definition 1.3].

$$L_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(a-\lambda)\|\,\|(a-\lambda)^{-1}\| = \frac{1}{\epsilon}\right\}.$$

By [8, Theorem 2.7(4)], $\sigma_\epsilon(a)$ is a nonempty compact subset of $\mathbb{C}$. For $\epsilon \in (0, 1)$, since $L_\epsilon(a)$ is a closed subset of $\sigma_\epsilon(a)$, $L_\epsilon(a)$ is a compact subset of $\mathbb{C}$ (see [13, Note 2.2 and Proposition 2.7]). The following correspondences arise naturally from the above definitions. For $a \in \mathcal{A}$,

$$C_a : (0, 1) \twoheadrightarrow \mathbb{C} \text{ defined by } C_a(\epsilon) = \sigma_\epsilon(a)$$

and

$$\mathcal{L}C_a : (0, 1) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{L}C_a(\epsilon) = L_\epsilon(a).$$

For $\epsilon \in (0, 1)$,

$$C_\epsilon : \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } C_\epsilon(a) = \sigma_\epsilon(a)$$

and

$$\mathcal{L}C_\epsilon : \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{L}C_\epsilon(a) = L_\epsilon(a).$$

Finally,

$$C : (0, 1) \times \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } C(\epsilon, a) = \sigma_\epsilon(a)$$

and

$$\mathcal{L}C : (0, 1) \times \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{L}C(\epsilon, a) = L_\epsilon(a),$$

where $(0, 1) \times \mathcal{A}$ is a metric space in which the metric is defined as in Equation (1.2).

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

The main aim of this article is to study the continuity of the correspondences defined above.

In this note, it is observed that the emptiness of the interior of level set $L_\epsilon(a)$ at given ϵ and a plays an important role in the continuity of the above correspondences.

Section 2 of this paper has the basic definitions and related results which are used in the subsequent sections. Section 3 deals with the continuity of the correspondences C_a, C_ϵ and C. In general, it is shown that the correspondences C_a and C_ϵ are upper hemicontinuous (Theorems 3.1 and 3.4). Further, we observe that emptiness of the interior of $L_\epsilon(a)$ at given ϵ and a turns out to be a necessary and sufficient condition for the lower hemicontinuity of C_a (Theorem 3.5). By assuming that C_a is continuous, we conclude that C_ϵ and C are continuous (Theorem 3.12). By proving a characterization for normal matrices in terms of the ϵ-condition spectrum (Theorem 3.18), it is found that C_ϵ is uniformly continuous on the set of normal matrices (Theorem 3.19).

Section 4 is devoted to the continuity property of $\mathcal{L}C_a, \mathcal{L}C_\epsilon$ and $\mathcal{L}C$. We will see that continuity of these correspondences relies entirely on the continuity of the correspondences C_a, C_ϵ and C. In general, $\mathcal{L}C_a, \mathcal{L}C_\epsilon$ are upper hemicontinuous (Theorem 4.3) but not lower hemicontinuous. An example is given to show that $\mathcal{L}C_a$ and $\mathcal{L}C_\epsilon$ are not lower hemicontinuous (Example 4.4). If the interior of $L_\epsilon(a)$ is empty at a and ϵ, then we prove that the correspondence $\mathcal{L}C$ is jointly upper hemicontinuous (Theorem 4.5).

In the rest of the paper, $B(a, r)$ denotes the open ball in the complex plane with center a and radius $r > 0$ and $\overline{B}(a, r)$ denotes the closure of $B(a, r)$. We denote the set of all nonscalar elements in a Banach algebra $\mathcal{A}$ by $\mathcal{A} \setminus \mathbb{C}e$.

2. Basic definitions and results

In this section, we present some basic definitions and results which are necessary for the main results of this paper given in subsequent sections. Since this section contains well established results, the reader who is familiar with the concepts of upper and lower hemicontinuity of a correspondence, the maximum modulus theorem for vector valued maps and condition spectra can omit it.

2.1. Hemicontinuity and its properties.

Definition 2.1 [1, page 558]. Let X be a topological space. A neighborhood of a subset A of X is any subset B for which there is an open subset V satisfying $A \subseteq V \subseteq B$.

In this note, we prove the continuity of condition spectra and level sets using upper and lower hemicontinuous concepts. The book [1] has a detailed study about the continuity of a correspondence.

Definition 2.2 [1, Definition 17.2]. A correspondence $\phi : X \twoheadrightarrow Y$ between topological spaces is:

(1) upper hemicontinuous at the point $x \in X$ if, for every neighborhood U of $\phi(x)$, there is a neighborhood V of x such that $z \in V$ implies that $\phi(z) \subseteq U$;

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

(2) lower hemicontinuous at $x \in X$ if, for every open set U with $U \cap \phi(x) \neq \emptyset$, there is a neighborhood V of x such that $z \in V$ implies that $\phi(z) \cap U \neq \emptyset$; and

(3) continuous at $x \in X$ if it is both upper and lower hemicontinuous at x.

The following is a characterization for the upper and lower hemicontinuity of a correspondence at a point using its graph.

Definition 2.3 [1, Definition 17.9]. A correspondence $\phi : X \twoheadrightarrow Y$ between two topological spaces is closed or has closed graph, if its graph

$$Gr\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$$

is a closed subset of $X \times Y$.

Theorem 2.4 [1, Theorem 17.20]. *Assume that the topological space X is first countable and that Y is metrizable. Then, for a correspondence $\phi : X \twoheadrightarrow Y$ and a point $x \in X$, the following statements are equivalent.*

(1) *The correspondence ϕ is upper hemicontinuous at x and $\phi(x)$ is compact.*

(2) *If a sequence $\{(x_n, y_n)\}$ in the graph of ϕ satisfies $x_n \to x$, then the sequence $\{y_n\}$ has a limit point in $\phi(x)$.*

Theorem 2.5 [1, Theorem 17.21]. *Assume that the topological space X is first countable and that Y is metrizable. Then, for a correspondence $\phi : X \twoheadrightarrow Y$ and a point $x \in X$, the following statements are equivalent.*

(1) *The correspondence ϕ is lower hemicontinuous at x.*

(2) *If $x_n \to x$, then, for $y \in \phi(x)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and elements $y_k \in \phi(x_{n_k})$ for each k such that $y_k \to y$.*

Excluding the extreme case $\epsilon = 1$, the level sets are subsets of the condition spectrum. Hence, in order to analyze the continuity of the level set of the condition spectrum correspondences, we use the subcorrespondence notion.

Definition 2.6 [1, page 564]. Let $\phi, \psi : X \twoheadrightarrow Y$ be correspondences between the topological spaces X and Y. If $\psi(x) \subseteq \phi(x)$ for each $x \in X$, then we say that ψ is a subcorrespondence of ϕ.

Theorem 2.7 explains that with some extra presumption the upper hemicontinuity of the subcorrespondence will be inherited from the upper hemicontinuity of the correspondence.

Theorem 2.7 [1, Corollary 17.18]. *Let $\phi, \psi : X \twoheadrightarrow Y$ be correspondences between the topological spaces X and Y such that ϕ is compact valued and ψ is a closed subcorrespondence of ϕ. If ϕ is upper hemicontinuous at $x \in X$, then ψ is also upper hemicontinuous at x.*

Here is a theorem which bridges the continuity of the compact-valued correspondence between the topological spaces X and Y and the continuity of a function from X to $K(Y)$, where $K(Y)$ denotes the space of nonempty compact subsets of Y endowed with its Hausdorff metric topology (defined as in Equation (1.1)).

