Moduli **2**, e11 (2025) 1–18
doi:10.1112/mod.2025.10003

Minimal Kinematics on $\mathcal{M}_{0,n}$

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Abstract

Minimal kinematics identifies likelihood degenerations where the critical points are given by rational formulas. These rest on the Horn uniformization of Kapranov–Huh. We characterize all choices of minimal kinematics on the moduli space $\mathcal{M}_{0,n}$. These choices are motivated by the CHY model in physics and they are represented combinatorially by 2-trees. We compute 2-tree amplitudes, and we explore extensions to non-planar on-shell diagrams, here identified with the hypertrees of Castravet–Tevelev.

1. Introduction

The moduli space $\mathcal{M}_{0,n}$ of n labeled points on the projective line $\mathbb{P}^1$ plays a prominent role in algebraic geometry and its interactions with combinatorics. It is equally important in physics, where it is used, for example, in the CHY model [CHY14] to compute scattering amplitudes. The space $\mathcal{M}_{0,n}$ is a very affine variety of dimension $n-3$, with coordinates given by the matrix

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & x_1 & x_2 & \cdots & x_{n-3} & 1 \end{bmatrix}. \tag{1}$$

More abstractly, $\mathcal{M}_{0,n}$ is the quotient of the open Grassmannian $\mathrm{Gr}(2,n)^o$ by the action of the torus $(\mathbb{C}^*)^n$. We write p_{ij} for the Plücker coordinates on $\mathrm{Gr}(2,n)$. These are the 2×2 subdeterminants of X, and being in $\mathrm{Gr}(2,n)^o$ means that $p_{ij}\neq 0$ for all $1\le i<j\le n$.

A basic ingredient in the CHY model is the following *scattering potential* on $\mathcal{M}_{0,n}$:

$$L = \sum_{1\le i<j\le n} s_{ij}\cdot \log(p_{ij}). \tag{2}$$

The coefficients s_{ij} are known as *Mandelstam invariants*. Using the conventions that $s_{ii}=0$ and $s_{ji}=s_{ij}$, the Mandelstam invariants must satisfy the *momentum conservation relations*

$$\sum_{j=1}^{n} s_{ij} = 0 \quad \text{for all } i\in\{1,2,\ldots,n\}. \tag{3}$$

Received 17 February 2024, revised 8 April 2025, accepted 8 April 2025.
2020 Mathematics Subject Classification 14J10, 52C35 (primary), 14D06, 05B35 (secondary).
Keywords: Moduli space, hyperplane arrangement, matroids, scattering amplitudes.

These relations ensure that L is well-defined on $\mathcal{M}_{0,n} = \mathrm{Gr}(2,n)^o/(\mathbb{C}^*)^n$, for any branch of the logarithm function. The following result on the critical points of L is well-known.

PROPOSITION 1.1. *For a general choice of s_{ij}, the scattering potential L has $(n-3)!$ complex critical points on $\mathcal{M}_{0,n}$. If the s_{ij} are real numbers, then all $(n-3)!$ critical points are real.*

We refer to [ST21, § 2] for a proof, computational aspects, and a statistics perspective.

Example 1.2 $(n=4)$. Up to an additive constant $s_{23}\cdot\log(-1)$, we have

$$L = s_{13}\cdot\log(x_1) + s_{23}\cdot\log(1-x_1). \tag{4}$$

This is the log-likelihood function for coin flips with bias x_1, when heads, respectively tails, were observed s_{13}, respectively s_{23}, times. The unique critical point of L is the maximum likelihood (ML) estimate:

$$\hat{x}_1 = \frac{s_{13}}{s_{13}+s_{23}}. \tag{5}$$

For $n \geq 5$ there is no simple formula because the *ML degree* is $(n-3)!$. Minimal kinematics [CE21, Early] provides an attractive alternative. The idea is to set a few s_{ij} to zero until we reach a model of ML degree 1 [ABFKST23, DMS21, Huh14]. This yields nice rational functions, such as (5), for all n.

Example 1.3 $(n=6)$. One choice of minimal kinematics is the codimension 3 subspace defined by $s_{24} = s_{25} = s_{35} = 0$. After this substitution, the scattering potential (2) becomes

$$\begin{aligned} L_T = {} & s_{13}\log(x_1) + s_{14}\log(x_2) + s_{15}\log(x_3) + s_{23}\log(x_1-1) + s_{34}\log(x_2-x_1) \\ & + s_{45}\log(x_3-x_2). \end{aligned}$$

Our task is to solve the scattering equations $\nabla L_T = 0$. Explicitly, these equations are

$$\frac{s_{13}}{x_1} + \frac{s_{23}}{x_1-1} - \frac{s_{34}}{x_2-x_1} = \frac{s_{14}}{x_2} + \frac{s_{34}}{x_2-x_1} - \frac{s_{45}}{x_3-x_2} = \frac{s_{15}}{x_3} + \frac{s_{45}}{x_3-x_2} = 0.$$

This system is easily solved, starting from the end. Back-substituting gives the critical point:

$$\hat{x}_1 = \frac{s_{13}+s_{14}+s_{15}+s_{34}+s_{45}}{s_{13}+s_{14}+s_{15}+s_{23}+s_{34}+s_{45}}, \quad \hat{x}_2 = \hat{x}_1\cdot\frac{s_{14}+s_{15}+s_{45}}{s_{14}+s_{15}+s_{34}+s_{45}}, \quad \hat{x}_3 = \hat{x}_2\cdot\frac{s_{15}}{s_{15}+s_{45}}.$$

The key observation is that the numerators and denominators are products of linear forms with positive coefficients. This is characteristic of all models of ML degree 1, thanks to a theorem of Huh [Huh14]. This goes back to Kapranov [Kap91], who coined the term *Horn uniformization*. We seek to find all Horn uniformizations of the moduli space $\mathcal{M}_{0,n}$. Assuming the representation in (1), the solution is given by graphs called 2-trees. Their vertices are indexed by $[n-1] = \{1,2,\ldots,n-1\}$. The following is our first main result in this article.

THEOREM 1.4. *Choices of minimal kinematics on $\mathcal{M}_{0,n}$ are in bijection with 2-trees on $[n-1]$.*

Here, we use the following formal definition of *minimal kinematics*. Fix the set of index pairs (i,j) in (2) whose corresponding 2×2 minor in the matrix X is non-constant. This is

$$S = \{(i,j) : 1 \leq i < j \leq n-1\} \setminus \{(1,2)\}. \tag{6}$$

For any subset T of S, we restrict to the kinematic subspace where $s_{ij} = 0$ for all $ij \in S\setminus T$:

$$L_T = \sum_{(i,j)\in T} s_{ij}\cdot\log(p_{ij}). \tag{7}$$

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We say that T exhibits *minimal kinematics* for $\mathcal{M}_{0,n}$ if the function (7) has exactly one critical point, which is hence rational in the s_{ij}, and T is inclusion-maximal with this property. With this definition, the minimal kinematics for $\mathcal{M}_{0,6}$ in Example 1.3 is exhibited by the subset

$$T = \{(1,3),(2,3),\ (1,4),(3,4),\ (1,5),(4,5)\}. \tag{8}$$

The proof of Theorem 1.4 is given in § 2. Section 3 features the Horn uniformization. Formulas for the unique critical point of L_T are given in Theorem 3.1 and Corollary 3.3.

In § 4 we introduce an *amplitude* m_T for any 2-tree T, and we compute m_T in Theorem 4.1. The following example explains why we use the term amplitude. It is aimed at readers from physics who are familiar with the *biadjoint scalar amplitude* m_n; see [CHYB14, § 3]. We recall that m_n is the integral of the *Parke–Taylor factor* $1/(p_{12}p_{23}\cdots p_{n-1,n}p_{n,1})$ over the moduli space $\mathcal{M}_{0,n}$, localized to the solutions to the scattering equations $\nabla L = 0$.

