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LEFT CAUCHY INTEGRAL BASES IN LINEAR TOPOLOGICAL SPACES

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1. **Introduction.** The purpose of this paper is to consider a representation for the elements of a linear topological space in the form of a σ-integral over a linearly ordered subset of V; this ordered subset is what will be called an L basis. The formal definition of an L basis is essentially an abstraction from ideas used, often tacitly, in proofs of many of the theorems concerning integral representations for continuous linear functionals on function spaces.

The L basis constructed in this paper differs in several basic ways from the integral basis considered by Edwards in [5]. Since the integrals used here are of Hellinger type rather than Radon type one has in the approximating sums for the integral an immediate and natural analogue to the partial sum operators of summation basis theory. Because of this one finds that the standard theorems of summation basis theory such as the Bessaga–Pelcynski weak basis theorem, the Banach–Newns–Arsove theorem and the Grynblum–Russo theorem can not only be shown to hold for L bases but can be proved by methods which are nearly identical to those used in the summation basis case. By use of an integral of Hellinger type one is also enabled to consider bases in nonlocally convex spaces. Another difference between the basis studied here and the Radon integral basis considered by Edwards is that the Hellinger integrals of the type used in this paper are well known to correspond to integration with respect to only finitely additive measures. A final difference between the L basis and the Radon integral basis of Edwards is that the theory of L bases does not include the theory of summation bases as a special case. The relationship between Schauder bases and L bases in separable Banach spaces is investigated in the final section of this paper.

2. **Preliminary results.** Throughout this paper all linear vector spaces are assumed, unless otherwise stated, to be infinite dimensional spaces over the real or complex number field. If V is a linear vector space, N will denote the null vector for V.

In this paper we shall be concerned with linearly ordered sets $(S, <)$ such that $<$ is antisymmetric, and S has largest and smallest elements relative to $<$. We shall call such an ordered set a generalized interval. If $S=(\{x_j : j\in J\}, <)$ is a generalized interval the initial and final elements of S will be denoted by $x_{j(0)}$ and $x_{j(f)}$ respectively and the interval itself by $[x_{j(0)}, x_{j(f)}]$. The notations $(x_{j(0)}, x_{j(f)})$, $[x_{j(0)}, x_{j(f)})$, and

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$(x_{j(0)}, x_{j(f)}]$ will be used in the same sense in which they are used in classical analysis. When topological properties of a generalized interval $(S, <)$ are considered in this paper it is always assumed that $(S, <)$ has been topologized with the order topology.

A subdivision of a generalized interval $(S, <)$ is a finite subset $\{x_{j(i)}\}_{i=1}^{p}$ of S having the property that $x_{j(0)}=x_{j(1)}<x_{j(2)}<\cdots<x_{j(p)}=x_{j(f)}$. The collection of all subdivisions of a given generalized interval will be denoted by $\mathscr{D}$. If D and E are subdivisions of a generalized interval and $D\subset E$ then E will be said to be a refinement of D. Most of the integration processes used in the remainder of this paper will involve an integral of the left Cauchy type over a generalized interval.

DEFINITION 1. Suppose S is a generalized interval, (V, τ) a linear Hausdorff space, f a function on S into the scalars for V, and G a function on S into V. If $D=\{x_{j(i)}\}_{i=1}^{p}$ is a subdivision of S let $\sum_D(\Delta f, G)$ denote $\sum_{i=1}^{p}[f(x_{j(i)})-f(x_{j(i-1)})]G(x_{j(i-1)})$. The statement that $I\in V$ is the left Cauchy integral of G with respect to $f(I=(L)\int_s (df)G)$ means that if the set, $\mathscr{D}$, of all subdivisions of S is ordered by refinement then $\lim_{\gamma} \sum_D(\Delta f, G)=I$; i.e. the net $\{\sum_D(\Delta f, G): D\in\mathscr{D}\}$ τ-converges to I.

We shall also on occasion need the right Cauchy integral of f with respect to $G((R)\int_s f\,dG)$ which is the limit under refinement of sums of the form $\sum_{i=1}^{p} f(x_{j(i)})[G(x_{j(i)})-G(x_{j(i-1)})]$.

All of the linearity properties of the usual Cauchy integrals clearly hold here and will be used without further comment. Also the usual integration by parts formula for Cauchy integrals is valid in this setting.