THEOREM 2.8 [1, Theorem 17.15]. *Let $\phi : X \twoheadrightarrow Y$ between topological spaces be a nonempty compact-valued correspondence from a topological space into a metrizable space. Then the function $f : X \to K(Y)$, defined by $f(x) = \phi(x)$, is continuous at $a \in X$ if and only if the correspondence ϕ is continuous at $a \in X$.*

In some parts of our work we concentrate on the limiting nature of the ϵ-condition spectrum sets as $\epsilon \to 1$. The following is the definition of limit superior, limit inferior and limit concepts involved in a sequence of sets.

DEFINITION 2.9 [2, Definition 1.1.1]. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space X. We say that the subset

$$\limsup_{n\to\infty} K_n := \left\{x \in X : \liminf_{n\to\infty} d(x, K_n) = 0\right\}$$

is the upper limit of the sequence K_n and that the subset

$$\liminf_{n\to\infty} K_n := \left\{x \in X : \lim_{n\to\infty} d(x, K_n) = 0\right\}$$

is its lower limit. A subset K is said to be the limit or the set limit of the sequence K_n if

$$K = \limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n =: \lim_{n\to\infty} K_n.$$

Note 2.10 [2, page 18]. It is clear from the definition that

$$\liminf_{n\to\infty} K_n \subseteq \limsup_{n\to\infty} K_n.$$

Note 2.11 [2, page 18]. If the sequence $\{K_n\}$ is decreasing, then $\lim_{n\to\infty} K_n$ exists and

$$\lim_{n\to\infty} K_n = \bigcap_{n\geq 0} \overline{K_n},$$

where $\overline{K_n}$ denotes the closure of K_n.

2.2. Basic results. We now list some lemmas and a theorem from Banach algebra and the vector valued maximum modulus principle, which will be applied frequently in the following sections.

LEMMA 2.12 [9, Lemma 5]. *Let $b \in \mathcal{A}$. Consider the sequence $\{b_n\}$ in $\mathcal{A}$ such that each b_n is invertible and there exists a positive number M such that $\|b_n^{-1}\| < M$ for all n. If $b_n \to b$, then b is invertible.*

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

Lemma 2.13 [13, Lemma 4.1]. *Let Ω_0 be a connected open subset of $\mathbb{C}$, let Ω be an open subset of Ω_0 and let X be a complex Banach space. For $i = 1, \ldots, n$, suppose we have the following.*

(1) $\psi_i : \Omega_0 \to X$ *are analytic vector valued maps.*
(2) $\prod_{i=1}^{n} \|\psi_i(\lambda)\| \leq M$ *for all* $\lambda \in \Omega$.
(3) $\prod_{i=1}^{n} \|\psi_i(\mu)\| < M$ *for some* $\mu \in \Omega_0$.

Then $\prod_{i=1}^{n} \|\psi_i(\lambda)\| < M$ *for all* $\lambda \in \Omega$.

Theorem 2.14 [8, Theorem 2.7(2)]. *Let $a \in \mathcal{A}$. If $0 < \epsilon_1 < \epsilon_2 < 1$, then $\sigma_{\epsilon_1}(a) \subseteq \sigma_{\epsilon_2}(a)$.*

Theorem 2.15 [8, Theorem 2.9]. *Let $\epsilon \in (0, 1)$. If $a \in \mathcal{A}$, then*

$$\sup\{|\lambda| : \lambda \in \sigma_\epsilon(a)\} \leq \frac{1+\epsilon}{1-\epsilon}\|a\|.$$

3. Continuity of ϵ-condition spectrum

This section contains the results on continuity and uniform continuity of the condition spectrum. We start this section by recalling the upper hemicontinuity of the correspondence C_ϵ.

Theorem 3.1 [8, Theorem 2.7(5)]. *The correspondence C_ϵ is upper hemicontinuous at $a \in \mathcal{A}$.*

Next, we establish a lemma about the graph of C which plays a crucial role in all our results.

Lemma 3.2. *The graph of the correspondence C is closed. Further, the correspondences C_ϵ and C_a are closed for fixed $\epsilon \in (0, 1)$ and fixed $a \in \mathcal{A}$.*

Proof. Consider the sequence $\{(\epsilon_n, a_n), \lambda_n\}$ in $Gr(C)$ and $((\epsilon_0, a), \lambda) \in ((0, 1) \times \mathcal{A}) \times \mathbb{C}$, where $((0, 1) \times \mathcal{A}) \times \mathbb{C}$ is a metric space with the metric

$$d(((\epsilon_1, a_1), \lambda), ((\epsilon_2, a_2), \mu)) = \|a_1 - a_2\| + |\epsilon_1 - \epsilon_2| + |\lambda - \mu|. \tag{3.1}$$

Suppose $((\epsilon_n, a_n), \lambda_n) \to ((\epsilon_0, a), \lambda)$ as $n \to \infty$. Then $\epsilon_n \to \epsilon_0, a_n \to a$ and $\lambda_n \to \lambda$ as $n \to \infty$. We need to prove that $\lambda \in \sigma_{\epsilon_0}(a)$. If $\lambda \in \sigma(a)$, then $\lambda \in \sigma_{\epsilon_0}(a)$. If $\lambda \notin \sigma(a)$, for $1/\|(a - \lambda)^{-1}\|$, there exists $n_0 \in \mathbb{N}$ such that

$$\|(a_n - \lambda_n) - (a - \lambda)\| < \frac{1}{\|(a - \lambda)^{-1}\|} \quad \text{for all } n \geq n_0.$$

Hence $(a_n - \lambda_n)$ is invertible for $n \geq n_0$. Consequently, there exists a subsequence $\{a_{n_k} - \lambda_{n_k}\}$ of $\{a_n - \lambda_n\}$ such that $(a_{n_k} - \lambda_{n_k})^{-1} \to (a - \lambda)^{-1}$ as $k \to \infty$. Hence

$$\|(a_{n_k} - \lambda_{n_k})\| \, \|(a_{n_k} - \lambda_{n_k})^{-1}\| \geq \frac{1}{\epsilon_{n_k}}$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

and

$$\|(a_{n_k}-\lambda_{n_k})^{-1}\| \to \|(a-\lambda)^{-1}\|, \quad \|(a_{n_k}-\lambda_{n_k})\| \to \|(a-\lambda)\|, \quad \frac{1}{\epsilon_{n_k}} \to \frac{1}{\epsilon_0} \quad \text{as } k \to \infty.$$

We have $\lambda \in \sigma_{\epsilon_0}(a)$. If $\epsilon_0 = 0$, then it is easy to see that $\lambda \in \sigma(a)$. If $\epsilon_0 = 1$ and $(a-\lambda)$ is invertible, then $\|a-\lambda\|\,\|(a-\lambda)^{-1}\| \geq 1$. In a similar fashion, we can prove that the correspondences C_ϵ and C_a are closed. □

The results about the upper, lower hemicontinuity and continuity of C_a for fixed $a \in \mathcal{A} \setminus \mathbb{C}e$ at given $\epsilon \in (0,1)$ begin from the forthcoming theorem. Before that, we draw attention to the following note.

Note 3.3. The continuity of C for any scalar element $a \in \mathcal{A}$ and $\epsilon \in (0,1)$ follows from Theorem 3.9. From this, the upper and lower hemicontinuity of C_a and C_ϵ at a are also assured. For this reason, we prove the continuity of C_a only for nonscalar $a \in \mathcal{A}$.