Example 1.5 $(n=6)$. The biadjoint scalar amplitude for $\mathcal{M}_{0,6}$ is the rational function

$$m_6 = \frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{12}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{123}} + \frac{1}{s_{23}s_{56}s_{234}} + \frac{1}{s_{34}s_{56}s_{234}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{12}s_{34}s_{345}}$$
$$+ \frac{1}{s_{12}s_{45}s_{123}} + \frac{1}{s_{12}s_{45}s_{345}} + \frac{1}{s_{16}s_{23}s_{234}} + \frac{1}{s_{16}s_{34}s_{234}} + \frac{1}{s_{16}s_{34}s_{345}} + \frac{1}{s_{16}s_{45}s_{345}} + \frac{1}{s_{23}s_{45}s_{123}}. \tag{9}$$

Here $s_{ijk} = s_{ij} + s_{ik} + s_{jk}$. In the physics literature this amplitude is usually denoted $m(\mathbb{I}_6, \mathbb{I}_6)$, where $\mathbb{I}_6 = (123456)$ is the standard cyclic order. See also [Cruz23, (2.7)] or [ST21, (23)].

The 14 summands in (9) correspond to the vertices of the 3-dimensional associahedron. The formula is unique since the nine planar kinematic invariants which appear form a basis of the dual kinematic space. With an eye towards more general situations, to achieve a unique formula for m_n modulo the relations (3), we may also use the basis S from (6). Thus, we set

$$s_{12} = -s_{13} - s_{14} - s_{15} - s_{23} - s_{24} - s_{25} - s_{34} - s_{35} - s_{45},$$
$$s_{16} = s_{23} + s_{24} + s_{25} + s_{34} + s_{35} + s_{45},\ \ s_{26} = s_{13} + s_{14} + s_{15} + s_{34} + s_{35} + s_{45},$$
$$s_{36} = -s_{13} - s_{23} - s_{34} - s_{35},\ \ s_{46} = -s_{14} - s_{24} - s_{34} - s_{45},\ \ s_{56} = -s_{15} - s_{25} - s_{35} - s_{45}.$$

We now restrict to the minimal kinematics in Example 1.3. For $s_{24} = s_{25} = s_{35} = 0$, we find

$$m_6|_{s_{24}=s_{25}=s_{35}=0} = \frac{(s_{13}+s_{14}+s_{15}+s_{34}+s_{45})\,(s_{14}+s_{15}+s_{45})\,s_{15}}{s_{23}\,(s_{13}+s_{14}+s_{15}+s_{23}+s_{34}+s_{45})\,s_{34}\,(s_{14}+s_{15}+s_{34}+s_{45})\,s_{45}\,(s_{15}+s_{45})}.$$

Up to relabeling, this is the amplitude m_{T_1} we associate with the 2-tree T_1 in (10) below. This illustrates Theorem 4.1. The factors are explained by the Horn matrix H_{T_1} in Example 3.2. We note that, modulo momentum conservation (3), our amplitude simplifies to

$$m_6|_{s_{24}=s_{25}=s_{35}=0} = \frac{(s_{12}+s_{23})\,(s_{45}+s_{56})\,(s_{34}+s_{456})}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{456}}.$$

In § 5 we venture into territory that is of great significance for both algebraic geometry and particle physics. Structures we attach to 2-trees generalize naturally to the hypertrees of Castravet and Tevelev [CT13]. By [Tev, Lemma 9.5.(3)], hypertrees are equivalent to the on-shell diagrams of Arkani-Hamed, Bourjaily, Cachazo, Postnikov and Trnka [ABCPT15]. Every hypertree T has an associated amplitude m_T. This rational function does not admit a Horn formula like that in (13) for 2-trees T. Indeed, now the ML degree for T is larger than 1.

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Here, we aim to reach minimal kinematics by restricting to subspaces of the kinematic space. This is seen in Example 5.8. The paper concludes with questions for future research.

2. 2-trees

We begin by defining the class of graphs referred to in Theorem 1.4. A *2-tree* is a graph T with $2n-5$ edges on the vertex set $[n-1]=\{1,2,\ldots,n-1\}$, which can be constructed inductively as follows. We start with the single edge graph $\{12\}$, which is the unique 2-tree for $n-1=2$. For $k=3,4,5,\ldots,n-1$, we proceed inductively as follows: we select an edge ij whose vertices i and j are in $\{1,\ldots,k-1\}$, and we introduce the two new edges ik and jk. There are $2k-5$ choices at this stage, so our process leads to $(2n-7)!!=1\cdot 3\cdot 5\cdot 7\cdots(2n-7)$ distinct 2-trees. Of course, many pairs of these 2-trees will be isomorphic as graphs.

The number of 2-trees up to isomorphism appears as the entry A054581 in the Online Encyclopedia of Integer Sequences (OEIS). That sequence begins with the counts

$$1,1,\mathbf{2},5,12,39,136,529,2171,9368,41534,\ldots \quad \text{for } n=4,5,\mathbf{6},7,8,9,10,11,12,13,\ldots$$

For instance, there are **2** distinct unlabeled 2-trees for $n=$ **6**. Representatives are given by

$$T_1=\{12,13,23,14,34,15,45\} \quad \text{and} \quad T_2=\{12,13,23,24,34,25,35\}. \tag{10}$$

We identify each 2-tree T with a subset of the set S in (6), by removing the initial edge 12. This thus specifies a function (7). For example, the 2-tree T_1 in (10) is identified with T in (8).

Theorem 1.4 has two directions. First, (7) has only one critical point when T is a 2-tree. Second, every maximal graph with this property is a 2-tree. We begin with the first direction.

Lemma 2.1. *Every 2-tree exhibits minimal kinematics.*

Proof. Let $m=2n-6$, and identify the coordinates on $(\mathbb{C}^*)^m$ with the pairs in T. We write X_T for the $(n-3)$-dimensional subvariety of $(\mathbb{C}^*)^m$ that is parametrized by the 2×2-minors p_{ij} with $(i,j)\in T$. Thus, X_T is a very affine variety, defined as the complement of an arrangement of m hyperplanes in $\mathbb{C}^{n-3}$. The scattering potential (7) is the log-likelihood function for X_T. The number of critical points, also known as the ML degree, equals the signed Euler characteristic of X_T; see [ABFKST23, HS14, ST21]. Therefore, our claim says that $|\chi(X_T)|=1$.

To prove this, we use the multiplicativity of the Euler characteristic. This states that, for any fibration $f:E\to B$, with fiber F, the following relation holds: $\chi(E)=\chi(F)\cdot\chi(B)$.

We proceed by induction on n. The base case is $n=4$, where $T=\{(1,3),(2,3)\}$. This set represents the unique 2-tree on [3], which is the triangle graph with vertices $1,2,3$. Here, the very affine variety is the affine line $\mathbb{C}^1$ with two points removed. In symbols, we have

$$X_T=\{\,(x_1,x_1-1)\in\mathbb{C}^2\,:\,x_1\neq 0,1\,\}=\{\,(p_{13},p_{23})\in(\mathbb{C}^*)^2\,:p_{13}-p_{23}=1\,\}\simeq\mathcal{M}_{0,4}.$$

This punctured curve satisfies $\chi(X_T)=-1$, so the base case of our induction is verified.

We now fix $n\geq 4$, and we assume that $X_T\subset(\mathbb{C}^*)^{2n-6}$ has Euler characteristic ± 1 for all 2-trees T with $n-1$ vertices. Note that $\dim(X_T)=n-3$. Let T' be any 2-tree with n vertices. The vertex n is connected to exactly two vertices i and j. Suppose $i<j$. The associated very affine variety $X_{T'}$ lives in $(\mathbb{C}^*)^{2n-4}$. Note that $\dim(X_{T'})=n-2$.

We write the coordinates on $(\mathbb{C}^*)^{2n-4}$ as $(\,p,\,p_{in},p_{jn})$, where $p\in(\mathbb{C}^*)^{2n-6}$. Consider the the map $\pi:(\mathbb{C}^*)^{2n-4}\to(\mathbb{C}^*)^{2n-6}$, $(\,p,\,p_{in},p_{jn})\mapsto p$ which deletes the last two coordinates.

