A CLASS OF c-GROUPS

A. R. CAMINA and T. M. GAGEN

(Received 18 December 1967; revised 25 March 1968)

In a paper by Polimeni [3] the concept of a c-group was introduced. A group is called a c-group if and only if every subnormal subgroup is characteristic. His paper claims to characterize finite soluble c-groups, which we will call fsc-groups. There are some errors in this paper; see the forthcoming review by K. W. Gruenberg in Mathematical Reviews. The following theorem is the correct characterization.

We are indebted to the referee for his suggestions which led to this generalization of our original result.

THEOREM. *Let G be a finite soluble group and L its nilpotent residual (i.e. the smallest normal subgroup of G such that G/L is nilpotent). Then G is an fsc-group if and only if*

(i) *G is a T-group (i.e. every subnormal subgroup is normal),*

(ii) *the Fitting subgroup F of G is cyclic and $F \supseteqq L$,*

(iii) *for every prime divisor p of the order of F, the Sylow p-subgroup of G is cyclic or quaternion,*

(iv) *if a Sylow 2-subgroup of G is quaternion, then $G/\langle u \rangle$ is a c-group, where $u^2 = 1$, $u \in F$.*

NOTE. Properties of T-groups will be used without comment, see Gaschütz [1].

PROOF: SUFFICIENCY. Clearly every subnormal subgroup of G is normal, (i). Condition (iii) implies that the Sylow 2-subgroups of G are either abelian or quaternion of order 8. For if a Sylow 2-subgroup S of G is non-abelian then the derived subgroup S' has order 2 and is contained in F by [1] and so by (iii), S is quaternion.

Since F is cyclic, every subgroup of F is characteristic in G and it follows from the lemma below that every subgroup of G containing F is characteristic. Let N be any normal subgroup of G. We have two cases, the first is when the Sylow 2-subgroups of G are abelian.

In this case we have that $N/N \cap F$ is a normal Hall subgroup of $NF/N \cap F$. For $F/N \cap F$ is a complement to $N/N \cap F$ in $NF/N \cap F$ and

$F/N \cap F$ is a Hall subgroup of $FN/N \cap F$ because if $p||F/F \cap N|$ the Sylow p-subgroups are cyclic. Also $N \cap F$ is characteristic in NF and so N is characteristic in NF which is characteristic in G. Thus N is characteristic in G.

The second case occurs when the Sylow 2-subgroups of G are quaternion. Let u be the involution lying in F. Now by (iv), $N\langle u\rangle$ is characteristic in G. If N does not contain $\langle u\rangle$, N has odd order and so N is characteristic in G. If N contains $\langle u\rangle$, all is well.

LEMMA. *Let M be a characteristic subgroup of a group G and C the centralizer of M in G. If M is finite and cyclic then every automorphism of G induces the identity automorphism on G/C.*

PROOF. Let $x \in M$, $g \in G$, and α be an automorphism of G. Because M is cyclic and finite there are integers r, s, t such that $x\alpha = x^r$, $x^g = x^s$, $x^{rt} = x$. Hence $x^{g\alpha} = ((x^r)^{g\alpha})^t = ((x^g)\alpha)^t = x^s = x^g$ and the result follows.

NECESSITY. Let K be a complement of L in G, see [1]. Then K is a Dedekind group, [1]. Let α be any automorphism of L. Now we define an automorphism α' of G as follows:

$$(kl)^{\alpha'} = kl^{\alpha} \quad \text{for all} \quad k \in K, l \in L.$$

This is an automorphism of G because every element of G induces a power automorphism on L, [1]. Hence every subgroup of L is characteristic in L and so L is cyclic as L is abelian, [1].

Let K_p be a Sylow p-subgroup of K and suppose that $K_p \cap F \neq 1$. Let $b \in K_p \cap F$ have order p. Then b is central in G. Let x be any generator of K_p and R a normal subgroup of G of index p which avoids x. There is an automorphism β of G which maps x to xb and fixes R elementwise since b is central in G. This automorphism maps the characteristic subgroup $L\langle x\rangle$ onto the subgroup $L\langle xb\rangle$ and it follows that b is a power of x. Thus K_p is cyclic or quaternion since K_p has a unique subgroup of order p. This proves (iii).