THEOREM 3.4. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\epsilon_0 \in (0,1)$, then C_a is upper hemicontinuous at ϵ_0.*

PROOF. Let $\epsilon_n \in (0,1)$ and $\lambda_n \in \sigma_{\epsilon_n}(a)$ such that $\epsilon_n \to \epsilon_0$. If there exists a subsequence $\{n_k\}$ such that $\epsilon_{n_k} < \epsilon_0$, then $\lambda_{n_k} \in \sigma_{\epsilon_0}(a)$. Since $\sigma_{\epsilon_0}(a)$ is compact, λ_n has a limit point $\lambda \in \sigma_{\epsilon_0}(a)$. By Theorem 2.4, C_a is upper hemicontinuous at ϵ_0.

Suppose there are only finitely many $\epsilon_n < \epsilon_0$. Then there exists a decreasing subsequence $\{\epsilon_{n_k}\}$ with $\epsilon_{n_k} > \epsilon_0$. Fix n_1; clearly, $\lambda_{n_k} \in \sigma_{\epsilon_{n_1}}(a)$ for all $n_k \geq n_1$. Since $\sigma_{\epsilon_{n_1}}(a)$ is compact, λ_{n_k} has a limit point λ in $\sigma_{\epsilon_{n_1}}(a)$. We prove that $\lambda \in \sigma_{\epsilon_0}(a)$. It is clear that the sequence $\{(\epsilon_{n_k}, \lambda_{n_k})\}$ from graph of C_a and $(\epsilon_{n_k}, \lambda_{n_k}) \to (\epsilon_0, \lambda)$ as $k \to \infty$. Since the graph of C_a is closed (Lemma 3.2), we have $(\epsilon_0, \lambda) \in Gr(C_a)$. This implies that $\lambda \in \sigma_{\epsilon_0}(a)$. By Theorem 2.4, C_a is upper hemicontinuous at ϵ_0. □

Unlike the upper hemicontinuity, the lower hemicontinuity of C_a for $a \in \mathcal{A}$ and at given ϵ_0 needs an extra assumption on the interior of $L_{\epsilon_0}(a)$.

THEOREM 3.5. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The correspondence C_a is lower hemicontinuous at $\epsilon_0 \in (0,1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

PROOF. Assume that the interior of $L_{\epsilon_0}(a)$ is empty. Let V be a nonempty open subset in $\mathbb{C}$ such that $\sigma_{\epsilon_0}(a) \cap V \neq \emptyset$. For any $\epsilon > \epsilon_0$, by Theorem 2.14, $\sigma_\epsilon(a) \cap V \neq \emptyset$.

If there exists $\epsilon \in (0,1)$ with $\epsilon < \epsilon_0$ such that $\sigma_\epsilon(a) \cap V \neq \emptyset$, then choose $\delta = (\epsilon_0 - \epsilon)/2$. By Theorem 2.14, $\sigma_\epsilon(a) \cap V \neq \emptyset$ for all $\epsilon \in (\epsilon_0 - \delta, \epsilon_0 + \delta)$, which yields the lower hemicontinuity of C_a at ϵ_0.

Suppose that, for every $\epsilon < \epsilon_0$,

$$\sigma_\epsilon(a) \cap V = \emptyset. \tag{3.2}$$

Now, for any $\mu \in \sigma_{\epsilon_0}(a) \cap V$,

$$\frac{1}{\epsilon_0} \leq \|a-\mu\|\,\|(a-\mu)^{-1}\| < \frac{1}{\epsilon_0 - \frac{1}{m}} \quad \text{for all } m > \frac{1}{\epsilon_0}.$$

This gives $\mu \in L_{\epsilon_0}(a)$. There exists $r > 0$ such that $B(\mu, r) \subseteq V$. By Equation (3.2),

$$\|a - \lambda\| \, \|(a - \lambda)^{-1}\| \leq \frac{1}{\epsilon_0} \quad \text{for all } \lambda \in B(\mu, r).$$

Since the interior of $L_{\epsilon_0}(a)$ is empty, there exists $\lambda_0 \in B(\mu, r)$ such that

$$\|a - \lambda_0\| \, \|(a - \lambda_0)^{-1}\| < \frac{1}{\epsilon_0}.$$

Take $\Omega = \Omega_0 = B(\mu, r)$ and apply Lemma 2.13 to the analytic vector valued maps ψ_1, ψ_2 from Ω_0 to $\mathcal{A}$ defined by $\psi_1(\lambda) = (a - \lambda)$ and $\psi_2(\lambda) = (a - \lambda)^{-1}$. This gives $\mu \notin L_{\epsilon_0}(a)$, which is a contradiction.

Conversely, we assume that C_a is lower hemicontinuous at ϵ_0. We prove that the interior of $L_{\epsilon_0}(a)$ is empty. Suppose that if the interior of $L_{\epsilon_0}(a)$ is nonempty, then there exists $\mu \in L_{\epsilon_0}(a)$ and $r > 0$ such that $B(\mu, r) \subsetneq L_{\epsilon_0}(a) \subseteq \sigma_{\epsilon_0}(a)$. Clearly, $B(\mu, r) \cap \sigma_\epsilon(a) = \emptyset$ for all $0 < \epsilon < \epsilon_0$. This is in contradiction to C_a being lower hemicontinuous at ϵ_0. $\square$

THEOREM 3.6. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The correspondence C_a is continuous at $\epsilon_0 \in (0, 1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

PROOF. This is immediate from Theorems 3.4 and 3.5. $\square$

COROLLARY 3.7. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The map $\mathfrak{C}_a : (0, 1) \to K(\mathbb{C})$, defined by $\mathfrak{C}_a(\epsilon) = \sigma_\epsilon(a)$, is continuous at $\epsilon_0 \in (0, 1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

PROOF. This is immediate from Theorems 3.6 and 2.8. $\square$

REMARK 3.8. The central theme of [13] is to identify and classify the Banach algebras $\mathcal{A}$ in which the interior of $L_\epsilon(a)$ is empty for any $a \in \mathcal{A}$ and $\epsilon \in (0, 1)$. If $\mathcal{A}$ is a finite-dimensional Banach algebra or $\mathcal{A}$ is a Banach algebra of continuous linear operators defined on the Banach space X, where X or X^*(dual space) is a complex uniformly convex Banach space, then the interior of $L_\epsilon(a)$ is empty for $\epsilon \in (0, 1)$ and for all $a \in \mathcal{A}$. Because of these facts, the correspondence C_a is continuous in those Banach algebras $\mathcal{A}$, for elements $a \in \mathcal{A} \setminus \mathbb{C}e$.