Let T be the 2-tree without the vertex n and the two edges in and jn. Then ij is an edge of T, so p_{ij} is among the coordinates of p. The restriction of π to $X_{T'}$ defines a fibration

$$\pi \,:\, X_{T'} \to X_T, \;\; (\,p,\, p_{in}, p_{jn}) \mapsto p.$$

Indeed, the fiber over $p \in X_T$ equals

$$F = \pi^{-1}(p) = \{\, (p_{in}, p_{jn}) \in (\mathbb{C}^*)^2 \,:\, p_{jn} - p_{in} = c\,\}, \qquad \text{where } c = p_{ij} \neq 0.$$

This is the punctured line above, i.e. $F \simeq \mathcal{M}_{0,4}$. We know that $\chi(F) = -1$. Using the induction hypothesis, and the multiplicativity of the Euler characteristic, we conclude that

$$\chi(X_{T'}) = \chi(F) \cdot \chi(X_T) = -\chi(X_T) = \pm 1.$$

This means that the 2-tree T' exhibits minimal kinematics, and the lemma is proved. □

From the induction step in the proof above, we also see how to write down the three-term linear equations that define X_T. We display these linear equations for the 2-trees with $n = 6$.

Example 2.2 $(n = 6)$. Consider the two 2-trees in (10). Each of them defines a very affine threefold of Euler characteristic -1. Explicitly, these two threefolds are given as follows:

$$X_{T_1} = V(\,p_{13} - p_{23} - 1,\; p_{14} - p_{34} - p_{13},\; p_{15} - p_{45} - p_{14}\,) \subset \;(\mathbb{C}^*)^6,$$
$$X_{T_2} = V(\,p_{13} - p_{23} - 1,\; p_{24} - p_{34} - p_{23},\; p_{25} - p_{35} - p_{23}\,) \subset \;(\mathbb{C}^*)^6.$$

We now come to the converse direction, which asserts that 2-trees are the only maximal graphs T satisfying $\chi(X_T) = \pm 1$. This will follow from known results on graphs and matroids.

Proof of Theorem 1.4. Consider any arrangement of $m+1$ hyperplanes in the real projective space $\mathbb{P}^d$ such that the intersection of all hyperplanes is empty. This data defines a matroid M of rank $d+1$ on $m+1$ elements. The complement of the hyperplanes is a very affine variety X of dimension d. Let $\mathbb{R}^d$ be the affine space obtained from $\mathbb{P}^d$ by removing any one of the hyperplanes. We are left with an arrangement of m hyperplanes in $\mathbb{R}^d$. By [HS14, Theorem 1.20], the number of bounded regions in that affine arrangement is equal to $|\chi(X)|$.

The number of bounded regions described above depends only on the matroid M, and it is known as the *beta invariant*. This is a result due to Zaslavsky [Zas75]. The beta invariant can be computed by substituting 1 into the reduced characteristic polynomial of M; see [Oxl82].

In our situation, we are considering arrangements of hyperplanes in $\mathbb{R}^d$ of the special types $\{x_i = 0\}$, $\{x_j = 1\}$, or $\{x_k = x_l\}$. These correspond to graphic matroids, and here the characteristic polynomial is essentially the chromatic polynomial of the underlying graph. Our problem is this: for which graphs does this affine hyperplane arrangement have precisely one bounded region? Or, more generally, which matroids have a beta invariant equal to one?

The answer to this question was given by Brylawski [Bry71, Theorem 7.6]: the beta invariant of a matroid M equals one if and only if M is series-parallel. This means that M is a graphic matroid, where the graph is series-parallel. We finally cite the following from Bodirsky et al. [BGKN07, p. 2092]: *A series-parallel graph on N vertices has at most $2N-3$ edges. Those having this number of edges are precisely the 2-trees.* Setting $N = n-1$, we can now conclude that the 2-trees T are the only subsets of S that exhibit minimal kinematics for $\mathcal{M}_{0,n}$. □

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3. Horn matrices

A remarkable theorem due to June Huh [Huh14] characterizes very affine varieties $X \subset (\mathbb{C}^*)^m$ that have ML degree 1. The unique critical point $\hat{p}$ of the log-likelihood function on X is given by the Horn uniformization, due to Kapranov [Kap91]. Huh's result was adapted to the setting of algebraic statistics by Duarte et al. in [DMS21]. For an exposition see also [HS14, § 3]. The *Horn uniformization* can be written in concise notation as follows:

$$\hat{p} = \lambda \star (Hs)^H. \tag{11}$$

Here, H is an integer matrix with m columns and λ is a vector in $\mathbb{Z}^m$. The pair (H, λ) is an invariant of the variety X, referred to as the *Horn pair* in [DMS21]. The coefficients s_{ij} in the log-likelihood function (aka Mandelstam invariants) form the column vector s of length m, so Hs is a vector of linear forms in the coordinates of s. The notation $(Hs)^H$ means that we regard each column of H as an exponent vector, and we form m Laurent monomials in the linear forms Hs. Finally, $\star$ denotes the Hadamard product of two vectors of length m.

The formula for $\hat{p}$ given in (11) is elegant, but one needs to get used to it. We encourage our readers to work through Example 3.2, where $(Hs)^H$ is shown for two Horn matrices H.

We now present the main result of this section, namely the construction of the *Horn matrix* $H = H_T$ for the very affine variety X_T associated with any 2-tree T on $[n-1]$. The matrix H_T has $3n-9$ rows and $m = 2n-6$ columns, one for each edge of T. It is constructed inductively as follows. If $n = 4$ with $T = \{13, 23\}$, which represents the triangle graph, then

$$H_T = \begin{array}{c} \begin{matrix} 13 & 23 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \end{array}.$$

Now, for $k \geq 5$, let T' be any 2-tree with k vertices, where vertex k is connected to i and j, and $T = T' \setminus \{ik, jk\}$ as in the proof of Lemma 2.1. Then the Horn matrix for T' equals

$$H_{T'} \;=\; \begin{array}{r} \\ 3k{-}12 \text{ rows} \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{matrix} 2k-\text{columns} & ik & jk \end{matrix} \\ \begin{bmatrix} H_T & \mathbf{h}_{ij} & \mathbf{h}_{ij} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & -1 & -1 \end{bmatrix} \end{array}$$

where $\mathbf{0}$ is the zero row vector and $\mathbf{h}_{ij}$ is the column of H_T that is indexed by the edge ij.

Theorem 3.1. *Given any 2-tree T on $[n-1]$, the $(3n-9) \times (2n-6)$ matrix H_T constructed above equals the Horn matrix H for the very affine variety X_T. There exists a sign vector $\lambda \in \{-1, +1\}^{2n-6}$ such that (11) is the unique critical point $\hat{p}$ of the scattering potential (7).*

Before proving this theorem, we illustrate the construction of H_T and the statement.

Example 3.2 ($n = 6$). We consider the two 2-trees that are shown in (10). In each case, the Horn matrix has nine rows and six columns. We find that the two Horn matrices are

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$$H_{T_1} = \begin{array}{c} \begin{matrix} 13 & 23 & 14 & 34 & 15 & 45 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array} \quad \text{and} \quad H_{T_2} = \begin{array}{c} \begin{matrix} 13 & 23 & 24 & 34 & 25 & 35 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}.$$

For $H = H_{T_i}$, the column vector Hs has nine entries, each a linear form in six s-variables. Each column of H specifies an alternating product of these linear forms, and these are the entries of $(Hs)^H$. By adjusting signs when needed, we obtain the six coordinates of $\hat{p}$.

For the second 2-tree T_2, the six coordinates of the critical point $\hat{p}$ are

$$\hat{p}_{13} = \frac{s_{13}}{s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35}} \qquad \hat{p}_{23} = -\frac{s_{23}+s_{24}+s_{25}+s_{34}+s_{35}}{s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35}}$$

$$\hat{p}_{24} = -\frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{24}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{24}+s_{34})}$$
$$\hat{p}_{34} = \frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{34}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{24}+s_{34})}$$
$$\hat{p}_{25} = -\frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{25}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{25}+s_{35})}$$
$$\hat{p}_{35} = \frac{(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})s_{35}}{(s_{13}+s_{23}+s_{24}+s_{34}+s_{25}+s_{35})(s_{25}+s_{35})}.$$

For the first 2-tree T_1, the six coordinates of the critical point $\hat{p}$ are

$$\hat{p}_{13} = \frac{s_{13}+s_{14}+s_{34}+s_{15}+s_{45}}{s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45}} \qquad \hat{p}_{23} = -\frac{s_{23}}{s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45}}$$

$$\hat{p}_{14} = \frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{34}+s_{45})}$$
$$\hat{p}_{34} = -\frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})s_{34}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})}$$
$$\hat{p}_{15} = \frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})s_{15}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})(s_{15}+s_{45})}$$
$$\hat{p}_{45} = -\frac{(s_{13}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{15}+s_{45})s_{45}}{(s_{13}+s_{23}+s_{14}+s_{34}+s_{15}+s_{45})(s_{14}+s_{34}+s_{15}+s_{45})(s_{15}+s_{45})}.$$

We note that these $\hat{p}_{ij}$ satisfy the trinomial equations given for X_{T_1}, respectively X_{T_2}, in Example 2.2.

