Proceedings of the Edinburgh Mathematical Society (1988) 31, 475–479

FITTING CLASSES OF CC-GROUPS

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(Received 10th February 1987)

1. Introduction

The theory of Fitting classes is, by now, a well established part of the theory of finite soluble groups. In contrast, Fitting classes have received rather scant attention in infinite groups, although some recent work of Beidleman and Karbe [2] and Beidleman, Karbe and Tomkinson [3] suggest that one can obtain results in this direction. The paper [2], cited above, in fact generalizes earlier work of Tomkinson [9] to the class of locally soluble FC-groups. The present paper is concerned with the theory of Fitting classes in a class of groups somewhat similar to the class of FC-groups, namely the class of CC-groups, introduced by Polovickiĭ in [6]. A group G is a CC-group if $G/C_G(x^G)$ is a Černikov group for all $x \in G$ where, as in the rest of this paper, we use the standard group theoretic notation of [7]. Recently, Alcázar and Otal [1] have shown how to generalize results of B. H. Neumann [5] to the class of CC-groups. The main purpose of the present note is to illustrate further how one can handle CC-groups, in an analogous manner to FC-groups, by using techniques similar to those used in [1] and [4].

Throughout this paper $\mathfrak{Y}$ will denote the class of locally soluble CC-groups. By lemma 1 of [1] each $\mathfrak{Y}$-group therefore has a local system of normal subgroups each of which is Černikov-by-free abelian of finite rank. The subgroups forming the local system here are the normal closures of finite subsets and hence are centre-by-Černikov, since the class of CC-groups is clearly closed under taking subgroups and homomorphic images.

By a *Fitting class* of $\mathfrak{Y}$-groups we shall mean a subclass $\mathfrak{F}$ of $\mathfrak{Y}$ with the following properties:

(i) If $G \in \mathfrak{F}$ and $H \operatorname{ser} G$ then $H \in \mathfrak{F}$.

(ii) If $G = \langle H_\lambda \mid \lambda \in \Lambda \rangle \in \mathfrak{Y}$ with $H_\lambda \in \mathfrak{F}$ and $H_\lambda \operatorname{ser} G$ then $G \in \mathfrak{F}$ also.

(Here Λ is an index set and $H \operatorname{ser} G$ means H is a serial subgroup of G, our definition of serial being precisely that given in Robinson [7].)

A subgroup V of $G \in \mathfrak{Y}$ will be called an $\mathfrak{F}$-injector of G if $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S whenever $S \operatorname{ser} G$. The set of maximal $\mathfrak{F}$-subgroups of G will be denoted by $\operatorname{Max}_{\mathfrak{F}} G$ and the set of $\mathfrak{F}$-injectors of G will be denoted by $\operatorname{Inj}_{\mathfrak{F}} G$. Our main result, proven in Section 3 of this paper, may then be stated as follows:

Theorem. *If $\mathfrak{F}$ is a Fitting class of $\mathfrak{Y}$-groups and if $G \in \mathfrak{Y}$ then $\operatorname{Inj}_{\mathfrak{F}} G \neq \varnothing$. Also the elements of $\operatorname{Inj}_{\mathfrak{F}} G$ form a unique local conjugacy class.*

(We recall that subgroups H, K of a group G are *locally conjugate* if there is a locally inner automorphism of G mapping H to K). Thus our theorem generalizes Theorems 3.2 and 3.3 of [**9**].

The layout of the paper is as follows. In Section 2 the characteristic of a Fitting class is defined and some recent results from [3] quoted. We then obtain a special case of the Theorem, by showing that certain centre-by-Černikov groups possess a unique conjugacy class of $\mathfrak{F}$-injectors for each Fitting class $\mathfrak{F}$. In Section 3 this allows us to invoke the inverse limit arguments of [**4**] to obtain the main result, the proof being somewhat similar to the corresponding one in [**9**].