THEOREM 3.9. *The function $\mathfrak{C} : (0, 1) \times \mathcal{A} \to K(\mathbb{C})$, defined by $\mathfrak{C}(\epsilon, b) = \sigma_\epsilon(b)$, is continuous at $(\epsilon_0, \lambda) \in (0, 1) \times \mathcal{A}$, where $\lambda \in \mathbb{C}$.*

PROOF. Consider a sequence $\{(\epsilon_n, a_n)\}$, where $a_n \in \mathcal{A}$ and $\epsilon_n \in (0, 1)$ such that $(\epsilon_n, a_n) \to (\epsilon_0, \lambda)$ as $n \to \infty$. This implies that $a_n \to \lambda, \epsilon_n \to \epsilon$ as $n \to \infty$. We claim that $\sigma_{\epsilon_n}(a_n) \to \sigma_\epsilon(\lambda)$ as $n \to \infty$ in the Hausdroff metric on $K(\mathbb{C})$. We first prove this

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

theorem for $\lambda = 0$. If $\lambda = 0$, then $\sigma_\epsilon(\lambda) = \{0\}$ for all $\epsilon \in (0, 1)$. We observe that

$$\begin{aligned}
H(\sigma_{\epsilon_n}(a_n), \sigma_\epsilon(\lambda)) &= H(\sigma_{\epsilon_n}(a_n), \{0\}) \\
&= \max\Big\{\sup_{\lambda \in \sigma_{\epsilon_n}(a_n)} d(\lambda, \{0\}), \sup_{\mu \in \{0\}} d(\mu, \sigma_{\epsilon_n}(a_n))\Big\} \\
&= \max\Big\{\sup_{\lambda \in \sigma_{\epsilon_n}(a_n)} \inf |\lambda|, \sup_{\mu \in \{0\}} \inf_{\lambda \in \sigma_{\epsilon_n}} |\mu - \lambda|\Big\} \\
&= \sup\{|\mu| : \mu \in \sigma_{\epsilon_n}(a_n)\} \\
&\leq \frac{1 + \epsilon_n}{1 - \epsilon_n} \|a_n\| \quad \text{[by Theorem 2.15]}.
\end{aligned}$$

Since $\|a_n\| \to 0$ and the sequence $\{1 + \epsilon_n/1 - \epsilon_n\}$ is bounded,

$$H(\sigma_{\epsilon_n}(a_n), \sigma_\epsilon(\lambda)) \to 0 \quad \text{as } n \to \infty.$$

Suppose $\lambda \neq 0$. Then we consider the sequence, $b_n = a_n - \lambda$. By our assumption, $b_n \to 0$, and by the above argument the proof of this theorem follows. □

Corollary 3.10. *Let $\lambda \in \mathbb{C}$ and $\epsilon_0 \in (0, 1)$. The correspondence C is continuous at $(\epsilon_0, \lambda) \in (0, 1) \times \mathcal{A}$. Furthermore, C_λ is continuous at ϵ_0.*

Proof. This follows from Theorems 3.9 and 2.8. □

The next lemma presents an upper bound for the perturbed condition spectrum. This lemma plays an essential role in proving the joint continuity of condition spectrum correspondence. Proof of this lemma is similar to [7, proof of Theorem 2.3(7)].

Lemma 3.11. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ with $\eta := \inf\{\|\mu - a\| : \mu \in \mathbb{C}\}$ and $\epsilon \in (0, 1)$. Let $n_0 \in \mathbb{N}$ such that $n_0\eta \geq 2$. If $b \in \mathcal{A}$ such that $\|b\| < \min\{(1 - \epsilon/n_0), \epsilon\eta\}$, then*

$$\sigma_\epsilon(a + b) \subseteq \sigma_{\epsilon + n_0\|b\|}(a).$$

Proof. If $b = 0$, then the result is immediate. Assume that $b \neq 0$. Suppose $\lambda \notin \sigma_{\epsilon + n_0\|b\|}(a)$. Then $\|a - \lambda\| \, \|(a - \lambda)^{-1}\| < 1/\epsilon + n_0\|b\|$. It follows that

$$\|(a - \lambda)^{-1}\| < \frac{1}{(\epsilon + n_0\|b\|)\|a - \lambda\|}. \tag{3.3}$$

By the definition of η and since $n_0\eta \geq 2$, we observe that

$$\|(a + b - \lambda) - (a - \lambda)\| = \|b\| < n_0\eta\|b\| < (\epsilon + n_0\|b\|)\eta \leq (\epsilon + n_0\|b\|)\|a - \lambda\|.$$

By Equation (3.3), $\|(a + b - \lambda) - (a - \lambda)\| < 1/\|(a - \lambda)^{-1}\|$. Hence, $\lambda \notin \sigma(a + b)$. Next

$$\begin{aligned}
\|(a + b - \lambda)^{-1} - (a - \lambda)^{-1}\| &= \|(a + b - \lambda)^{-1}((a - \lambda) - (a + b - \lambda))(a - \lambda)^{-1}\| \\
&\leq \|(a + b - \lambda)^{-1}\| \, \|b\| \, \|(a - \lambda)^{-1}\|.
\end{aligned}$$

By Equation (3.3),

$$\|(a + b - \lambda)^{-1} - (a - \lambda)^{-1}\| < \frac{\|(a + b - \lambda)^{-1}\| \, \|b\|}{(\epsilon + n_0\|b\|)\|a - \lambda\|}.$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

Now,

$$\begin{aligned}\|(a+b-\lambda)^{-1}\|\,\|(a+b-\lambda)\| &= \|(a+b-\lambda)\|\,\|(a+b-\lambda)^{-1}-(a-\lambda)^{-1}+(a-\lambda)^{-1}\|\\ &< \|(a+b-\lambda)\|\frac{\|(a+b-\lambda)^{-1}\|\,\|b\|}{(\epsilon+n_0\|b\|)\|a-\lambda\|}+\frac{\|(a+b-\lambda)\|}{(\epsilon+n_0\|b\|)\|a-\lambda\|}.\end{aligned}$$

Moving the first part on the right-hand side of the inequality to the left and combining, it follows that

$$\|(a+b-\lambda)^{-1}\|\,\|(a+b-\lambda)\| < \frac{\|(a+b-\lambda)\|}{(\epsilon+n_0\|b\|)\|a-\lambda\|-\|b\|}. \tag{3.4}$$

Hence

$$\begin{aligned}(\epsilon+n_0\|b\|)\|a-\lambda\|-\|b\| &= \epsilon\|a-\lambda\|+n_0\|b\|\,\|a-\lambda\|-\|b\|\\ &\geq \epsilon\|a-\lambda\|+n_0\eta\|b\|-\|b\|(\because \|a-\lambda\|\geq\eta)\\ &= \epsilon\|a-\lambda\|+(n_0\eta-1)\|b\|\\ &\geq \epsilon\|a-\lambda\|+\|b\|(\because n_0\eta\geq 2)\\ &> \epsilon\|a-\lambda\|+\epsilon\|b\|(\because 1>\epsilon)\\ &> \epsilon\|a+b-\lambda\|.\end{aligned}$$

Substituting this into Equation (3.4) implies that $\lambda \notin \sigma_\epsilon(a+b)$. □

Now, our goal is to show that C and C_ϵ are continuous by assuming that C_a is continuous for every $a \in \mathcal{A}$. But here we demonstrate the proof of continuity of $\mathfrak{C}$ and $\mathfrak{C}_\epsilon$ (which are defined in Theorem 3.12) by presuming that $\mathfrak{C}_a$ (defined in Theorem 3.12) is continuous for all $a \in \mathcal{A}$. The main idea of the proof can be found in [7, Theorem 4.3].

Theorem 3.12. *Suppose that the function*

$$\mathfrak{C}_a : (0,1) \to K(\mathbb{C}) \text{ defined by } \mathfrak{C}_a(\epsilon) = \sigma_\epsilon(a)$$

is continuous at ϵ_0 for every $a \in \mathcal{A}$. Then the function

$$\mathfrak{C}_{\epsilon_0} : \mathcal{A} \to K(\mathbb{C}) \text{ defined by } \mathfrak{C}_{\epsilon_0}(a) = \sigma_{\epsilon_0}(a)$$

is continuous at a with respect to the norm on $\mathcal{A}$, and the function

$$\mathfrak{C} : (0,1)\times\mathcal{A} \to K(\mathbb{C}) \text{ defined by } \mathfrak{C}(\epsilon,a) = \sigma_\epsilon(a)$$

is continuous at (ϵ_0, a) with respect to the metric defined by Equation (1.2).