Proof of Theorem 3.1. We start by reviewing the Horn uniformization (11) in the version proved by Huh [Huh14]. Let $X \subset (\mathbb{C}^*)^m$ be any very affine variety. The following are equivalent:

(i) the variety X has ML degree 1;

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(ii) there exists $\lambda = (\lambda_1, \dots, \lambda_m) \in (\mathbb{C}^*)^m$ and a matrix $H = (h_{ij})$ in $\mathbb{Z}^{\ell\times m}$ with zero column sums and left kernel A, such that the monomial map

$$(\mathbb{C}^*)^\ell \to (\mathbb{C}^*)^m, \quad \mathbf{q} = (q_1, \dots, q_\ell) \mapsto \left(\lambda_1 \prod_{i=1}^{\ell} q_i^{h_{i1}}, \dots, \lambda_m \prod_{i=1}^{\ell} q_i^{h_{im}}\right),$$

maps the A-discriminantal variety Δ_A dominantly onto X.

Constructing a Horn uniformization proves that the variety X has ML degree 1. The corresponding Horn map (11) is precisely the unique critical point of the log-likelihood function. The following argument thus also serves as an alternative proof of Theorem 1.4.

Fix a 2-tree T on $[n-1]$, with associated variety $X = X_T$ in $(\mathbb{C}^*)^m$, where $m = 2n-6$. Let $H = H_T$ be the $(3n-9)\times(2n-6)$ matrix constructed previously. Its left kernel is given by

$$A = \begin{Bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{Bmatrix}.$$

This matrix has $n-3$ rows. Its toric variety is the $(n-4)$-dimensional linear space

$$X_A = \{\, (t_1, t_1, t_1,\ t_2, t_2, t_2,\ \dots,\ t_{n-3}, t_{n-3}, t_{n-3}) : t_1, t_2, \dots, t_{n-3} \in (\mathbb{C}^*)^{3n-9} \,\} \quad \text{in } \mathbb{P}^{3n-10}.$$

The A-discriminantal variety is the variety projectively dual to X_A. Here, it is the linear space

$$\Delta_A = \{\, q = (q_1, q_2, \dots, q_{3n-9}) \ :\ q_{3i+1} + q_{3i+2} + q_{3i+3} = 0 \ \text{ for } \ i = 0, 1, \dots, n-4 \,\}.$$

Disregarding λ for now, the monomial map in item (ii) above takes $q \in \Delta_A$ to the vector

$$p = (\, p_{13}, p_{23},\ \dots,\ p_{ik},\ p_{jk},\ \dots\,) = \left(\frac{q_1}{q_3}, \frac{q_2}{q_3}, \dots,\ p_{ij}\cdot\frac{q_{3k-8}}{q_{3k-6}},\ p_{ij}\cdot\frac{q_{3k-7}}{q_{3k-6}}, \dots\right).$$

Using the trinomial equations that define Δ_A, we can write this as follows

$$p = \left(\frac{q_1}{-(q_1+q_2)}, \frac{q_2}{-(q_1+q_2)}, \dots,\ p_{ij}\cdot\frac{q_{3k-8}}{-(q_{3k-8}+q_{3k-7})},\ p_{ij}\cdot\frac{q_{3k-7}}{-(q_{3k-8}+q_{3k-7})}, \dots\right).$$

The vector p satisfies the equation $p_{13} + p_{23} + 1 = 0$ and it satisfies all subsequent equations $p_{ik} + p_{jk} + p_{ij} = 0$ that arise from the construction of the 2-tree T.

In a final step, we need to adjust the signs of the coordinates in order for p to satisfy the equations $p_{ik} - p_{jk} - p_{ij} = 0$ that cut out X_T; see e.g. Example 2.2. This is done by replacing $\hat{p}_{lm} = \lambda_{lm} p_{lm}$ for appropriate signs $\lambda_{lm} \in \{-1, +1\}$, or, equivalently, by replacing p with the Hadamard product $\hat{p} = \lambda \star p$ for an appropriate sign vector $\lambda \in \{-1, +1\}^{2n-6}$.

The signs can be constructed recursively as follows. We set $\lambda_{13} = -1, \lambda_{23} = 1$ and, with the same notation as earlier, we define $\lambda_{ik} = -\lambda_{ij}$ and $\lambda_{jk} = \lambda_{ij}$. Indeed, if $(\hat{p}_{ij}, \hat{p}_{ik}, \hat{p}_{jk})$ satisfy $p_{ik} + p_{jk} + p_{ij} = 0$, then $(\lambda_{ij}\hat{p}_{ij}, \lambda_{ik}\hat{p}_{ik}, \lambda_{jk}\hat{p}_{jk})$ satisfy $p_{ik} - p_{jk} - p_{ij} = 0$. This now gives the desired birational map from Δ_A onto X_T. That map furnishes the rational formula for the unique critical point $\hat{p}$. To obtain the version $\hat{p} = \lambda \star (Hs)^H$ seen in (11), we note that the map $s \mapsto Hs$ parametrizes the linear space Δ_A. This completes the proof. □

We conclude this section by making the coordinates of $\hat{p} = \lambda \star (Hs)^H$ more explicit. Given any 2-tree T and any edge ij of T, we write $[s_{ij}]$ for the sum of all Mandelstam invariants s_{lm} where

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lm is any descendant of the edge ij in T. Here *descendant* refers to the transitive closure of the parent-child relation in the iterative construction of T: the new edges ik and jk are children of the old edge ij. In this case we call $\{i,j,k\}$ a *triangle* of the 2-tree T. This triangle is an *ancestral triangle* of an edge lm of T if lm is a descendant of the edge ik. The entries of the vector Hs are the linear forms $[s_{ij}]$ and their negated sums $-[s_{ik}]-[s_{jk}]$.

Corollary 3.3. *The evaluation of the Plücker coordinate p_{lm} at the critical point of L_T equals*

$$\hat{p}_{lm} = \prod \lambda_{ik} \frac{[s_{ik}]}{[s_{ik}]+[s_{jk}]}, \tag{12}$$

where the product runs over all ancestral triangles $\{i,j,k\}$ of the edge lm, and $\lambda_{ik} \in \{-1,+1\}$ is the sign chosen in the proof of Theorem 3.1.

This is a corollary of Theorem 3.1. The proof is given by inspecting the Horn matrix H_T. It is instructive to rewrite the rational functions $\hat{p}_{ij}$ in Example 3.2 using the notation in (12).

4. Amplitudes

In this section we take a step towards particle physics. We define an amplitude m_T for any 2-tree T. This is a rational function in the Mandelstam invariants s_{ij}. When T is planar, m_T is a degeneration of the biadjoint scalar amplitude m_n. We saw this in Example 1.5. Cachazo and Early in [CE21] introduced minimal kinematics as a means to study such degenerations.

We shall express the amplitude m_T in terms of the Horn matrix $H = H_T$. Given any 2-tree T, the entries of the column vector s are the $2n-6$ Mandelstam invariants s_{ij}. The entries of the column vector Hs are $3n-9$ linear forms in s. Our main result is the following theorem.

Theorem 4.1. *Fix a 2-tree T on $[n-1]$. The amplitude m_T associated with T equals*

$$m_T = \prod_{\{i<j<k\}} \frac{[s_{ik}]+[s_{jk}]}{[s_{ik}]\cdot[s_{jk}]}. \tag{13}$$

The product is over all triangles in T. This rational function of degree $3-n$ is the product of $n-3$ of the linear forms in Hs divided by the product of the other $2n-6$ linear forms in Hs.