The usual definition of bounded variation is immediately applicable to functions on a generalized interval S into a metric space. Given such a function F its variation on S will be denoted by $V_S F$. Among the functions of bounded variation, one class will play a central role in this paper. This is the class of complex valued step functions. A step function on a generalized interval is defined in the same way as a step function on a number interval. There are three classes of step functions with which we will be primarily concerned.

DEFINITION 2. Suppose S is a generalized interval and $x_k\in S$. Then τ_{x_k}, ρ_{x_k}, and θ_{x_k} are the functions defined by:

$$\tau_{x_k}(x_j)=\begin{cases}0, & x_j\le x_k\\ 1, & x_j>x_k\end{cases},\ x_j\in S,\qquad \rho_{x_k}(x_j)=\begin{cases}1, & x_j<x_k\\ 0, & x_j\ge x_k\end{cases},\quad x_j\in S,$$

and

$$\theta_{x_k}(x_j)=\begin{cases}0, & x_j\ne x_k\\ 1, & x_j=x_k\end{cases},\quad x_j\in S.$$

If S is a generalized interval we will denote the uniform closure of the complex linear combinations of the τ_{x_k} functions on S by Q_L^S. It is easily shown that Q_L^S with the sup norm topology is a Banach space. If S is a real number interval

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$[a, b]$ then $Q_L^{[a,b]}$ coincides with the class of all quasicontinuous functions on $[a, b]$ which are left continuous on $(a, b]$ and anchored at a. One property of the τ_{x_k} and ρ_{x_k} functions which will be used in the remainder of this paper is contained in Theorem 1; the proof being an immediate consequence of Definitions 1 and 2.

THEOREM 1. *Suppose S is a generalized interval and $x_k \in S$. If G is a function on S into a linear Hausdorff space and $x_k < x_{j(f)}$ then $(L) \int_S (d\tau_{x_k})G$ exists and is equal to $G(x_k)$. If f is a complex valued function on S and $x_k > x_{j(0)}$ then $(R) \int_S f\, d\rho_{x_k}$ exists and is equal to $-f(x_k)$.*

For a further discussion of integrals of the Cauchy type the reader is referred to Hildebrandt [6, Ch. II].

3. L **bases in linear Hausdorff spaces.** If one considers the left Cauchy integral as a form of generalized summation then it is rather natural to consider constructions such as the following.

DEFINITION 3. Suppose V is a linear Hausdorff space and $B = (\{x_j : j \in J\}, <)$ is a generalized interval of distinct elements of V with $x_{j(f)} = N$. Let M denote the collection of all scalar valued bounded left anchored functions on B. The statement that B is an L basis for V means that there exists a unique scalar valued map ϕ on $V \times B$ such that if y is in V then:

(1) $\phi(y, -) \in M$, and

(2) $\{\sum_D [\Delta\phi(y, -), \mathscr{I}] : D \in \mathscr{D}\}$ is bounded and $(L) \int_B [d\phi(y, x_j)]\mathscr{I}(x_j) = y$, where $\mathscr{I}$ is the identity operator on V.

NOTATION. If $x_j \in B$, the linear functional on V, $\phi(-, x_j)$ will be referred to as the x_j-th coordinate functional.

It should be noted that the order on B is not assumed to have any relation to any given structure in V and that the order topology on B need have no relation to the given topology on V.

It is obvious from Definitions 3 and 1 that if B is an L basis for V then $B' = B - \{x_{j(f)}\}$ is fundamental in V. It follows from Theorem 1 that if $x_k \in B'$ then $\phi(x_k, -) = \tau_{x_k}$. Since $\{\tau_{x_k} : x_k \in B'\}$ is a linearly independent set and since the left Cauchy integral is linear in the integrator position it follows from the uniqueness of ϕ that B' is also linearly independent in V.