Proof. Suppose $a = \lambda$ for some $\lambda \in \mathbb{C}$. Then the conclusion follows from Theorem 3.9.

Assume that $a \in \mathcal{A}\setminus\mathbb{C}e$ and $\epsilon_0 \in (0,1)$. For $r > 0$, consider the open ball

$$B(\sigma_{\epsilon_0}(a), r) := \{D \in K(\mathbb{C}) : H(D, \sigma_{\epsilon_0}(a)) < r\}.$$

Since $\mathfrak{C}_a$ is continuous at ϵ_0, there exists $\delta \in (0,1)$ with $\epsilon_0 + \delta < 1$ such that

$$\sigma_\epsilon(a) \in B\Big(\sigma_{\epsilon_0}(a), \frac{r}{2}\Big) \quad \text{for all } \epsilon \in (\epsilon_0-\delta, \epsilon_0+\delta). \tag{3.5}$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

Take $\eta := \inf\{\|\mu - a\| : \mu \in \mathbb{C}\}$. Note that $\eta > 0$. For any

$$b \in B\Big(a, \frac{\eta}{2}\Big) := \Big\{b \in \mathcal{A} : \|a - b\| < \frac{\eta}{2}\Big\}$$

and $\lambda \in \mathbb{C}$, $\|b - \lambda\| \geq \eta/2$. Hence $\inf_{b \in B(a,(\eta/2))}\{\inf\{\|b - \lambda\| : \lambda \in \mathbb{C}\}\} \geq \eta/2$. Let $n_0 \in \mathbb{N}$ such that $n_0(\eta/2) > 2$. Choose $\nu \in E$ such that $\nu < \min\{\epsilon_0/4n_0, \delta/4n_0, 1 - \epsilon_0/4n_0, \epsilon_0\eta/4\}$. We claim that if $(\epsilon, b) \in B((\epsilon_0, a), \nu)) := \{(\epsilon, b) \in (0, 1) \times \mathcal{A} : \|a - b\| + |\epsilon_0 - \epsilon| < \nu\}$, then $\sigma_\epsilon(b) \in B(\sigma_{\epsilon_0}(a), r)$. Let $(\epsilon, b) \in B((\epsilon_0, a), \nu))$. We calculate $H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b))$. Take $c = b - a$. We observe that

$$\epsilon_0 - \nu - n_0\|c\| \geq \frac{\epsilon_0}{2}\Big(1 - \frac{1}{2n_0}\Big) > 0 \tag{3.6}$$

$$\epsilon_0 + \nu + n_0\|c\| \leq \epsilon_0 + \frac{1 - \epsilon_0}{4n_0} + n_0\frac{1 - \epsilon_0}{4n_0} < 1 \tag{3.7}$$

$$\epsilon_0 - \nu + \|c\| < \epsilon_0 + (\epsilon - \epsilon_0) = \epsilon < 1 \tag{3.8}$$

and

$$\epsilon + \|c\| < \epsilon + \nu + (\epsilon_0 - \epsilon) < \epsilon_0 + \nu < 1. \tag{3.9}$$

By Equations (3.6) and (3.7), the sets $\sigma_{\epsilon_0 - \nu - n_0\|c\|}(a)$ and $\sigma_{\epsilon_0 + \nu + n_0\|c\|}(a)$ are well defined. Now,

$$\begin{aligned}
\sigma_{\epsilon_0 - \nu - n_0\|c\|}(a) &= \sigma_{\epsilon_0 - \nu - n_0\|c\|}(b - c) && \text{[by assumption } c = b - a\text{]}\\
&\subseteq \sigma_{\epsilon_0 - \nu}(b) && \text{[by Lemma 3.11]}\\
&\subseteq \sigma_{\epsilon_0 - \nu + \|c\|}(b) && \text{[by Theorem 2.14]}\\
&\subseteq \sigma_\epsilon(b) && \text{[by Equation (3.8) and by Theorem 2.14]}\\
&= \sigma_\epsilon(a + c) && \text{[by assumption } c = b - a\text{]}\\
&\subseteq \sigma_{\epsilon + n_0\|c\|}(a) && \text{[by Lemma 3.11]}\\
&\subseteq \sigma_{\epsilon_0 + \nu + n_0\|c\|}(a) && \text{[by Equation (3.9) and by Theorem 2.14].}
\end{aligned}$$

Finally, from the above calculation,

$$\sigma_{\epsilon_0 - \nu - n_0\|c\|}(a) \subseteq \sigma_\epsilon(b) \subseteq \sigma_{\epsilon_0 + \nu + n_0\|c\|}(a).$$

Now,

$$\begin{aligned}
H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b)) &= \max\Big\{\sup_{\lambda \in \sigma_{\epsilon_0}(a)} d(\lambda, \sigma_\epsilon(b)), \sup_{\mu \in \sigma_\epsilon(b)} d(\mu, \sigma_{\epsilon_0}(a))\Big\}\\
&\leq \max\Big\{\sup_{\lambda \in \sigma_{\epsilon_0}(a)} d(\lambda, \sigma_{\epsilon_0 - \nu - n_0\|c\|}(a)), \sup_{\mu \in \sigma_{\epsilon_0 + \nu + n_0\|c\|}(a)} d(\mu, \sigma_{\epsilon_0}(a))\Big\}\\
&\leq \max\{H(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0 - \nu - n_0\|c\|}(a)), H(\sigma_{\epsilon_0 + \nu + n_0\|c\|}(a), \sigma_{\epsilon_0}(a))\}. \quad (3.10)
\end{aligned}$$

We observe that

$$|\epsilon_0 - (\epsilon_0 - \nu - n_0\|c\|)| = \nu + n_0\|c\| < (n_0 + 1)\nu < (2n_0)\nu < (2n_0)\frac{\delta}{4n_0} = \frac{\delta}{2}.$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

Similarly, $|\epsilon_0 - (\epsilon_0 + \nu + n_0\|c\|)| = \nu + n_0\|c\| < \delta/2$. By Equation (3.5),

$$H(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0-\nu-n_0\|c\|}(a)) \leq \frac{r}{2} \quad \text{and} \quad H(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0+\nu+n_0\|c\|}(a)) \leq \frac{r}{2}.$$

By Equation (3.10), $H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b)) < r$. This proves the theorem. □

Corollary 3.13. *Suppose C_a is continuous at $\epsilon_0 \in (0,1)$. Then C_{ϵ_0} is continuous at $a \in \mathcal{A}$ with respect to the norm on $\mathcal{A}$. Further, the correspondence C is jointly continuous at (ϵ_0, a).*

Proof. This theorem follows from Theorems 3.12 and 2.8. □

Remark 3.14. For a fixed $a \in \mathcal{A}$, the assumption that $\mathfrak{C}_a$ and C_a are continuous at $\epsilon_0 \in (0,1)$ can be replaced by the assumption that the interior of $L_{\epsilon_0}(a)$ is empty for given ϵ_0 and a. After this replacement, the proof of Theorem 3.12 and Corollary 3.13 follows from Corollary 3.7 and Theorems 3.6 and 3.9.

