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RICCI CURVATURE OF SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract

Recently, Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. Afterwards, we dealt with similar problems for submanifolds in complex space forms.

In the present paper, we obtain sharp inequalities between the Ricci curvature and the squared mean curvature for submanifolds in Sasakian space forms. Also, estimates of the scalar curvature and the k-Ricci curvature respectively, in terms of the squared mean curvature, are proved.

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1. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi) Y = -g(X, Y)\xi + \eta(Y)X, \quad \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

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A plane section π in $T_p\tilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]

$$\begin{aligned}(1.1)\qquad \tilde{R}(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\},\end{aligned}$$

for any tangent vector fields X, Y, Z on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [1]).

Let M be an n-dimensional submanifold of a Sasakian space form $\tilde{M}(c)$ of constant ϕ-sectional curvature c. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M. Then the equation of Gauss is given by

$$\begin{aligned}(1.2)\qquad \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) \\ - g(h(X, Z), h(Y, W)),\end{aligned}$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space T_pM. We denote by H the mean curvature vector, that is

$$(1.3)\qquad H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

We also set

$$(1.4)\qquad h_{ij}^r = g(h(e_i, e_j), e_r)$$

and

$$(1.5)\qquad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field X to M, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX, respectively. We write

$$(1.6)\qquad \|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$.

Define the *Ricci curvature* Ric_L of L at X by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{1.7}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We simply called such a curvature a *k-Ricci curvature*.

The *scalar curvature* τ of the k-plane section L is given by

$$\tau(L) = \sum_{1 \le i < j \le k} K_{ij}. \tag{1.8}$$

For each integer k, $2 \le k \le n$, the *Riemannian invariant* Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \quad p \in M, \tag{1.9}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannian manifold, the *relative null space* of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM \mid h(X, Y) = 0, \text{ for all } Y \in T_pM\}. \tag{1.10}$$

2. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [4]). We prove similar inequalities for certain submanifolds of a Sasakian space form.

A submanifold M normal to ξ in a Sasakian space form $\tilde{M}(c)$ is called a *C-totally real submanifold*. It follows that ϕ maps any tangent space of M into the normal space, that is $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

THEOREM 2.1. *Let M be an n-dimensional C-totally real submanifold of a* $(2m+1)$-*dimensional Sasakian space form $\tilde{M}(c)$. Then*:

(i) *For each unit vector $X \in T_pM$, we have*

$$\mathrm{Ric}(X) \le \frac{1}{4}\{(n-1)(c+3) + n^2\|H\|^2\}. \tag{2.1}$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of* (2.1) *if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of* (2.1) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

PROOF. (i) Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1} = \xi$, such that $e_1, \dots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)(c+3)/4. \tag{2.2}$$

From (2.2), we get

$$\begin{aligned} n^2\|H\|^2 &= 2\tau + \sum_{r=n+1}^{2m}\left[(h^r_{11})^2 + (h^r_{22} + \dots + h^r_{nn})^2 + 2\sum_{i<j}(h^r_{ij})^2\right] \\ &\quad - 2\sum_{r=n+1}^{2m}\sum_{2\le i<j\le n} h^r_{ii}h^r_{jj} - n(n-1)\frac{c+3}{4} \\ &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m}\left[(h^r_{11} + \dots + h^r_{nn})^2 + (h^r_{11} - h^r_{22} - \dots - h^r_{nn})^2\right] \\ &\quad + 2\sum_{r=n+1}^{2m}\sum_{i<j}(h^r_{ij})^2 - 2\sum_{r=n+1}^{2m}\sum_{2\le i<j\le n} h^r_{ii}h^r_{jj} - n(n-1)\frac{c+3}{4}. \end{aligned} \tag{2.3}$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=n+1}^{2m}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{c+3}{4}, \tag{2.4}$$

and consequently

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m}\sum_{2\le i<j\le n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{(n-1)(n-2)}{2}\frac{c+3}{4}. \tag{2.5}$$

Substituting (2.5) in (2.3), one gets

$$n^2\|H\|^2 \ge 2\tau + \frac{n^2}{2}\|H\|^2 + 2\sum_{r=n+1}^{2m}\sum_{j=2}^{n}(h^r_{1j})^2 - 2\sum_{2\le i<j\le n} K_{ij} - 2(n-1)\frac{c+3}{4}. \tag{2.6}$$

Therefore, $n^2\|H\|^2/2 \ge 2\,\mathrm{Ric}(X) - 2(n-1)(c+3)/4$ or equivalently (2.1).

(ii) Assume $H(p) = 0$. Equality holds in (2.1) if and only if

$$\begin{cases} h^r_{12} = \dots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \dots + h^r_{nn}, \quad r \in \{n+1, \dots, 2m\}. \end{cases} \tag{2.7}$$

Then $h^r_{1j} = 0$, for every $j \in \{1, \ldots, n\}$, $r \in \{n+1, \ldots, 2m\}$, that is $X \in \mathcal{N}_p$.