It is easily shown from Theorem 1 that if V is a finite dimensional inner product space and $\{x_i\}_{i=1}^p$ is a basis for V then $B = \{x_i\}_{i=1}^p \bigcup \{x_{p+1} = N\}$ is an L basis for V if $<$ is taken to be the order induced by the indexing set. In this case ϕ will be given by $\phi(y, x_j) = \sum_{i=1}^p \langle y, x_i \rangle \tau_{x_i}(x_j)$. A more interesting example however is given by:

EXAMPLE 1. In l^1 let $B' = \{\delta n : n \in \omega\}$ be the usual Schauder basis and let $<$ be any generalized interval order for $B = B' \bigcup \{N\}$ such that N is the last element of

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B. If z is in l^1 and p and q are positive integers then $\sum_{n=p+1}^{q} z(n)\tau_{\delta_n}$ is of bounded variation on $(B, <)$, being a finite linear combination of step functions, and

$$V_B\left(\sum_{n=p+1}^{q} z(n)\tau_{\delta_n}\right) \le \sum_{n=p+1}^{q} |z(n)|.$$

It follows therefore that $\sum_{n=1}^{\infty} z(n)\tau_{\delta_n}$ is a function of bounded variation and $\lim_{p\to\infty} V_B\,[\sum_{n=1}^{\infty} z(n)\tau_{\delta_n} - \sum_{n=1}^{p} z(n)\tau_{\delta_n}]$ is 0. Denote this function by $\phi(z, -)$. Theorem 15.1 [6, p. 69] can be shown to hold for the integral of Definition 1, and from this theorem and Theorem 1 it follows that

$$z = (L)\int_B [d\phi(z, x_j)]\mathscr{I}(x_j).$$

Theorem 1 together with integration by parts implies that the integral representation just obtained is unique, and B is an L basis for l^1.

An example of an L basis in a nonseparable space is suggested by the arguments used by Kaltenborn [7], to obtain a representation for the continuous linear functionals on the quasicontinuous functions on a number interval.

EXAMPLE 2. Suppose $[a, b]$ is a closed number interval and let V denote $Q_L^{[a,b]}$ with the sup norm topology. Let $B=\{\tau_t\colon a\le t\le b\}$ be ordered by the relation $<$, defined by $\tau_r<\tau_s$ if and only if $r<s$. Then $(B, <)$ is a generalized interval in V with $\tau_b=N$. Finally, define ϕ by

$$\phi(f, \tau_t) = f(t), \quad \forall f\in V, a \le t \le b.$$

Suppose that $D=\{\tau_{t(i)}\}_{i=0}^{p}$ is a subdivision of B and f is in V. By a straightforward algebraic manipulation it can be shown that

$$\sum_D (\Delta\phi(f, -)\mathscr{I}) = \sum_{i=1}^{p} f(t_i)[\tau_{t(i)} - \tau_{t(i-1)}].$$

It then follows by the same argument as that given in [7, pp. 704–705], that the net $\{\sum_D (\Delta\phi(f, -), \mathscr{I})\colon D\in\mathscr{D}\}$ converges uniformly to f, and this net is clearly bounded. The evaluation functionals are continuous linear functionals on $Q_L^{[a,b]}$. If t is in $(a, b]$ then the restriction of the t-th evaluation functional to B is ρ_t. It therefore follows in the same way as in Example 1 that ϕ satisfies the uniqueness requirement of Definition 3.

It is easily seen that the L basis constructed in Example 2 has the property that $B'=\{\tau_t\colon a\le t\le b\}$ is topologically free in $Q_L^{[a,b]}$. B' is not, however, a generalized basis as defined by Arsove and Edwards in [2], since one can show that the functionals biorthogonal to B' annihilate the continuous functions on $[a, b]$.

It would also seem reasonable to ask whether it is necessarily true that if B is an L basis with continuous coordinate functionals then B' is topologically free. For a Schauder basis this question is trivial, but that is not the case for an L basis. One does however have the following result:

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THEOREM 2. *Suppose V is a linear Hausdorff space, and B is an L basis for V with continuous coordinate functionals. If V has the property that every barrel is a neighborhood of N then B' is topologically free.*

Proof. Let C be the linear space whose elements are all of the functions on B of the form $\phi(y, -)$, for y in V, and suppose that C has the sup norm topology. Let $\mathscr{T}$ denote the one-to-one map on V onto C defined by $\mathscr{T}y=\phi(y, -)$. By hypothesis each element of $S=\{\phi(-, x_j): x_j \in B\}$ is a continuous linear functional on V, and S is pointwise bounded, and it then follows that $\mathscr{T}$ is a continuous map. The image under $\mathscr{T}$ of each element in the linear manifold spanned by B is a left continuous left anchored step function on B, and since B is fundamental in V it follows that C is contained in Q_L^B. It follows from Theorem 15.1 [6, p. 69], that every right anchored scalar valued function f which is of bounded variation on B may be extended to a continuous functional ψ_f on C. Then $\psi_f \circ \mathscr{T}$ is a continuous functional on V, and for each y in V,

$$(\psi_f \circ \mathscr{T})y = (L)\int_B [d\phi(y, x_j)]f(x_j).$$

Hence if $x_j \in B'$, θ_{x_j} generates a continuous functional on V and so B' is topologically free.