The following lemma gives a fine picture about the growth of the condition spectrum when the value of ϵ approaches 1. With the aid of this lemma, we look at the limiting behaviour of $\sigma_\epsilon(a)$ as $\epsilon \to 1$.

Lemma 3.15. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $K \subsetneq \mathbb{C}$ is compact with $K \cap L_1(a) = \emptyset$, then there exists an $\epsilon \in (0,1)$ such that $K \subsetneq \sigma_\epsilon(a)$.*

Proof. Take $\lambda \in K$. We first prove that there exists $\delta' \in (0,1)$ such that $\lambda \in \sigma_{\delta'}(a)$. If $\lambda \in \sigma(a)$, then $\lambda \in \sigma_{\delta'}(a)$ for all $0 < \delta' < 1$. If $\lambda \notin \sigma(a)$, then

$$\|a - \lambda\| \, \|(a - \lambda)^{-1}\| = M$$

for some $M > 1$ (since $K \cap L_1(a) = \emptyset$). Take $\delta' = 1/M$. It follows that $\lambda \in \sigma_{1/M}(a)$. Hence, for any $0 < \delta' < \delta < 1$,

$$\|a - \lambda\| \, \|(a - \lambda)^{-1}\| > \frac{1}{\delta}.$$

Hence λ is an interior point of $\sigma_\delta(a)$. There exists an $r_\lambda > 0$ such that $B(\lambda, r_\lambda) \subsetneq \sigma_\delta(a)$. Thus the collection $\{B(\lambda, r_\lambda) : \lambda \in K\}$ is an open cover for K. Since K is compact, we have $K \subseteq \bigcup_{i=1}^n B(\lambda_i, r_{\lambda_i})$. Since each $B(\lambda_i, r_{\lambda_i}) \subsetneq \sigma_{\delta_i}(a)$, by taking $\epsilon = \max\{\delta_i : i = 1 \text{ to } n\}$, we get $K \subsetneq \sigma_\epsilon(a)$. □

Note 3.16. For any fixed $a \in \mathcal{A}$, by Theorem 2.14 and Note 2.11, we have $\lim_{\epsilon\to 0} \sigma_\epsilon(a) = \sigma(a)$.

Consider $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a)$ has either one element or has more than two elements, then [13, Theorems 3.5 and 3.8] tell us that $L_1(a)$ has at most one element, and if $\sigma(a) = \{\lambda_1, \lambda_2\}$ for some complex numbers λ_1 and λ_2, then, by [13, Lemma 3.4], the elements of $L_1(a)$ belongs to the perpendicular bisector of the line segment joining λ_1, λ_2. These facts imply that $L_1(a)$ has empty interior. Thus, by Theorem 3.17, we obtain that $\sigma_\epsilon(a)$ grows to $\mathbb{C}$ as $\epsilon \to 1$.

Theorem 3.17. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. Consider the sequence $\{\epsilon_n\}$, where $\epsilon_n \in (0,1)$. If $\lim_{n\to\infty} \epsilon_n = 1$, then $\lim_{n\to\infty} \sigma_{\epsilon_n}(a) = \mathbb{C}$.*

Proof. Let $\lambda \in \mathbb{C} \setminus L_1(a)$. There exists a compact set K such that $\lambda \in K$ and $K \cap L_1(a) = \emptyset$. By Lemma 3.15, $K \subset \sigma_\epsilon(a)$ for some $\epsilon \in (0,1)$. Since $\lim_{n\to\infty} \epsilon_n = 1$,

$$d(\lambda, \sigma_{\epsilon_n}(a)) = 0 \quad \text{for all } n \geq n_0$$

for some $n_0 \in \mathbb{N}$. Hence $\lambda \in \liminf_{n\to\infty} \sigma_{\epsilon_n}(a)$.

If $\lambda \in L_1(a)$, then, by Lemma 3.15, there exists a sequence $\{\lambda_k\}$ and a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that $\lambda_k \in \sigma_{\epsilon_{n_k}}(a)$ and $|\lambda - \lambda_k| < 1/n_k$. From this, it follows that $\lim_{k\to\infty} d(\lambda, \sigma_{\epsilon_{n_k}}(a)) = 0$. For given $\delta > 0$, there exists $n_k > 0$ such that $1/n_k < \delta$. By Theorem 2.14, for any n which satisfies $n_k \leq n$ such that $d(\lambda, \sigma_{\epsilon_n}(a)) \leq d(\lambda, \sigma_{\epsilon_{n_k}}(a)) < 1/n_k < \delta$, we have $\lambda \in \liminf_{n\to\infty} \sigma_{\epsilon_n}(a)$. The proof of the theorem follows from Definition 2.9 and Note 2.10. □

Next, we noticed that the condition spectrum map defined on the set of all normal matrices enjoys uniform continuity for a fixed $\epsilon \in (0,1)$. In this regard, Theorem 3.18 identifies the condition spectrum set for a given normal matrix. Theorem 3.19 deals with uniform continuity.

In the following theorem, $M_n(\mathbb{C})$ denotes the Banach algebra of the set of all $n \times n$ matrices. If $A \in M_n(\mathbb{C})$, then the norm is defined as $\|A\| = s_{\max}(A)$, where $s_{\max}(A)$ denotes the maximum singular value of A (see [6, Example 5.6.6]).

Theorem 3.18. *Suppose A is normal and $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Then*

$$\sigma_\epsilon(A) = \bigcup_{i,j\in \underline{k}} \overline{B}\Big(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\Big), \tag{3.11}$$

where $\underline{k} = \{1, 2, \ldots, k\}$.

Proof. Assume that A is normal and $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Then $A = U^*DU$, where $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. We observe that

$$\begin{aligned}
\lambda \in \sigma_\epsilon(A) \setminus \sigma(A) &\Leftrightarrow \|(A-\lambda)\|\,\|(A-\lambda)^{-1}\| \geq \frac{1}{\epsilon} \\
&\Leftrightarrow \frac{\max_{i\in\underline{k}} |\lambda - \lambda_i|}{\min_{j\in\underline{k}} |\lambda - \lambda_j|} \geq \frac{1}{\epsilon} \\
&\Leftrightarrow \frac{|\lambda - \lambda_m|}{|\lambda - \lambda_l|} \geq \frac{1}{\epsilon} \quad (\text{for some } l, m \in \underline{k}).
\end{aligned}$$

Consider the inequality

$$\epsilon|\lambda - \lambda_m| \geq |\lambda - \lambda_l|.$$

Apply $\lambda = x + iy$, $\lambda_m = a_r + ia_i$ and $\lambda_l = b_r + ib_i$ and square both sides. This gives

$$\epsilon^2[(x-a_r)^2 + (y-a_i)^2] \geq (x-b_r)^2 + (y-b_i)^2.$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

Rearrange and expand to get

$$(1-\epsilon^2)x^2 + (1-\epsilon^2)y^2 - 2(b_r - \epsilon^2 a_r)x - 2(b_i - \epsilon^2 a_i)y + |\lambda_l|^2 - \epsilon^2|\lambda_m|^2 \le 0.$$