(iii) The equality case of (2.1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h^r_{ij} = 0, & i \neq j,\ r \in \{n+1, \ldots, 2m\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, & i \in \{1, \ldots, n\},\ r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{2.8}$$

We distinguish two cases

(a) $n \neq 2$, then p is a totally geodesic point;

(b) $n = 2$, it follows that p is a totally umbilical point.

The converse is trivial. □

In the following we will consider submanifolds M tangent to the Reeb vector field ξ.

THEOREM 2.2. *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and M an n-dimensional submanifold tangent to ξ. Then:*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$\mathrm{Ric}(X) \leq \left\{(n-1)(c+3) + (3\|PX\|^2 - 2)(c-1)/2 + n^2\|H\|^2\right\}/4. \tag{2.9}$$

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of* (2.9) *if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of* (2.9) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

PROOF. Let $X \in T_pM$ be a unit tangent vector X at p, orthogonal to ξ. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)(c+3)/4 - (3\|P\|^2 - 2n + 2)(c-1)/4. \tag{2.10}$$

From (2.10), we get

$$\begin{aligned} n^2\|H\|^2 &= 2\tau + \sum_{r=n+1}^{2m+1}\left[(h^r_{11})^2 + (h^r_{22} + \cdots + h^r_{nn})^2 + 2\sum_{i<j}(h^r_{ij})^2\right] \\ &\quad - 2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj} - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n + 2)\frac{c-1}{4} \\ &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}\left[(h^r_{11} + \cdots + h^r_{nn})^2 + (h^r_{11} - h^r_{22} - \cdots - h^r_{nn})^2\right] \end{aligned} \tag{2.11}$$

$$+2\sum_{r=n+1}^{2m+1}\sum_{i<j}(h_{ij}^r)^2-2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}h_{ii}^rh_{jj}^r-n(n-1)\frac{c+3}{4}$$
$$-(3\|P\|^2-2n+2)\frac{c-1}{4}.$$

From the equation of Gauss, we find

$$\sum_{2\le i<j\le n}K_{ij}=\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}[h_{ii}^rh_{jj}^r-(h_{ij}^r)^2]+\frac{(n-1)(n-2)}{2}\frac{c+3}{4}$$
$$+(3\|P\|^2-3\|Pe_1\|^2-2n+4)\frac{c-1}{8}. \tag{2.12}$$

Substituting (2.12) in (2.11), as in the proof of Theorem 2.1 one gets

$$n^2\|H\|^2/2\ge 2\,\mathrm{Ric}(X)-2(n-1)(c+3)/4-(3\|PX\|^2-2)(c-1)/4,$$

which is equivalent to (2.9).

The proofs of (ii) and (iii) are similar to their corresponding statements of Theorem 2.1. In this case, since ξ is tangent to M, it follows that a totally umbilical point is totally geodesic. □

A submanifold M tangent to ξ is said to be *invariant* (respectively *anti-invariant*) if $\phi(T_pM)\subset T_pM$, for every $p\in M$ (respectively $\phi(T_pM)\subset T_p^{\perp}M$, for every $p\in M$).

COROLLARY 2.3. *Let M be an n-dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then*:

(i) *For each unit vector $X\in T_pM$ orthogonal to ξ, we have*

$$\mathrm{Ric}(X)\le\{(n-1)(c+3)+(c-1)/2\}/4. \tag{2.13}$$

(ii) *A unit tangent vector $X\in T_pM$ orthogonal to ξ satisfies the equality case of* (2.13) *if and only if $X\in\mathcal{N}_p$.*

(iii) *The equality case of* (2.13) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

COROLLARY 2.4. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then*:

(i) *For each unit vector $X\in T_pM$ orthogonal to ξ, we have*

$$\mathrm{Ric}(X)\le\{(n-1)(c+3)-(c-1)+n^2\|H\|^2\}/4. \tag{2.14}$$

(ii) *If $H(p)=0$, then a unit tangent vector $X\in T_pM$ orthogonal to ξ satisfies the equality case of* (2.14) *if and only if $X\in\mathcal{N}_p$.*

(iii) *The equality case of* (2.14) *holds identically for all unit tangent vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

A submanifold M tangent to ξ is called a *contact CR-submanifold* [8] if there exists a pair of orthogonal differentiable distributions $\mathscr{D}$ and $\mathscr{D}^\perp$ on M, such that:

(i) $TM = \mathscr{D} \oplus \mathscr{D}^\perp \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ.

(ii) $\mathscr{D}$ is invariant by ϕ, that is $\phi(\mathscr{D}_p) \subset \mathscr{D}_p$, for every $p \in M$.