In order for this theorem to make sense, we first need a definition of the amplitude m_T. We shall work in the framework of beyond-planar MHV amplitudes developed by Arkani-Hamed et al. in [ABCPT15]. Our point of departure is the observation that every 2-tree on $[n-1]$ defines an on-shell diagram. Here we view T as a list of $n-2$ triples ijk, starting with 123 and ending with $12n$. The last triple $12n$ is special because it uses the vertex n. Moreover, it does not appear in the formula (13). For a concrete example, we identify the 2-trees T_1 and T_2 in (10) with the following two on-shell diagrams, one planar and one non-planar:

$$T_1 : \{123, 134, 145, 126\} \qquad \text{and} \qquad T_2 : \{123, 234, 235, 126\}. \tag{14}$$

With each triple ijk in T we associate the row vector $p_{jk}e_i - p_{ik}e_j + p_{ij}e_k$, and we define M_T to be the $(n-2)\times n$ matrix whose rows are these vectors for all triples in T. For instance,

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$$M_{T_2} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} p_{23} & -p_{13} & p_{12} & 0 & 0 & 0 \\ 0 & p_{34} & -p_{24} & p_{23} & 0 & 0 \\ 0 & p_{35} & -p_{25} & 0 & p_{23} & 0 \\ p_{26} & -p_{16} & 0 & 0 & 0 & p_{12} \end{bmatrix} \end{array}.$$

Note that the kernel of M_T coincides with the row span of the matrix X in (1). This implies that there exists a polynomial $\Delta(M_T)$ of degree $n-3$ in the Plücker coordinates such that the maximal minor of M_T obtained by deleting columns i and j is equal to $\pm p_{ij} \cdot \Delta(M_T)$.

LEMMA 4.2. *For any 2-tree T, the gcd of the maximal minors of the matrix M_T equals*

$$\Delta(M_T) = \prod_{ij} p_{ij}^{v_T(ij)-1}, \tag{15}$$

where the product is over all edges of T, and $v_T(ij)$ is the number of triangles containing ij.

Proof. The rightmost maximal square submatrix of M_T is lower triangular. Its determinant equals the product of the p_{ij} where $\{i,j,k\}$ runs over all triangles in the 2-tree. By construction of T, the number of occurrences of p_{ij} is $v_T(ij)-1$. We divide this product by p_{12} to get $\Delta(M_T)$, since the triangle $\{1,2,n\}$ has to be disregarded for a 2-tree on $[n-1]$. □

Following [ABCPT15, (2.15)], we define the *integrand* associated with the 2-tree T to be

$$\mathcal{I}_T = \frac{\Delta(M_T)^2}{\prod_{ijk\in T} p_{ij}p_{ik}p_{jk}}. \tag{16}$$

This is a rational function of degree $-n$ in the Plücker coordinates. Lemma 4.2 now implies the following corollary.

COROLLARY 4.3. *Given any 2-tree T, the associated integrand equals*

$$\mathcal{I}_T = \prod_{ij} p_{ij}^{v_T(ij)-2}. \tag{17}$$

The integrands for the 2-trees T_1 and T_2 from our running example in (10) and (14) are

$$\mathcal{I}_{T_1} = \frac{1}{p_{16}p_{26}p_{23}p_{34}p_{45}p_{15}} \qquad \text{and} \qquad \mathcal{I}_{T_2} = \frac{p_{23}}{p_{16}p_{26}p_{13}p_{24}p_{34}p_{25}p_{35}}.$$

Note that $\mathcal{I}_{T_1}$ is a Parke–Taylor factor for $n=6$. We are now using the unconventional labeling $1,n,2,3,4,\ldots,n-1$ for the vertices of the n-gon. Any triangulation of this n-gon is a 2-tree T. Such 2-trees are called *planar*. To be precise, the triangulation consists of the three edges of the triangle $\{1,n,2\}$ together with the $2n-6$ edges given by the 2-tree T.

COROLLARY 4.4. *For any planar 2-tree T, our integrand equals the Parke–Taylor factor.*

$$\mathcal{I}_T = \frac{1}{p_{1n}\,p_{n2}\,p_{23}\,p_{34}\,\cdots\,p_{n-2,n-1}\,p_{n-1,1}} = PT(1,n,2,3,\ldots,n-1). \tag{18}$$

Proof. This was observed in [ABCPT15, § 3]. It follows directly from Corollary 4.3. □

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We now define the *amplitude* associated with a 2-tree T to be the following expression:

$$m_T = -\frac{(\mathcal{I}_T)^2}{\mathrm{Hess}(L_T)}(\hat{p}). \tag{19}$$

Here $\mathrm{Hess}(L_T)$ is the determinant of the Hessian of the log-likelihood function in (7). The numerator and the denominator are evaluated at the critical point $\hat{p}$ which is given by Theorem 3.1.

Example 4.5 $(n=6)$. Let $T=T_2$ be the non-planar 2-tree in our running example. Then

$$\begin{aligned}
-\mathrm{Hess}(L_T)(\hat{p}) &= \frac{(s_{13}+s_{23}+s_{24}+s_{25}+s_{34}+s_{35})^7\,(s_{24}+s_{34})^3\,(s_{25}+s_{35})^3}{s_{13}\,(s_{23}+s_{24}+s_{25}+s_{34}+s_{35})^5\,s_{24}\,s_{34}\,s_{25}\,s_{35}}\\
&= \frac{([s_{13}]+[s_{23}])^7}{[s_{13}]^1\,[s_{23}]^5}\cdot\frac{([s_{24}]+[s_{34}])^3}{[s_{24}]^1\,[s_{34}]^1}\cdot\frac{([s_{25}]+[s_{35}])^3}{[s_{25}]^1\,[s_{35}]^1}.
\end{aligned}$$

The numerator $(\mathcal{I}_T)^2(\hat{p})$ is the same expression but with each exponent increased by one. Therefore m_T is equal to the ratio given by the Horn matrix H_T, as promised in Theorem 4.1.

Proof of Theorem 4.1. Fix a 2-tree T on $[n-1]$. For any edge ij of T, we write a_{ij} for the number of descendants of that edge. For any triangle $\{i,j<k\}$ of T, we set $b_k = a_{ik}+a_{jk}+1$.

A key combinatorial lemma about 2-trees is that $a_{ij}+1$ equals the sum of the integers $2(2-v_T(lm))$ where lm runs over the set $\mathrm{dec}_T(ij)$ of descendants of the edge ij. To be precise,

$$a_{ij}+1 = \sum_{lm\in\mathrm{dec}_T(ij)} 2(2-v_T(lm)) \qquad \text{for all edges } ij\neq 12. \tag{20}$$

The initial edge 12 is excluded. Recall that $v_T(lm)$ is the number the triangles containing the edge lm. We prove (20) by induction on the construction of T, after checking it for $n\leq 5$. Indeed, suppose a new vertex n enters the 2-tree, with edges rn and sn. Then $v_T(rs)$ increases by 1 and $v_T(rn)=v_T(sn)=1$. Otherwise v_T is unchanged. For all ancestors ij of rs, the right hand side and the left hand side of (20) increase by 2. For $ij\in\{rn,sn\}$, both sides are 2. For all other edges ij, the two sides remain unchanged. Hence (20) is proved.

We now evaluate $(\mathcal{I}_T)^2$ at $\hat{p}$ by plugging (12) into the square of (17). This gives

$$(\mathcal{I}_T)^2(\hat{p}) = \left(\prod_{lm}(\hat{p}_{lm})^{v_T(lm)-2}\right)^2 = \prod_{lm}\prod_{ijk}\left(\frac{[s_{ik}]+[s_{jk}]}{[s_{ik}]}\right)^{2(2-v_T(lm))},$$

where the inner product is over ancestral triangles ijk of lm. Switching the products yields

$$\begin{aligned}
(\mathcal{I}_T)^2(\hat{p}) &= \prod_{ijk}\left[\prod_{lm\in\mathrm{dec}_T(ik)}\left(\frac{[s_{ik}]+[s_{jk}]}{[s_{ik}]}\right)^{2(2-v_T(lm))}\cdot\prod_{lm\in\mathrm{dec}_T(jk)}\left(\frac{[s_{ik}]+[s_{jk}]}{[s_{jk}]}\right)^{2(2-v_T(lm))}\right]\\
&= \prod_{ijk}\left[\left(\frac{[s_{ik}]+[s_{jk}]}{[s_{ik}]}\right)^{a_{ik}+1}\cdot\left(\frac{[s_{ik}]+[s_{jk}]}{[s_{jk}]}\right)^{a_{jk}+1}\right].
\end{aligned}$$

where the product is over all triangles ijk of T. In conclusion, we have derived the formula

$$(\mathcal{I}_T)^2(\hat{p}) = \prod_{ijk}\frac{(\,[s_{ik}]+[s_{jk}]\,)^{b_k+1}}{[s_{ik}]^{a_{ik}+1}\,[s_{jk}]^{a_{jk}+1}}, \tag{21}$$

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In order to prove Theorem 4.1, we must show that the Hessian at the critical point equals

$$\mathrm{Hess}(L_T)(\hat{p}) = -\prod_{ijk} \frac{(\,[s_{ik}]+[s_{jk}]\,)^{b_k}}{[s_{ik}]^{a_{ik}}\,[s_{jk}]^{a_{jk}}}. \tag{22}$$

The proof is organized by an induction on k, where the Mandelstam invariants s_{ij} are transformed as we deduce the desired formula for k from the corresponding formula for $k-1$.