Now complete the square

$$\begin{aligned}
(1-\epsilon^2)\Big(x - \frac{b_r - \epsilon^2 a_r}{1-\epsilon^2}\Big)^2 &+ (1-\epsilon^2)\Big(y - \frac{b_i - \epsilon^2 a_i}{1-\epsilon^2}\Big)^2 \\
&\le \frac{(b_r - \epsilon^2 a_r)^2 + (b_i - \epsilon^2 a_i)^2}{(1-\epsilon^2)} - (|\lambda_l|^2 - \epsilon^2|\lambda_m|^2) \\
&= \frac{|\lambda_l|^2 + \epsilon^4|\lambda_m|^2 - 2\epsilon^2(b_r a_r + b_i a_i) - (|\lambda_l|^2 - \epsilon^2|\lambda_m|^2)(1-\epsilon^2)}{(1-\epsilon^2)} \\
&= \frac{\epsilon^2(|\lambda_m|^2 + |\lambda_l|^2) - 2\epsilon^2(a_r b_r + a_i b_i)}{(1-\epsilon^2)} \\
&= \frac{\epsilon^2}{(1-\epsilon^2)}|\lambda_m - \lambda_l|^2.
\end{aligned}$$

From here,

$$\Big|\lambda - \frac{\lambda_l - \epsilon^2\lambda_m}{1-\epsilon^2}\Big| \le \frac{\epsilon}{1-\epsilon^2}|\lambda_m - \lambda_l|.$$

Hence, $\lambda \in \sigma_\epsilon(A) \setminus \sigma(A)$ if and only if

$$\lambda \in \bigcup_{i,j\in \underline{k}, i\neq j} \overline{B}\Big(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\Big).$$

Since $\sigma_\epsilon(A)$ has no isolated points,

$$\sigma_\epsilon(A) = \bigcup_{i,j\in \underline{k}} \overline{B}\Big(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\Big). \qquad \square$$

Theorem 3.19. *Let A and B are two normal matrices. If $\epsilon \in (0,1)$, then*

$$H(\sigma_\epsilon(A), \sigma_\epsilon(B)) \le \frac{1+\epsilon}{1-\epsilon}\|A - B\|.$$

Proof. By [3, Theorem 17, Lecture 18], it is known that $H(\sigma(A), \sigma(B)) \le \|A - B\|$. Let $\mu \in \sigma_\epsilon(A)$. Since A and B are normal, by Equation (3.11), $\mu = (\mu_i - \epsilon^2\mu_j/1-\epsilon^2) + \delta_1$ for some $\delta_1 \le (\epsilon/1-\epsilon^2)|\mu_i - \mu_j|$ for some $\mu_i, \mu_j \in \sigma(A)$. For any $\lambda_k, \lambda_l \in \sigma(B)$, consider the scalar

$$\lambda = \frac{\lambda_k - \epsilon^2\lambda_l}{1-\epsilon^2} + \frac{\epsilon}{1-\epsilon^2}|\lambda_k - \lambda_l|.$$

Clearly, $\lambda \in \sigma_\epsilon(B)$ and

$$\begin{aligned}
|\lambda - \mu| &\le \Big|\frac{(\mu_i - \lambda_k) + \epsilon^2(\lambda_l - \mu_j)}{1-\epsilon^2}\Big| + \frac{\epsilon}{1-\epsilon^2}(|\mu_i - \lambda_k| + |\mu_j - \lambda_l|) \\
&= \frac{1+\epsilon}{1-\epsilon}\|A - B\| \quad \text{(By Theorem 17, Lecture 18 in [3]).} \qquad \square
\end{aligned}$$

Question 3.20. Can an analog of Theorem 3.19 be obtained for normal elements of any C^* algebra?

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

4. Continuity of level sets of ϵ-condition spectrum

This section presents results for the continuity of level sets of ϵ-condition spectra and the necessity of an empty interior for level sets in continuity. The upper hemicontinuity of the level set correspondence $\mathcal{LC}_a$ follows in a similar way to the upper hemicontinuity of the condition spectrum correspondence C_a without needing any further assumptions.

Note that, if $a = \lambda$ for some $\lambda \in \mathbb{C}$, then $L_\epsilon(a) = \emptyset$ for any $\epsilon \in (0, 1)$. Thus it is trivial that the correspondence $\mathcal{LC}_a, \mathcal{LC}_\epsilon$ and $\mathcal{LC}$ are continuous at a and $\epsilon \in (0, 1)$. For this reason, we concentrate on the nonscalar elements in $\mathcal{A}$ only.

LEMMA 4.1. *The graph of the correspondence $\mathcal{LC}$ is closed. Further, the correspondences $\mathcal{LC}_a$ and $\mathcal{LC}_\epsilon$ are closed for fixed $a \in \mathcal{A}$ and $\epsilon \in (0, 1)$.*

PROOF. Consider the sequence $\{(\epsilon_n, a_n), \lambda_n\}$ in $Gr(\mathcal{LC})$ and $((\epsilon_0, a), \lambda) \in ((0, 1) \times \mathcal{A}) \times \mathbb{C}$. Here $((0, 1) \times \mathcal{A}) \times \mathbb{C}$ is a metric space whose metric is defined in Equation (3.1). Suppose $((\epsilon_n, a_n), \lambda_n) \to ((\epsilon_0, a), \lambda)$ as $n \to \infty$. Then $\epsilon_n \to \epsilon_0, a_n \to a$ and $\lambda_n \to \lambda$ as $n \to \infty$. Clearly,

$$\|(a_n - \lambda_n)^{-1}\| = \frac{1}{\epsilon_n \|(a_n - \lambda_n)\|}$$

and $\epsilon_n \|(a_n - \lambda_n)\| \to \epsilon_0 \|a - \lambda\|$. If we take $b_n = (a_{n_k} - \lambda_{n_k})$ and $b = a - \lambda$, then, by Lemma 2.12, $a - \lambda$ is invertible. Since $\|(a_n - \lambda_n)^{-1}\| \to \|(a - \lambda)^{-1}\|$, $\lambda \in L_{\epsilon_0}(a)$. Hence, the graph of $\mathcal{LC}$ is closed. Similarly, we can prove that $\mathcal{LC}_a$ and $\mathcal{LC}_\epsilon$ are closed. □

REMARK 4.2. For $\epsilon \in (0, 1)$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, it is evident that $L_\epsilon(a) \subsetneq \sigma_\epsilon(a)$. By Lemma 4.1, we understand that the correspondences $\mathcal{LC}_a, \mathcal{LC}_\epsilon$ and $\mathcal{LC}$ are closed subcorrespondences of C_a, C_ϵ and C, respectively.

THEOREM 4.3. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ and $\epsilon_0 \in (0, 1)$. The correspondence $\mathcal{LC}_a$ is upper hemicontinuous at ϵ_0 and $\mathcal{LC}_{\epsilon_0}$ is upper hemicontinuous at a.*

PROOF. This follows from Remark 4.2 and Theorems 2.7, 3.4 and 3.1. □

In general, the correspondence $\mathcal{LC}_a$ and $\mathcal{LC}_\epsilon$ need not be lower hemicontinuous even though the interior of $L_\epsilon(a)$ is empty for given a and ϵ. By looking at a particular Banach algebra $\mathcal{A}$ and an appropriate element $a \in \mathcal{A}$, we furnish an example to show that the correspondence $\epsilon \mapsto L_\epsilon(a)$ is not lower hemicontinuous at some ϵ_0 and that, for a fixed $\epsilon_0 \in (0, 1)$, the correspondence $b \mapsto L_\epsilon(b)$ is not lower hemicontinuous at a. The following example has been found with the help of one of the famous pioneering illustrations that were constructed by Shargorodsky in [12]. This example shows that the level set of a pseudospectrum need not be empty.