In several important respects the theory of L bases is rather similar to the theory of Schauder bases. For example Theorem 2 [1], holds also for L bases. A similar theorem has been given in [5] for the Radon integral basis.

THEOREM 3. *Suppose V is a complete linear metric space and B is an L basis for V. Then the coordinate functionals for B are continuous.*

Proof. For each subdivision D of B let $\mathscr{T}_D$ be the linear operator on V defined by

$$\mathscr{T}_D y = \sum_D (\Delta\phi(y, -), \mathscr{I}).$$

For each y in V, $\{\mathscr{T}_D y: D \in \mathscr{D}\}$ is bounded, hence a new paranorm, $!!'$, may be defined on V by the relation

$$!y!' = \sup_{D \in \mathscr{D}} !\mathscr{T}_D y!, \quad \forall y \in V. \tag{1}$$

Clearly $!y! \le !y!'$, $\forall y \in V$, and if each of D and E is a subdivision of B then

$$!\mathscr{T}_D y - \mathscr{T}_E y! \le 2\,!y!', \quad \forall y \in V. \tag{2}$$

If $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in V relative to $!!'$, it follows from equations (1), (2), and consideration of $\mathscr{T}_{D_0}$ and $\mathscr{T}_{D_k}$, where $D_0=\{x_{j(0)}, x_{j(f)}\}$, and $D_k =\{x_{j(0)}, x_k, x_{j(f)}\}$, that $\{\phi(y_n, -)\}_{n=1}^{\infty}$ is a pointwise Cauchy sequence on B. Let θ denote the scalar valued function on B which is the limit of this sequence. It can now be shown by the same type of argument as that given in the proof of Theorem

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1 [12, p. 207], that $(L)\int_B [d\theta(x_j)]\mathscr{I}(x_j)$ exists, that $\{y_n\}_{n=1}^{\infty}$ converges to the value of this integral in the !!′-topology, and that $\phi(-, x_k)$ is continuous for all x_k in B.

There are also analogues for L bases to the weak basis theorems of Dieudonné [4, Proposition 5], and Bessaga and Pelczyński [3, Theorem 4], for Schauder bases.

THEOREM 4. *Suppose (V, τ) is a barrelled space and B is a $\sigma(V, V^*)L$ basis for V such that the coordinate functionals are continuous. Then B is an L basis for V with the τ topology.*

Proof. By hypothesis, if D is a subdivision of B, $\mathscr{T}_D$ is a continuous linear operator on V. From Definition 3, $\{\mathscr{T}_D y: D \in \mathscr{D}\}$ is a $\sigma(V, V^*)$ bounded and hence τ bounded set for each y in V. Hence $\{\mathscr{T}_D: D \in \mathscr{D}\}$ is a τ equicontinuous collection. The span of B is τ dense in V since it is $\sigma(V, V^*)$ dense. It then follows by the same argument used in the proof of Lemma 1 [8, p. 72], that $\{\mathscr{T}_D y: D \in \mathscr{D}\}$ τ-converges to y for every y in V.

THEOREM 5. *Suppose (V, τ) is a Fréchet space and B is a $\sigma(V, V^*)L$ basis for V. Then B is an L basis for V with the τ topology.*

The proof of this theorem follows in essentially the same manner as in Lemma 2 [8]. The differences in the proof here are essentially the same as the differences between the proofs of Theorem 3 and Theorem 1 [12, p. 207].

Finally, one has a theorem analogous to Theorem 3.1 [9].