As a warm-up, it is instructive to examine the case $k=4$, where the Hessian is a 1×1 matrix. The entry of that matrix is the second derivative of (4) evaluated at (5). We find

$$\mathrm{Hess}(L_T)(\hat{p}) = \frac{\partial^2 L}{\partial x_1^2}(\hat{p}) = -\frac{s_{13}}{\hat{x}_1^2} - \frac{s_{23}}{(\hat{x}_1-1)^2} = -\frac{(s_{13}+s_{23})^3}{s_{13}s_{23}}.$$

This equation matches (22), and it serves as the blueprint for the identity in (24) below.

Our first step towards (22) is to get rid of the minus sign. To this end, we write $\mathcal{H}$ for the Hessian matrix of the negated scattering potential L_T, evaluated at $\hat{p}$. Its entries are

$$\mathcal{H}_{ii} = \sum_{\ell=1}^{n} \frac{s_{i\ell}}{\hat{p}_{i\ell}^2} \qquad \text{and} \qquad \mathcal{H}_{ij} = -\frac{s_{ij}}{\hat{p}_{ij}^2} \quad \text{for } \ i\neq j.$$

We shall prove that $\det(\mathcal{H})$ equals the product on the right-hand side of (22). This will be done by downward induction. Let k be the last vertex, connected to earlier vertices i,j. We display the rows and columns of the Hessian that are indexed by the triangle $\{i,j,k\}$:

$$\mathcal{H} = \begin{Bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots + \frac{s_{ij}}{\hat{p}_{ij}^2} + \frac{s_{ik}}{\hat{p}_{ik}^2} & \cdots & -\frac{s_{ij}}{\hat{p}_{ij}^2} & \cdots & -\frac{s_{ik}}{\hat{p}_{ik}^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & -\frac{s_{ij}}{\hat{p}_{ij}^2} & \cdots & \cdots + \frac{s_{ij}}{\hat{p}_{ij}^2} + \frac{s_{jk}}{\hat{p}_{jk}^2} & \cdots & -\frac{s_{jk}}{\hat{p}_{jk}^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -\frac{s_{ik}}{\hat{p}_{ik}^2} & 0 & -\frac{s_{jk}}{\hat{p}_{jk}^2} & 0 & \frac{s_{ik}}{\hat{p}_{ik}^2} + \frac{s_{jk}}{\hat{p}_{jk}^2} \end{Bmatrix}.$$

Since k is a terminal node in the 2-tree T, its two edges satisfy a variant of (5), namely

$$\hat{p}_{ik} = \frac{s_{ik}}{s_{ik}+s_{jk}} \cdot \hat{p}_{ij} \quad \text{and} \quad \hat{p}_{jk} = \frac{s_{jk}}{s_{ik}+s_{jk}} \cdot \hat{p}_{ij}. \tag{23}$$

Using these identities, the lower right entry of the matrix $\mathcal{H}$ can be written as follows:

$$\mathcal{H}_{kk} = \frac{s_{ik}}{\hat{p}_{ik}^2} + \frac{s_{jk}}{\hat{p}_{jk}^2} = \frac{s_{ik}}{\frac{s_{ik}^2\hat{p}_{ij}^2}{(s_{ik}+s_{jk})^2}} + \frac{s_{jk}}{\frac{s_{jk}^2\hat{p}_{ij}^2}{(s_{ik}+s_{jk})^2}} = \frac{(s_{ik}+s_{jk})^3}{s_{ik}\,s_{jk}} \cdot \frac{1}{\hat{p}_{ij}^2}. \tag{24}$$

By factoring out $\mathcal{H}_{kk}$ from the last row, the lower right entry becomes 1. We add multiples of the last row to rows i and j, so as to cancel their last entries. The resulting upper left block with one fewer row and one fewer column is the Hessian matrix of the scattering potential for the 2-tree which is obtained from T by removing vertex k and its two incident edges ik and jk. However, in the new matrix s_{ij} is now replaced by $s_{ij}+s_{ik}+s_{jk}$. Note that this sum equals $[s_{ij}]$ if $v_T(ij)=2$. Proceeding inductively, more and more terms get added, and eventually each Mandelstam invariant s_{lm} is replaced by the corresponding sum $[s_{lm}]$.

In the end, we find that the determinant of $\mathcal{H}$ is equal to the product of the quantities

$$\frac{[s_{ik}]}{\hat{p}_{ik}^2} + \frac{[s_{jk}]}{\hat{p}_{jk}^2} = \frac{([s_{ik}]+[s_{jk}])^3}{[s_{ik}]\,[s_{jk}]} \cdot \frac{1}{\hat{p}_{ij}^2}, \tag{25}$$

where ijk ranges over all triangles of the 2-tree T. A combinatorial argument like that presented above shows that this product is equal to (22). This completes the proof. □

Remark 4.6. Our proof rests on the fact that the Hessian is the product of the expressions (25). Another way to get this is to directly triangularize the scattering equations $\nabla L_T = 0$. Using Corollary 3.3 and the trinomials defining X_T, we see that $\nabla L_T = 0$ is equivalent to

$$\frac{[s_{ik}]}{p_{ik}} + \frac{[s_{jk}]}{p_{jk}} = 0 \quad \text{for all triangles } ijk \text{ of } T.$$

This is a triangular system of $n-3$ equations in the unknowns $x_1, x_2, \ldots, x_{n-3}$. The Jacobian of this system is upper triangular, and its determinant is the product of the expressions (25).

5. On-shell diagrams and hypertrees

On-shell diagrams were developed by Arkani-Hamed, Bourjaily, Cachazo, Postnikov and Trnka [ABCPT16]. They encode rational parts, or leading singularities [Cacha08], occurring in scattering amplitudes for $\mathcal{N} = 4$ Super Yang–Mills theory [ABCPT16, § 4.7]. These are defined using the spinor-helicity formalism on a product of Grassmannians, $\mathrm{Gr}(2,n) \times \mathrm{Gr}(2,n)$. We focus on the special case of MHV on-shell diagrams, where leading singularities are rational functions in the coordinates p_{ij} ($= \langle ij \rangle$, in physics notation [ABCPT15, § 2.3]) of a single Grassmannian $\mathrm{Gr}(2,n)$. The resulting discontinuities of MHV amplitudes are, for us, CHY integrands.

In [ABCPT15, § 2.2], an identification was proposed between on-shell diagrams and certain collections of $n-2$ triples in $[n] = \{1, \ldots, n\}$. Among these on-shell diagrams are the hypertrees of Castravet–Tevelev [CT13]. This identification is again noted in [Tev, Lemma 9.5].

In algebraic geometry, hypertrees represent effective divisors on the moduli space $\mathcal{M}_{0,n}$. In physics, one can also consider on-shell diagrams for higher Grassmannians $\mathrm{Gr}(k,n)$; cf. [FGPW15]. The analogs to hypertrees are now $(n-k)$-element collections of $(k+1)$-sets in $[n]$. These define effective divisors on the configuration spaces $X(k,n) = \mathrm{Gr}(k,n)^o/(\mathbb{C}^*)^n$. It would be interesting to examine these through the lens of [CT13, Tev]. In this paper, we stay with $k = 2$.

We define a *hypertree* to be a collection T of $n-2$ triples $\Gamma_1, \ldots, \Gamma_{n-2}$ in $[n]$ such that:

(a) each $i \in [n]$ appears in at least two triples; and
(b) $|\bigcup_{i \in S} \Gamma_i| \geq |S| + 2$ for all non-empty subsets $S \subseteq [n-2]$.