(iii) $\mathscr{D}^\perp$ is anti-invariant by ϕ, that is $\phi(\mathscr{D}_p^\perp) \subset T_p^\perp M$, for every $p \in M$.

COROLLARY 2.5. *Let* M *be an* n*-dimensional contact* CR*-submanifold of a Sasakian space form* $\tilde{M}(c)$. *Then*:

(i) *For each unit vector* $X \in \mathscr{D}_p$, *we have*

$$\operatorname{Ric}(X) \le \left\{(n-1)(c+3) + (c-1)/2 + n^2\|H\|^2\right\}/4.$$

(ii) *For each unit vector* $X \in \mathscr{D}_p^\perp$, *we have*

$$\operatorname{Ric}(X) \le \left\{(n-1)(c+3) - (c-1) + n^2\|H\|^2\right\}/4.$$

3. k-Ricci curvature

In this section, we prove a relationship between the k-Ricci curvature and the squared mean curvature for submanifolds in Sasakian space forms.

First, we state an inequality between the scalar curvature and the squared mean curvature for C-totally real submanifolds.

THEOREM 3.1. *Let* M *be an* n*-dimensional* C*-totally real submanifold of a Sasakian space form* $\tilde{M}(c)$. *Then we have*

$$\|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{c+3}{4}. \tag{3.1}$$

PROOF. We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1} = \xi\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_r = \left(h_{ij}^r\right),\ i, j = 1, \dots, n;\ r = n+2, \dots, 2m, \quad \operatorname{trace} A_r = \sum_{i=1}^{n} h_{ii}^r = 0.$$

From (2.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - n(n-1)\frac{c+3}{4}. \tag{3.2}$$

On the other hand, since $0 \le \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j$, we obtain

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \le n \sum_{i=1}^{n} a_i^2, \tag{3.3}$$

which implies $\sum_{i=1}^{n} a_i^2 \ge n\|H\|^2$. We have from (3.2)

$$n^2\|H\|^2 \ge 2\tau + n\|H\|^2 - n(n-1)(c+3)/4, \tag{3.4}$$

which is equivalent to (3.1). □

Using Theorem 3.1, we obtain the following.

THEOREM 3.2. *Let M be an n-dimensional C-totally real submanifold M of a Sasakian space form $\tilde{M}(c)$. Then, for any integer k, $2 \le k \le n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \ge \Theta_k(p) - (c+3)/4. \tag{3.5}$$

PROOF. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. It follows from (1.7) and (1.8) that

$$\tau(L_{i_1 \cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \operatorname{Ric}_{L_{i_1 \cdots i_k}}(e_i) \quad \text{and} \quad \tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \le i_1 < \cdots < i_k \le n} \tau(L_{i_1 \cdots i_k}). \tag{3.6}$$

Combining (1.9) and (3.6), we find that $\tau(p) \ge n(n-1)\Theta_k(p)/2$, which together with (3.1) gives us (3.5). □

Next, we obtain analogous estimates for submanifolds tangent to ξ.

THEOREM 3.3. *Let $\tilde{M}(c)$ be a Sasakian space form and M an n-dimensional submanifold tangent to ξ. Then we have*

$$\|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{(3\|P\|^2 - 2n + 2)(c-1)}{4n(n-1)}. \tag{3.7}$$

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PROOF. We choose an orthonormal basis $\{e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_r = (h^r_{ij}), \quad i, j = 1, \ldots, n, \ r = n+2, \ldots, 2m+1, \quad \operatorname{trace} A_r = \sum_{i=1}^{n} h^r_{ii} = 0.$$

From (2.10), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h^r_{ij})^2 - n(n-1)(c+3)/4 \\ - (3\|P\|^2 - 2n + 2)(c-1)/4.$$

Since, by (3.4), we have $\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2$, it follows that

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n + 2)\frac{c-1}{4},$$

which is equivalent to (3.7). □

From (3.6) and (3.7), we obtain the following theorem.

THEOREM 3.4. *Let $\tilde{M}(c)$ be a Sasakian space form and M an n-dimensional submanifold tangent to ξ. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{(3\|P\|^2 - 2n + 2)(c-1)}{4n(n-1)}. \tag{3.8}$$

COROLLARY 3.5. *Let M be an n-dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have $\Theta_k(p) \leq (c+3)/4 + (c-1)/(4n)$.*

COROLLARY 3.6. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have $\|H\|^2(p) \geq \Theta_k(p) - (c+3)/4 + (c-1)/(2n)$.*

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COROLLARY 3.7. *Let M be an n-dimensional contact CR-submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{(3h-n+1)(c-1)}{2n(n-1)},$$

where $2h = \dim \mathscr{D}$.

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