THEOREM 6. *Suppose V is a linear Hausdorff space and B is a generalized interval of distinct elements of V with $x_{j(f)}=N$. A sufficient condition that B be an L basis with continuous coordinate functionals for its closed linear span is that for each neighborhood U of N there exists a neighborhood W_U of N such that if $D=\{x_{j(i)}\}_{i=0}^{p}$ is a subdivision of B, $E=\{x_{j(k)}\}_{i=0}^{q}$ is a refinement of D, and $\sum_{k=0}^{q-1} t_k x_{j(k)}$, $x_{j(k)} \in E$, is in W_U then*

$$(3) \qquad \sum_{i=0}^{p-1}\Big(\sum_{k\in s_i} t_k\Big)x_{j(i)} \in U, \quad x_{j(i)} \in D,$$

where

$$s_i = \{k: x_{j(i)} \le x_{j(k)} < x_{j(i+1)}, x_{j(i)} \text{ and } x_{j(i+1)} \in D, x_{j(k)} \in E\}.$$

If $\overline{\text{sp}B}$ is barrelled or has the t-property this condition is also necessary.

Proof. The necessity of this condition is an immediate consequence of Definition 3 and the continuity of the coordinate functionals.

Conversely, suppose U is a neighborhood of N. If y is in $\text{sp}B \cap W_U$ there exists a subdivision $D=\{x_{j(i)}\}_{i=0}^{p+1}$ of B such that $y=\sum_{i=0}^{p} t_i x_{j(i)}$. Since D is a refinement of D_0 it follows that

$$(4) \qquad \Big(\sum_{i=0}^{p} t_i\Big)x_{j(0)} \in U.$$

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If x_k is an arbitrary element of $(x_{j(0)}, x_{j(f)})$ it may be assumed that x_k is in D. Therefore D refines D_k and it follows that

$$\left(\sum_{i:x_{j(i)}<x_k} t_i\right)(x_k - x_{j(0)}) - \left(\sum_{i=0}^{p} t_i\right)x_k \in U. \tag{5}$$

Relations (4) and (5) imply that $B-\{x_{j(f)}\}$ is linearly independent, and consequently that $\rho_{x_{j(f)}}$ and ρ_{x_k} may be extended to continuous linear functionals $\psi_{x_{j(f)}}$ and ψ_{x_k} on $\overline{\text{sp}B}$. Let ϕ be the function $\tau_{x_{j(0)}}\psi_{x_j}$. Then for each subdivision D, $\mathscr{T}_D$ is continuous on $\overline{\text{sp}B}$. Since the collection $\{\mathscr{T}_D: D \in \mathscr{D}\}$ is equicontinuous on sp$B$ by the hypothesis of the theorem it follows that $\{\mathscr{T}_D: D \in \mathscr{D}\}$ is equicontinuous on $\overline{\text{sp}B}$. It follows as in Theorem 4 that B is an L basis for $\overline{\text{sp}B}$.

4. **Relations between L bases and Schauder bases.** We have seen in the discussion of Example 2 that an L basis need not be a generalized basis. This can occur even in a separable Banach space as can be seen by considering Q_L^R, where R denotes the rational numbers in [0, 1] with the usual ordering. The construction of Example 2 goes through without change for Q_L^R, and since there exist nontrivial continuous functions in Q_L^R the resulting L basis B has the property that B' is not a Schauder basis for Q_L^R.

There do exist, however, L bases which are Schauder bases. We have seen in Example 1 that $B=\{\delta_n\}_{n=0}^{\infty} \cup \{N\}$ is an L basis for l^1 with any generalized interval order on B. In particular this is true for what will be called the natural order on B, namely the order defined by $\delta_p<\delta_q$ if and only if $p<q$ and $\delta_p<N$ for all p. This observation leads to a theorem.

THEOREM 7. *Suppose $\{b_i\}_{i=0}^{\infty}$ is a bounded sequence in a linear Hausdorff space V having the property that every barrell is a neighborhood of N. If $\{b_i\}_{i=0}^{\infty} \cup \{N\}$ with the natural order is an L basis with continuous coordinate functionals for V then $\{b_i\}_{i=0}^{\infty}$ is a Schauder basis for V.*

Proof. Let l_i denote $[\phi(-, b_{i+1})-\phi(-, b_i)]$ for each nonnegative integer i. Then $\{(b_i, l_i)\}_{i=0}^{\infty}$ is a biorthogonal collection. For each nonnegative integer p let T_p be the linear operator defined by $T_p y=\sum_{i=0}^{p} l_i(y)b_i$. T_p can also be written in the form

$$T_p y = \mathscr{T}_{D_{p+1}}y+[\phi(y, b_{p+1})-\phi(y, N)]b_{p+1}, \quad \forall y \in V,$$

where D_{p+1} is the subdivision $\{b_0, b_1, \ldots, b_{p+1}, N\}$. It follows from the proof of Theorem 2 that $\lim_{p\to\infty} \phi(y, b_p)$ is $\phi(y, N)$. Therefore, since $\{b_i\}_{i=0}^{\infty}$ is bounded, the sequence $\{T_p y\}_{p=0}^{\infty}$ converges to y for each y in V.