Hypertrees have the same number of triples as 2-trees, but 2-trees are not hypertrees because some i appears in only one triple. However, if axiom (a) for hypertrees is dropped but (b) is kept, one obtains all nonzero leading singularities of MHV amplitudes. In particular, 2-trees satisfy (b), and one might view them as hypertrees in a weak sense. A hypertree is called *irreducible* if the inequality in (b) is strict for $2 \leq |S| \leq n-3$; see [CT13, Definition 1.2].

Example 5.1 ($n = 6$). The 2-trees in (14) are not hypertrees. The following is a hypertree:

$$T = \{\, 123, 345, 156, 246 \,\}, \tag{26}$$

but it is not a 2-tree. The hypertree T corresponds to the octahedral on-shell diagram in [CEGM19, Figure 1.1]. See also [CT13, Figure 2] and [HYZZ18, § 5.2]. This hypertree is irreducible, in the sense defined above, and it will serve as our running example throughout this section.

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Many of the concepts for 2-trees from the previous sections make sense for hypertrees T. We define the $(n-2)\times n$ matrix M_T as in § 4. The rows of M_T are the vectors $p_{jk}e_i - p_{ik}e_j + p_{ij}e_k$ for $\{i,j,k\}\in T$. These span the kernel of the $2\times n$ matrix X in (1). The gcd of the maximal minors of M_T is a polynomial $\Delta(M_T)$ of degree $n-3$ in the Plücker coordinates p_{ij}. The equation $\Delta(M_T)=0$ defines a divisor in $\mathcal{M}_{0,n}$, namely the *hypertree divisor*. If T is a 2-tree then the hypertree divisor is a union of Schubert divisors $\{p_{ij}=0\}$, as seen in Lemma 4.2. For the geometric application in [CT13], this case is uninteresting. Instead, Castravet and Tevelev focus on hypertree divisors that are irreducible; see [CT13, Theorem 1.5].

Example 5.2 (Irreducible hypertree). The hypertree T in (26) is irreducible. Its matrix is

$$M_T = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \\ \begin{bmatrix} p_{23} & -p_{13} & p_{12} & 0 & 0 & 0 \\ 0 & 0 & p_{45} & -p_{35} & p_{34} & 0 \\ p_{56} & 0 & 0 & 0 & -p_{16} & p_{15} \\ 0 & p_{46} & 0 & -p_{26} & 0 & p_{24} \end{bmatrix} \end{array}.$$

The hypertree divisor for T is an irreducible surface in the threefold $\mathcal{M}_{0,6}$. It is defined by

$$\Delta(M_T) = p_{12}p_{35}p_{46} \; - \; p_{13}p_{26}p_{45}.$$

This polynomial is irreducible in the coordinate ring of $\mathrm{Gr}(2,6)$. See also [CT13, Figure 2]. The corresponding on-shell diagram appears in [ABCPT15, (2.18)] where different labelings are used.

For every hypertree T, we define the *CHY integrand* $\mathcal{I}_T$ by the formula in (16). This is a rational function of degree $-n$ in the Plücker coordinates p_{ij}. The *scattering potential* for T is the log-likelihood function L_T in (7) where the sum is over all pairs (i,j) that are contained in some triple of T. Thus we set $s_{ij}=0$ in (2) for all non-edges ij of T. The hypertree T in (26) has three non-edges, namely 14, 25 and 36. Furthermore, we use the same formulas as in (19) to define the *hypertree amplitude* for T. To be precise, we set

$$m_T = \sum_{\hat{p}} \frac{(\mathcal{I}_T)^2}{\mathrm{Hess}(L_T)}(\hat{p}), \tag{27}$$

where the sum ranges over all critical points $\hat{p}$ of the scattering potential L_T. This is a rational function of degree $3-n$ in the Mandelstam invariants s_{ij} where (i,j) appears in T.

Remark 5.3. Our definition of L_T for hypertrees T differs from that for 2-trees in (7). The difference arises from the restriction to the basis S in (6) which reflects the gauge fixing in (1). In the new definition, L_T has no critical point when T is a 2-tree. For instance, let $n=4$ and $T=\{123,124\}$. This T has one non-edge, namely 34, which is not in $S=\{13,23\}$. The new definition requires us to set $s_{34}=-s_{13}-s_{23}$ to zero, so that $s_{23}=-s_{13}$. In this case, the function in (4) becomes $L_T = s_{13}\cdot\log(x_1/(1-x_1))$, which has no critical point. Minimal kinematics only arises when s_{34} remains an unknown, resulting in the critical point (5).

Example 5.4 (Hypertree amplitude). We compute the amplitude m_T for the hypertree T in (26). The generic $n=6$ scattering potential L has six critical points $\hat{p}$, but the restricted scattering

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potential L_T has only two. The expression (27) also makes sense for L. We have

$$m_T = \sum_{\hat{p}} \frac{1}{\mathrm{Hess}(L)} \left(\frac{(p_{12}p_{35}p_{46} \, - \, p_{13}p_{26}p_{45})^2}{(p_{12}p_{13}p_{23})(p_{34}p_{35}p_{45})(p_{15}p_{16}p_{56})(p_{24}p_{26}p_{46})} \right)^2 (\hat{p}). \tag{28}$$

This evaluates to a rational function in the Mandelstam invariants. For generic s_{ij}, we find

$$\begin{aligned} m_T = {} & \frac{1}{s_{16}s_{24}s_{35}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{13}s_{26}s_{45}} + \frac{1}{s_{15}s_{23}s_{46}} + \frac{1}{s_{12}s_{35}s_{46}} + \frac{1}{s_{15}s_{26}s_{34}} + \frac{1}{s_{12}s_{34}s_{56}} \\ & + \frac{1}{s_{12}s_{35}s_{124}} + \frac{1}{s_{24}s_{35}s_{124}} + \frac{1}{s_{12}s_{56}s_{124}} + \frac{1}{s_{24}s_{56}s_{124}} + \frac{1}{s_{13}s_{24}s_{56}} + \frac{1}{s_{15}s_{34}s_{125}} + \frac{1}{s_{12}s_{46}s_{125}} \\ & + \frac{1}{s_{15}s_{46}s_{125}} + \frac{1}{s_{13}s_{26}s_{134}} + \frac{1}{s_{26}s_{34}s_{134}} + \frac{1}{s_{12}s_{34}s_{125}} + \frac{1}{s_{34}s_{56}s_{134}} + \frac{1}{s_{15}s_{23}s_{145}} + \frac{1}{s_{15}s_{26}s_{145}} \\ & + \frac{1}{s_{23}s_{45}s_{145}} + \frac{1}{s_{26}s_{45}s_{145}} + \frac{1}{s_{13}s_{56}s_{134}} + \frac{1}{s_{16}s_{35}s_{235}} + \frac{1}{s_{23}s_{46}s_{235}} + \frac{1}{s_{35}s_{46}s_{235}} + \frac{1}{s_{13}s_{24}s_{245}} \\ & + \frac{1}{s_{16}s_{24}s_{245}} + \frac{1}{s_{16}s_{23}s_{235}} + \frac{1}{s_{16}s_{45}s_{245}} + \frac{1}{s_{13}s_{45}s_{245}}. \end{aligned}$$

We next impose the constraints $s_{14} = s_{25} = s_{36} = 0$ coming from the hypertree T. Some poles now become spurious. By collecting distinct maximal nonzero residues and then canceling spurious poles, we obtain the Feynman diagram expansion for the hypertree amplitude:

$$\begin{aligned} m_T = {} & \frac{1}{s_{16}s_{24}s_{35}} + \frac{1}{s_{16}s_{23}s_{45}} + \frac{1}{s_{13}s_{26}s_{45}} + \frac{1}{s_{15}s_{23}s_{46}} + \frac{1}{s_{12}s_{35}s_{46}} + \frac{1}{s_{13}s_{24}s_{56}} + \frac{1}{s_{12}s_{34}s_{56}} \\ & + \frac{1}{s_{15}s_{26}s_{34}} + \frac{s_{15} + s_{45}}{s_{15}s_{23}s_{26}s_{45}} + \frac{s_{12} + s_{24}}{s_{12}s_{24}s_{35}s_{56}} + \frac{s_{12} + s_{15}}{s_{12}s_{15}s_{34}s_{46}} + \frac{s_{13} + s_{34}}{s_{13}s_{26}s_{34}s_{56}} + \frac{s_{13} + s_{16}}{s_{13}s_{16}s_{24}s_{45}} \\ & + \frac{s_{16} + s_{46}}{s_{16}s_{23}s_{35}s_{46}}. \end{aligned}$$

This is the rational function (27), where we sum over the two critical points of L_T. This amplitude has 12 poles, 24 compatible pairs, and 14 Feynman diagrams, in bijection with the faces of a rhombic dodecahedron shown in Figure 1. By comparison, the amplitude in (9) has 9 poles and is a sum over 14 Feynman diagrams, given combinatorially by the associahedron.