EXAMPLE 4.4. Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm

$$\|x\|_* = |x_0| + \sup_{n \neq 0} |x_n| \quad \text{where } x = (\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots),$$

https://doi.org/10.1017/S1446788719000338 Published online by Cambridge University Press

where the box represents the zeroth coordinate of an element in $\ell_\infty(\mathbb{Z})$. For $M > 2$, consider an operator $A \in B(\ell_\infty(\mathbb{Z}))$ such that

$$A(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots) = \bigg(\ldots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{M}}, x_2, x_3, \ldots\bigg).$$

We first show that the interior of $L_{\epsilon_0}(A)$ is empty for $\epsilon_0 = 1/M + 1$. It is proved in [12, Theorem 3.1] that

$$\|(A-\lambda)^{-1}\| = M \quad \text{for } |\lambda| < \min\Big\{\frac{1}{M}, \frac{1}{2} - \frac{1}{M}\Big\}. \tag{4.1}$$

Take $r = \min\{(1/M), (1/2) - (1/M)\}$. From [7, Example 4.9], we have $\sigma(A) = \{z \in \mathbb{C} : |z| = 1\}$ and

$$\|(A-\lambda)^{-1}\| \geq M \quad \text{for } |\lambda| < 1. \tag{4.2}$$

It is easy to see, with unit vectors $y = (y_k)_{k=-\infty}^{\infty}$ such that

$$y_k = \begin{cases} 1 & \text{for } k = 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $z = (z_k)_{k=-\infty}^{\infty}$ such that

$$z_k = \begin{cases} 1 & \text{for } k = 1, 4, \\ -\overline{\lambda} & \text{for } k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < |\lambda| < 1$,that

$$\|A\| = \frac{1}{M} + 1 \quad \text{and} \quad \|A - \lambda\| \geq \frac{1}{M} + 1 + |\lambda|^2 \quad \text{for } 0 < |\lambda| < 1. \tag{4.3}$$

By Equations (4.1) and (4.3), we get $\|A\|\,\|A^{-1}\| = M + 1$ and, by Equations (4.2) and (4.3),

$$\|A-\lambda\|\,\|(A-\lambda)^{-1}\| \geq M + 1 + M|\lambda|^2 > M + 1 \quad \text{for } 0 < |\lambda| < 1. \tag{4.4}$$

Thus $B(0, 1) \cap L_{\epsilon_0}(A) = \{0\}$. Hence the interior of $L_{\epsilon_0}(A)$ is empty in the set $B(0, 1)$. By [13, Corollary 4.3], the interior of $L_{\epsilon_0}(A)$ is empty in the set $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$.

Now, we show that the correspondence $\epsilon \mapsto L_\epsilon(A)$ is not lower hemicontinuous at ϵ_0.

For any $\epsilon > 1/M + 1$, it is clear that $L_\epsilon(A) \cap B(0, r) = \emptyset$ but $L_{\epsilon_0}(A) \cap B(0, r) \neq \emptyset$. Hence the correspondence $\mathcal{LC}_A$ is not lower hemicontinuous at ϵ_0.

Next, we prove the the correspondence $\mathcal{LC}_{\epsilon_0}$ is not lower hemicontinuous at A.

Let $\delta > 0$ and consider the set $\{S \in B(\ell_\infty(\mathbb{Z})) : \|A - S\| < \delta\}$. Choose $N \in \mathbb{N}$ such that $N > M > 2$ and $(1/M) - (1/N) < \delta$. Take $B \in B(\ell_\infty(\mathbb{Z}))$ such that

$$B(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots) = \bigg(\ldots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{N}}, x_2, x_3, \ldots\bigg).$$

By [7, Example 4.9], $\sigma(B) = \{z \in \mathbb{C} : |z| = 1\}$. Applying $A = B$ and $M = N$ in Equations (4.1)–(4.4) gives

$$\|(B-\lambda)^{-1}\| = N \quad \text{for } |\lambda| < \min\left\{\frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\}$$

and

$$\|B-\lambda\|\,\|(B-\lambda)^{-1}\| \geq N+1 > M+1 \quad \text{for } 0 \leq |\lambda| < 1.$$

Thus $L_{\epsilon_0}(B) \cap B(0,r) = \emptyset$ but $\|A-B\| < \delta$ and $L_{\epsilon_0}(A) \cap B(0,r) \neq \emptyset$. Hence the correspondence $\mathcal{LC}_{\epsilon_0}$ is not lower hemicontinuous at A.

Next, we pay attention to the continuity of correspondence $\mathcal{LC}$. We show that an empty interior of $L_\epsilon(a)$ is a sufficient condition for the continuity.

Theorem 4.5. *Let (ϵ_0, a) in $(0,1) \times \mathcal{A}$. If the interior of $L_{\epsilon_0}(a)$ is empty, then $\mathcal{LC}$ is jointly upper hemicontinuous at (ϵ_0, a).*

Proof. It is assumed that the interior of $L_{\epsilon_0}(a)$ is empty. By Remark 3.14, C is jointly continuous. In particular, C is jointly upper hemicontinuous at (ϵ_0, a). Since $\mathcal{LC}$ is the closed subcorrespondence of C, it follows, by Theorem 2.7, that $\mathcal{LC}$ is jointly upper hemicontinuous at (ϵ_0, a). □

Acknowledgements

The authors are grateful to the anonymous referees and the editor(s) for their critical comments leading to improvements in the manuscript.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3rd edn (Springer, Berlin, 2006).

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, reprint of the 1990 edition, Modern Birkhäuser Classics (Birkhäuser, Boston, MA, 2009).

[3] R. Bhatia, *Notes on Functional Analysis*, Texts and Readings in Mathematics, 50 (Hindustan Book Agency, New Delhi, 2009).

[4] L. Burlando, 'Continuity of spectrum and spectral radius in Banach algebras', in: *Functional Analysis and Operator Theory (Warsaw, 1992)*, Banach Center Publications, 30 (Institute of Mathematics Polish Academy of Sciences, Warsaw, 1994), 53–100.

[5] J. Globevnik, 'Norm-constant analytic functions and equivalent norms', *Illinois J. Math.* **20**(3) (1976), 503–506.

[6] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd edn (Cambridge University Press, Cambridge, 2013).

[7] A. Krishnan and S. H. Kulkarni, 'Pseudospectrum of an element of a Banach algebra', *Oper. Matrices* **11**(1) (2017), 263–287.

[8] S. H. Kulkarni and D. Sukumar, 'The condition spectrum', *Acta Sci. Math. (Szeged)* **74**(3–4) (2008), 625–641.

[9] J. D. Newburgh, 'The variation of spectra', *Duke Math. J.* **18** (1951), 165–176.

[10] C. E. Rickart, *General Theory of Banach Algebras*, The University Series in Higher Mathematics (van Nostrand, Princeton, NJ, 1960).

[11] E. Shargorodsky, 'On the level sets of the resolvent norm of a linear operator', *Bull. Lond. Math. Soc.* **40**(3) (2008), 493–504.

[12] E. Shargorodsky, 'On the definition of pseudospectra', *Bull. Lond. Math. Soc.* **41**(3) (2009), 524–534.

[13] D. Sukumar and S. Veeramani, 'Level sets of the condition spectrum', *Ann. Funct. Anal.* **8**(3) (2017), 314–328.

D. SUKUMAR, Department of Mathematics,
Indian Institute of Technology, Hyderabad, India
e-mail: suku@iith.ac.in

S. VEERAMANI, Department of Mathematics,
School of Advanced Sciences,
Vellore Institute of Technology, Vellore, India
e-mail: ma13p1005@iith.ac.in