2. The characteristic of a Fitting class

If $\mathfrak{F}$ is a Fitting class of $\mathfrak{Y}$-groups then as in [3] we define the *finite characteristic* of $\mathfrak{F}$ to be the set of primes p such that $C_p \in \mathfrak{F}$, where C_p is the cyclic group of order p. The finite characteristic will be denoted by $C_f(\mathfrak{F})$. The *infinite characteristic*, $C_i(\mathfrak{F})$, of $\mathfrak{F}$ is defined to be $\{\infty\}$ if $\mathfrak{F}$ contains the infinite cyclic group; otherwise $C_i(\mathfrak{F}) = \varnothing$, the empty set. The *characteristic* of $\mathfrak{F}$ is defined to be $C_f(\mathfrak{F}) \cup C_i(\mathfrak{F})$ and is denoted by $C(\mathfrak{F})$. Hence $C(\mathfrak{F})$ is a subset of $\mathbb{P} \cup \{\infty\}$, where $\mathbb{P}$ denotes the set of all primes.

If $G \in \mathfrak{Y}$ and $x \in G$ then clearly x^G is a soluble Černikov-by-cyclic group. The following is therefore a straightforward consequence of [**3**, lemma 2.1].

Lemma 2.1. *Let $G \in \mathfrak{Y}$*

(*i*) *If G contains an element of prime order p then there are subgroups $K \lhd H \operatorname{sn} G$ such that $H/K \cong C_p$.*

(*ii*) *If G contains an element of infinite order then there are subgroups $K \lhd H \operatorname{sn} G$ such that $H/K \cong C_\infty$.*

A standard argument (see [**3**, Theorem 2.2, **3**, Theorem 2.4]) now shows:

Lemma 2.2. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{Y}$-groups*

(*i*) *If there is a group $G \in \mathfrak{F}$ which contains a p-element then $C_p \in \mathfrak{F}$.*

(*ii*) *If there is a group $G \in \mathfrak{F}$ which contains an element of infinite order then $C_\infty \in \mathfrak{F}$.*

(*iii*) *If $C_\infty \in \mathfrak{F}$ then $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$.*

(*iv*) *Either $C(\mathfrak{F}) = \pi \subseteq \mathbb{P}$ and every $\mathfrak{F}$-group is a π-group or $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$.*

(*v*) *If $C(\mathfrak{F}) = \pi \subseteq \mathbb{P}$ then $\mathfrak{F}$ contains all locally nilpotent π-groups in $\mathfrak{Y}$.*

(*vi*) *If $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$ then $\mathfrak{F}$ contains all locally nilpotent $\mathfrak{Y}$-groups.*

(Of course the proof of the latter two facts depend on the fact that all subgroups of a locally nilpotent group are serial.)

These results concerning characteristic will only be required in Černikov-by-(free abelian of finite rank) $\mathfrak{Y}$-groups. Such groups are of course $\mathfrak{S}_1$-groups (in the sense of [**3**]). The full results are included here for the sake of completeness.

To obtain results in the Černikov-by-(free abelian of finite rank) case we require the following preliminary result on serial subgroups:

Lemma 2.3. *Let G be a centre-by-Černikov group. Then every serial subgroup of G is ascendant.*

Proof. Let Z denote the centre of G and let $S \operatorname{ser} G$. We show that $SZ/Z \operatorname{ser} G/Z$ from which it follows that $S \lhd SZ \operatorname{asc} G$. In fact, if $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ is a series of G containing S then there is a series of G containing SZ consisting of the subgroups $(\bigcap_{\tau>\sigma} V_\tau Z, \Lambda_\sigma Z, V_\sigma Z : \sigma \in \Sigma)$. This follows from elementary set theory. The fact that $\Lambda_\sigma Z \lhd \bigcap_{\tau>\sigma} V_\tau Z$ follows because $[\Lambda_\sigma Z, \bigcap_{\tau>\sigma} V_\tau Z] \leqq \bigcap_{\tau>\sigma}[\Lambda_\sigma, V_\sigma] \leqq \bigcap_{\tau>\sigma} V_\tau = \Lambda_\sigma$. □

If $\mathfrak{F}$ is a Fitting class of $\mathfrak{Y}$-groups and $G \in \mathfrak{Y}$ we shall let $G_\mathfrak{F}$ denote the $\mathfrak{F}$-radical of G. Then $G_\mathfrak{F}$ is the largest normal $\mathfrak{F}$-subgroup of G. The following result is important for our purposes:

Theorem 2.4. *Let G be an extension of a central finitely generated abelian group Z by a soluble Černikov group and let $\mathfrak{F}$ be a Fitting class of $\mathfrak{Y}$-groups. Then*

(*a*) *G has a unique conjugacy class of $\mathfrak{F}$-injectors.*

(*b*) *If $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$ then G has only finitely many $\mathfrak{F}$-injectors.*

(*c*) *If $C(\mathfrak{F}) = \pi \subseteq \mathbb{P}$ then the cardinality of* $\operatorname{Inj}_\mathfrak{F} G$ *is the same as the cardinality of $|T : N_T(V)|$ where V is a specific $\mathfrak{F}$-injector of G and T is the subgroup of G consisting of elements of finite order.*

Proof. (a) Since G is centre-by-Černikov it is a CC-group and G' is Černikov, by [7, Theorem 4.21, Corollary 2]. Hence the set T of elements of finite order in G forms a normal subgroup of G. Since $T \cap Z$ is finite and TZ/Z is Černikov it follows that T is Černikov. If $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$ and if H/Z is the divisible part of G/Z then $H \in \mathfrak{F}$ by 2.2(vi). Hence $G/G_\mathfrak{F}$ is finite and the result now follows by a special case of [3, Theorem 4.4]. Part (b) clearly follows from this since the $\mathfrak{F}$-injectors must contain $G_\mathfrak{F}$. If, on the other hand, $C(\mathfrak{F}) = \pi \subseteq \mathbb{P}$, then T has an $\mathfrak{F}$-injector V, by [3, Theorem 4.4]. In fact $V \in \operatorname{Inj}_\mathfrak{F} G$. For, if $A \operatorname{ser} G$ then by 2.3, $A \operatorname{asc} G$. If $V \cap A \leqq X \leqq A$ with $X \in \mathfrak{F}$ then X is a π-group by 2.2(i). Hence $X \leqq T$ so $X \leqq A \cap T$. Since $A \cap T \operatorname{asc} T$ and $V \in \operatorname{Inj}_\mathfrak{F} T$ it follows that $V \cap (A \cap T) \in \operatorname{Max}_\mathfrak{F}(A \cap T)$. Hence $V \cap A = X$ and then it follows that G has $\mathfrak{F}$-injectors. Since the $\mathfrak{F}$-injectors must all lie in T, conjugacy also follows, by [3, Theorem 4.4]. The remainder of part (c) is now evident, since any two $\mathfrak{F}$-injectors of G are actually $\mathfrak{F}$-injectors of T. □

3. Proof of the theorem

It is convenient to split the proof of the main theorem in two pieces, namely a proof of existence and a proof of local conjugacy of the injectors. The main technical result required in our proofs is [8, Theorem 2.1] which requires that we make certain sets into compact topological spaces. In general this would not be possible, but for Černikov groups such a topology always exists (see [4]) and it is this fact which is important here.

Theorem 3.1. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{Y}$-groups. Then the $\mathfrak{Y}$-group G possesses $\mathfrak{F}$-injectors.*

Proof. The group G has a local system, $\mathscr{L}$, consisting of soluble normal subgroups, each of which is Černikov-by-(free abelian of finite rank) and finitely generated as a G-operator group. Hence each $N \in \mathscr{L}$ is an extension of a central finitely generated abelian group by a soluble Černikov group. It follows that each $N \in \mathscr{L}$ has a unique conjugacy class of $\mathfrak{F}$-injectors. If $N \in \mathscr{L}$ let $A(N)$ denote the set of $\mathfrak{F}$-injectors of N. Notice that if $C(\mathfrak{F}) = \mathbb{P} \cup \{\infty\}$ then each $A(N)$ is a finite set and the proof follows as in [9, Theorem 3.2]. So we may assume $C(\mathfrak{F}) = \pi \subseteq \mathbb{P}$, by 2.2(iv). Let $N \in \mathscr{L}$ and $V \in \mathrm{Inj}_{\mathfrak{F}} N$. By 2.4(iii), $A(N)$ is in 1–1 correspondence with the set of cosets of $N_N(V)$ in N via the correspondence:

$$V^t \leftrightarrow tN_N(V) \qquad (t \in N).$$

The sets $A(N)$ may be partially ordered by defining $A(N) \leqq A(M)$ if and only if $N \leqq M$ whenever $N, M \in \mathscr{L}$. Clearly, if $W \in A(M)$ and $A(N) \leqq A(M)$ then $W \cap N \in A(N)$ so we may define a projection $\pi_{MN}: A(M) \to A(N)$ by

$$(W)\pi_{MN} = W \cap N.$$

If $N \in \mathscr{L}$ the set $A(N)$ can be made into a compact topological T_1-space as follows: The group $\bar{N} = N/Z(N)$ is Černikov so has the coset topology, defined in [4], and $\bar{N}$ is then a compact, topological T_1-space whose closed sets are in fact finite unions of right cosets of subgroups, by [4, Lemma 2.1]. Let $N^* = N_N(V)$ denote the set of cosets of $N_N(V)$ in N. Since $Z(N) \leqq N_N(V)$ there is a natural map $\Theta_N: \bar{N} \to N^*$, defined by $nZ(N) \mapsto nN_N(V)$, for $n \in N$. The set N^* is given the quotient topology induced by Θ_N and hence N^* is also a compact topological T_1-space. Since $A(N)$ is in 1–1 correspondence with N^* there is an induced topology on $A(N)$ and in this way $A(N)$ is a compact topological T_1-space.

The topology induced on $A(N)$ is independent of the choice of $V \in \mathrm{Inj}_{\mathfrak{F}} N$. This follows using an argument similar to that given in [4, Theorem 3.8]. Also if $N \leqq M$ with $M, N \in \mathscr{L}$ and $W \in A(M)$ then there is a natural map

$$\beta_{MN}: \frac{M}{N_M(W)} \to \frac{N}{N_N(W \cap N)}.$$

For, by a Frattini argument, $M = N \cdot N_M(W \cap N)$ so if $x \in M$ then $x = yt$ for some $y \in N$ and $t \in N_M(W \cap N)$ and one can define β_{MN} by:

$$(xN_M(W))\beta_{MN} = yN_N(W \cap N).$$

It is easy to see that β_{MN} is well defined and that it is a closed, continuous map. It follows that $\pi_{MN}: A(M) \to A(N)$ is also closed and continuous since the topology induced on both $A(M)$ and $A(N)$ does not depend on the choice of injectors. Hence $\varprojlim A(N)$ is non-empty by [8, Theorem 2.1]. If $(V_N) \in \varprojlim A(N)$ then for $N \leqq M$, we have $V_M \cap N = V_N$ so that $V = \bigcup_{N \in \mathscr{L}} V_N$ is a subgroup of G. Moreover $V \in \mathfrak{F}$ since $V \cap N = V_N \lhd V$ so V is generated by normal $\mathfrak{F}$-subgroups. Finally if $S \operatorname{ser} G$ then $S \cap N \operatorname{ser} N$ for each $N \in \mathscr{L}$. Suppose $V \cap S \leqq X \leqq S$ with $X \in \mathfrak{F}$. Then $V \cap S \cap N = X \cap N$ since $V \cap N \in \mathrm{Inj}_{\mathfrak{F}} N$ and it is now clear that $V \cap S \in \mathrm{Max}_{\mathfrak{F}} S$. Hence V is the required $\mathfrak{F}$-injector. □

https://doi.org/10.1017/S0013091500037676 Published online by Cambridge University Press

The local conjugacy of the $\mathfrak{F}$-injectors is easy to establish using the argument given be Alcázar and Otal [1, Theorem 1] and the result established in 2.4; we therefore simply state:

Theorem 3.2. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{Y}$-groups. Then the $\mathfrak{F}$-injectors of $G \in \mathfrak{Y}$ are locally conjugate in G.*

In conclusion, we remark that one can now extend many of the standard results from the theory of Fitting classes in finite soluble groups to the class $\mathfrak{Y}$.

Acknowledgement. The author would like to thank the referee for simplifying the proof of 2.3 and for correcting other parts of Section 2.

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