We remark that if $K_2 \leqq F$, K_2 is normal in G and is a direct factor of G. Then K_2 is cyclic. We note that if K_2 is quaternion then $|F \cap K_2| = 2$. For if c, d are generators of K_2, $c^2 = d^2 = u \in F$ we suppose that $c \in F$. If R is a normal 2-complement, the mapping γ which maps d to cd and fixes $R\langle c\rangle$ elementwise is an automorphism of G since c is central in $R\langle c\rangle$. This is a contradiction since $R\langle d\rangle$ is characteristic in G.

We are left with proving the necessity of (iv). We may assume that F is of even order and thus that the Sylow 2-subgroups of G are quaternion as we have already proved in (iii). Let u be the involution lying in F. Then $\langle u\rangle$ lies in the Frattini subgroup of G and so the Fitting subgroup of $G/\langle u\rangle$

https://doi.org/10.1017/S1446788700007035 Published online by Cambridge University Press

is $F/\langle u\rangle$, by [2]. Thus if $G/\langle u\rangle$ satisfies all the conditions for the theorem, then $G/\langle u\rangle$ will be a *c*-group. Conditions (i) and (ii) are clearly satisfied. Condition (iv) is vacuous since a Sylow 2-subgroup of $G/\langle u\rangle$ is abelian. Condition (iii) will hold for all odd primes. Since we have shown that $|F/\langle u\rangle|$ is odd, we are done. This completes the proof of the theorem.

The extension of a cyclic group of odd order by the automorphism which inverts it shows that any cyclic group of odd order can be the nilpotent residual of an *fsc*-group.

Any finite abelian group H can be embedded in a *fsc*-group. First we show that any finite abelian group A can be embedded in the automorphism group of a cyclic group of coprime order. This is well known but we have no reference so we include the proof. Now A is a direct product of cyclic subgroups. Let $x_1, \cdots, x_r$ be generators of these cyclic subgroups. Choose distinct primes $p_i \equiv 1 \pmod{|A|}$ for $i = 1, \ldots, r$. This is possible by Dirichlet's Theorem. For each $i = 1, \cdots, r$ choose ρ_i to be a primitive n_i-th root of unity mod p_i, where x_i has order n_i. Let L be the abelian group of order $\prod_{i=1}^{r} p_i$ and let $y_1, \cdots, y_r$ be elements of L of order $p_1, \cdots, p_r$ respectively. Embed A in the automorphism group of L as follows

$$y_i^{x_j} = y_i^{\rho_{ij}} \quad \text{where} \quad \rho_{ij} = 1 \quad \text{if} \quad i \neq j \quad \text{and} \quad \rho_{ii} = \rho_i .$$

Now let G be the extension of L by A described above. G satisfies the conditions of the theorem and so G is an *fsc*-group. We remark that if H is any Dedekind group whose Sylow 2-subgroups are quaternion it can similarly be embedded in an *fsc*-group. We merely extend L by H, where L is a cyclic group whose automorphism group contains H/H', H acting on L as H/H'. Of course it is not possible to embed any Dedekind group in an *fsc*-group since it was remarked earlier in the proof that a Sylow 2-subgroup of an *fsc*-group is either abelian or quaternion.

References

[1] W. Gaschütz, 'Gruppen, in denen das Normalteilersein transitiv ist', *Journal für die Reine und Angew. Math.* 198 (1957) 87—92.

[2] W. Gaschütz, 'Über die Φ-Untergruppe endlicher Gruppen', *Math. Zeitschr.* 58 (1953) 160—170.

[3] A. D. Polimeni, 'Finite solvable *c*-groups', *J. Austral. Math. Soc.* 7 (1967) 172—176.

University of Illinois
Urbana, Illinois