Conversely, there exist linear topological spaces which admit a Schauder basis B' such that $B' \cup \{N\}$ cannot be ordered to form an L basis. Since the coordinate functionals for an L basis B in a complete linear metric space V are continuous it follows that there exists a continuous linear functional on V, namely $\phi(-, x_{j(f)})$,

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which takes on the value one at each element of B'. The existence of such a continuous linear functional is then a necessary condition for a Schauder basis B' in a complete linear metric space to have the property that $B' \cup \{N\}$ can be ordered to form an L basis. A bounded Schauder basis in a Banach space having the property that there exists a continuous linear functional which is one at each element of the basis has been called a basis of type P^* by Singer [11, Proposition 3]. This notation will be used here for bases in any linear Hausdorff space. As a consequence of the preceding argument and Theorem 4.1 [10], it may be concluded that if V is a reflexive Fréchet space no bounded Schauder basis in V generates an L basis.

Now suppose that V is a barrelled space and $B'=\{b_i\}_{i=0}^{\infty}$ is a Schauder basis of type P^* for V. Denote the ith coordinate functional for B' by l_i. Let $B=B' \cup \{N\}$ be given the natural order. Let ϕ be the function on $V \times B$ defined by

$$\phi(y, x_j) = \begin{cases} 0, & x_j = b_0 \\ \sum_{k<1} l_k(y), & x_j = b_i, \quad \forall y \in V, \\ \sum_{k=0}^{\infty} l_k(y), & x_j = N. \end{cases}$$

It is easily shown that if $\{\mathcal{T}_D : D \in \mathcal{D}\}$ is pointwise bounded on V then B is an L basis for V. One condition which guarantees this is for $\phi(y, -)$ to be of bounded variation for each y. This condition however implies that $\sum_{i=0}^{\infty} |l_i(y)|$ converges for each y. A second condition which guarantees the pointwise boundedness of $\{\mathcal{T}_D y : D \in \mathcal{D}\}$ is: if ψ is a continuous linear functional on V then $\{\psi(b_i)\}_{i=0}^{\infty}$ is a sequence of bounded variation. Since the collection $\{D_n\}_{n=0}^{\infty}$ is cofinal in $\mathcal{D}$ this condition implies that ψ/B is of bounded variation and the weak pointwise boundedness of $\{\mathcal{T}_D : D \in \mathcal{D}\}$ follows. It might be noted that the net $\{\mathcal{T}_{D_n}\}_{n=0}^{\infty}$,which is cofinal in $\{\mathcal{T}_D : D \in \mathcal{D}\}$, is always pointwise bounded if B' is of type P^*.

We conclude with an example which illustrates the material of this section. Let ω^* denote the set consisting of the nonnegative integers and ∞, and suppose ω^* is ordered with the usual order. Let V denote $Q_L^{\omega^*}$ with the sup norm topology. By the same arguments as in Example 2, or by Theorem 6, it can be shown that $B=\{\tau_t : t \in \omega^*\}$ is an L basis for V. The order on B induced by ω^* is the natural order on B so that by Theorem 7 $B'=\{\tau_t\}_{t=0}^{\infty}$ is a Schauder basis for $Q_L^{\omega^*}$. However, it can also be easily shown by a direct application of Theorem 5 [12, p. 211], that B' is a Schauder basis for $Q_L^{\omega^*}$. It can also be shown by the same type of argument used in [7] that if ψ is a continuous linear functional on V then ψ/B is of bounded variation, and since B' is bounded this implies that $\{\psi(\tau_t)\}_{t=0}^{\infty}$ is a sequence of bounded variation. It is also easily shown that B' is a basis of type P^* and so satisfies the conditions discussed in the previous paragraph. This again shows that B is an L basis for V.

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