It was shown in [ABCPT15, § 3.2] that, for any on-shell diagram T, the CHY integrand $\mathcal{I}_T$ decomposes as a sum of Parke–Taylor factors $PT(\alpha_1, \ldots, \alpha_n)$. Therefore, m_T is a linear combination of biadjoint amplitudes $m(\alpha, \beta)$ as α, β range over pairs of cyclic orders on $[n]$ that are both cyclic shuffles of the triples in T. See [CHYB14] for details of this construction. Such a decomposition with seven terms is shown in [ABCPT15, (3.12)] for the integrand $\mathcal{I}_T$ in (28).

This example raises several questions for future research. The first concerns the ML degree of an arbitrary hypertree T. By this we mean the number of complex critical points of the function L_T. The hypertree T in (26) has $\mathrm{MLdegree}(T) = 2$.

Question 5.5. Can we find a formula for the ML degree of an arbitrary on-shell diagram, and in particular for an arbitrary hypertree T? How is that ML degree related to the geometry of the hypertree divisor $\{\Delta(M_T) = 0\}$ on the moduli space $\mathcal{M}_{0,n}$?

Remark 5.6. We computed the ML degrees for the hypertrees in the Opie–Tevelev database https://people.math.umass.edu/~tevelev/HT˙database/database.html. For instance, for $n = 9$,

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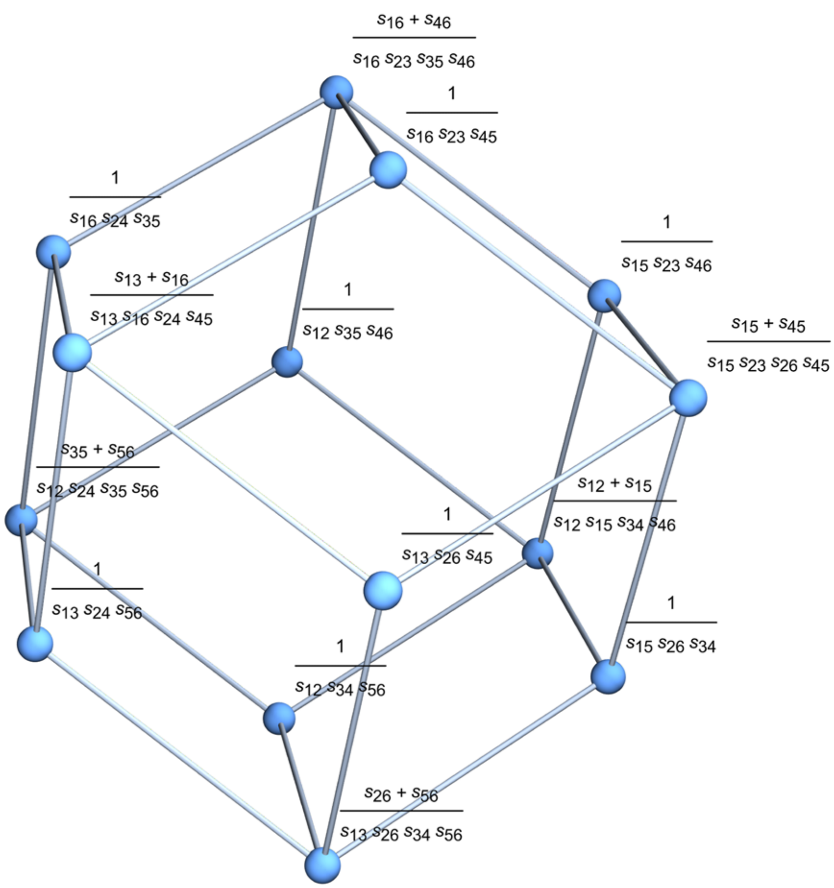

FIGURE 1. Combinatorics of the octahedral hypertree amplitude m_T.

the ML degrees range from 8 to 16. For a concrete example consider the hypertree $T = \{123, 129, 456, 789, 147, 258, 367\}$. Here the ML degree equals 10, i.e. the function L_T has 10 critical points. By contrast, the general scattering potential L in (2) has 720 critical points, and its $m(\alpha, \beta)$ expansion involves 3185 Feynman diagrams. After imposing $s_{ij} = 0$ for every non-edge ij, we find that 297 distinct maximal residues remain in the amplitude m_T. This is still a considerable amount of structure to be found from only 10 critical points.

Returning to the title of this paper, we ought to be looking for minimal kinematics.

Question 5.7. For any hypertree T, how can we best reach ML degree 1 by restricting the scattering potential L_T to a subspace of kinematic space? In particular, can we always reach a Horn uniformization formula (11) for the critical points $\hat{p}$ by setting some multiple-particle poles $s_{ij\cdots k}$ to zero? This would lead to a formula like (13) for the specialized amplitude m_T.

The following computation shows that we can indeed do this for our running example.

Example 5.8. In Example 5.4 we set the three-particle pole $s_{234} = s_{23} + s_{34} + s_{24}$ to zero, in addition to $s_{14} = s_{25} = s_{36} = 0$. This results in a dramatic simplification of the amplitude:

$$m_T = \frac{(s_{12} + s_{45})\,(s_{13} + s_{46})\,(s_{26} + s_{35})}{s_{12}s_{13}s_{26}s_{35}s_{45}s_{46}}.$$

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The ML degree is now 1. By Huh's theorem [Huh14], the critical point is given by a Horn pair (H,λ). In short, the subspace $\{s_{14}=s_{25}=s_{36}=s_{234}=0\}$ exhibits minimal kinematics.

Finally, all of our questions extend naturally from $\mathrm{Gr}(2,n)$ to $\mathrm{Gr}(k,n)$. Using physics acronyms, we seek to extend our amplitudes m_T from CHY theory [CHY14] to CEGM theory [CEGM19]. For example, the CEGM potential on the 4-dimensional space $X(3,6)=\mathrm{Gr}(3,6)^o/(\mathbb{C}^*)^6$ is

$$L=\sum_{1\le i<j<k\le 6}\log(p_{ijk})\cdot\mathfrak{s}_{ijk}.$$

This log-likelihood function is known to have 26 critical points; see e.g. [ST21, Proposition 5]. We now restrict to the kinematic subspace $\{\mathfrak{s}_{135}=\mathfrak{s}_{235}=\mathfrak{s}_{246}=\mathfrak{s}_{256}=\mathfrak{s}_{356}=\mathfrak{s}_{245}=0\}$. Then the ML degree drops from 26 to 1. In short, this subspace exhibits minimal kinematics.

Question 5.9. Can we characterize minimal kinematics for the configuration space $X(k,n)=\mathrm{Gr}(k,n)^o/(\mathbb{C}^*)^n$? What plays the role that 2-trees play in Theorem 1.4? Can we determine the ML degree of on-shell diagrams for $k\ge 3$? Ambitiously, we seek an all k and n peek [CE21]. Furthermore, we ask the same question for the realization space of any rank k matroid on $[n]$.

Revisiting the case $k=2$ once more, here is a complexity question about graphs for $\mathcal{M}_{0,n}$.

Question 5.10. The 2-trees are exactly the maximal graphs of treewidth 2. Can we compute the number of critical points of the scattering potential function for a graph of treewidth t?

Conflicts of Interest

None.

Journal Information

Moduli is published as a joint venture of the Foundation Compositio Mathematica and the London Mathematical Society. As not-for-profit organisations, the Foundation and Society reinvest 100% of any surplus generated from their publications back into mathematics through their charitable activities.

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