UNIQUENESS OF SOLUTIONS OF IMPROPERLY POSED PROBLEMS FOR SINGULAR ULTRAHYPERBOLIC EQUATIONS

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1. Introduction

In [1], Owen gave sufficient conditions for the uniqueness of certain mixed problems having elliptic and hyperbolic nature for the ultrahyperbolic equation. Recently, Diaz and Young [2] has obtained necessary and sufficient conditions for the uniqueness of solutions of the Dirichlet and Neumann problems involving the more general ultrahyperbolic equation

$$\Delta u - D_j(a_{jk}D_k u) + cu = 0$$

The purpose of this paper is to present corresponding uniqueness conditions for the Dirichlet and Neumann problems for the singular ultrahyperbolic equation

$$Lu \equiv u_{tt} + (\alpha/t)u_t + \Delta u - D_j(a_{jk}D_k u) + cu = 0 \tag{1}$$

for all values of the parameter α, $-\infty < \alpha < \infty$. The symbol Δ denotes the Laplace operator in the variables $x_1, \cdots, x_m$, D_j indicates partial differentiation with respect to the variable y_j $(1 \leqq j \leqq n)$, and the summation convention is adopted for repeated indices including $(\partial_i u)^2$, where ∂_i denotes differentiation with respect to the variable x_i.

The boundary value problems will be considered in the domain $Q = X^* \times Y$ where X^* is the parallelepiped defined by $0 < t < T$, $0 < x_i < a_i (1 \leqq i \leqq m)$, and Y is a bounded domain in the space $y_1, \cdots, y_n$. The parallelepiped defined by $0 < x_i < a_i$ $(1 \leqq i \leqq m)$ will be denoted by X. For brevity, we write $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_n)$, and denote a point in Q by (t, x, y).

Throughout this paper, we assume that the coefficients a_{jk} and c depend only on the variables $y_1, \cdots, y_n$, and are continuous functions of these variables with

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$c \geqq 0$ in Y. As in [2] and [4], we also assume that the matrix (a_{jk}) is symmetric, positive definite, and that a_{jk}, c and the domain Y are sufficiently regular in order to allow the application of the divergence theorem and to ensure the existence of a complete set of eigenfunctions of class $C^2(Y) \cap C^1(\bar{Y})$ for the eigenvalue problems that will be needed below. By a solution of a boundary value problem considered here we shall mean a function $u \in C^2(Q) \cap C^1(\bar{Q})$ which satisfies the differential equation and the boundary condition of the problem.

The results given here include not only those obtained in [2], but also those derived by Dunninger and Zachmanoglov [3], [4], Sigillito [5] and Young [6] in the case of the normal hyperbolic equation.

2. The Dirichlet problem

We consider first the homogeneous Dirichlet problem

$$Lu = 0 \text{ in } Q, \quad u = 0 \text{ on } \partial Q \tag{2}$$

Corresponding to various ranges of the parameter α, we shall prove uniqueness of solution by showing that every solution of the problem vanishes identically in Q. We begin by stating a lemma which characterizes every smooth solution of the equation (1) for $\alpha \neq 0$.

LEMMA. *If $\alpha \neq 0$, then every solution u of* (1) *belonging to C^2 for $t > 0$ and to C^1 for $t \geqq 0$ satisfies the condition $u_t(0, x, y) = 0$.*

This lemma can be proved by following, almost step for step, the method employed by Fox [7] in establishing the same property for the corresponding singular normal hyperbolic equation in the case that (a_{jk}) is the identity matrix, using the domain Q.

THEOREM 1. *If $\alpha > 0$, then every solution of the problem* (2) *vanishes identically in Q.*

PROOF. Let u be a solution of the problem (2). We integrate the identity

$$\begin{aligned} 0 = 2u_t Lu = {} & [u_t^2 - (\partial_i u)^2 + a_{jk} D_j u D_k u + cu^2]_t \\ & + 2\partial_i(u_t \partial_i u) - 2D_j(a_{jk} u_t D_k u) \\ & + (2\alpha/t) u_t^2 \end{aligned}$$

over Q and use the divergence theorem to obtain

$$\begin{aligned} & \int_{X \times Y} (u_t^2 - (\partial_i u)^2 + a_{jk} D_j u D_k u + cu^2) \Big|_{t=0}^{t=T} dx dy \\ & + \int_{\delta Q} (2u_t \partial_i u v_i - 2u_t a_{jk} D_k u v_j^*) \, dS \end{aligned} \tag{3}$$

$$+ 2\alpha \int_Q (u_t^2/t)\, dtdxdy = 0$$

where v_i and v_j^* denote the components of the outward normal vector on ∂X and ∂Y, respectively. By the lemma and the fact that u vanishes on ∂Q, (3) reduces to

$$\int_{X\times Y} u_t^2(T,x,y)dxdy + 2\alpha \int_Q (u_t^2/t)dtdxdy = 0 \tag{4}$$

Since $\alpha > 0$, this implies that $u_t \equiv 0$ in Q, that is, u is independent of t. But $u = 0$ on $t = 0$, hence $u \equiv 0$ in Q.

THEOREM 2. *Let λ_r $(r = 1,2,\cdots)$ be the eigenvalues of the problem*

$$\begin{aligned} &D_j(a_{jk}D_k v) - cv + \lambda v = 0 \text{ in } Y \\ &v = 0 \text{ on } \partial Y. \end{aligned} \tag{5}$$

If $\alpha \leqq 0$, then every solution of the problem (2) vanishes identically in Q if and only if

$$J_{(1-\alpha)/2}(\mu^{\frac{1}{2}}T) \neq 0 \tag{6}$$

for any real number $\mu \neq 0$ and nonzero integers $p_1,\cdots,p_m$ such that

$$\mu + \sum_{i=1}^{m} (p_i\pi/a_i)^2 = \lambda_r, \tag{7}$$

where $J_\alpha(t)$ is the Bessel's function of the first kind of order α.

PROOF. Suppose that there exist an eigenvalue λ_s of (5), a real number $\mu_s \neq 0$, and nonzero integers $q_1,\cdots,q_m$ satisfying (7) such that

$$J_{(1-\alpha)/2}(\mu_s^{\frac{1}{2}}T) = 0. \tag{8}$$

Let v_s be an eigenfunction of (5) corresponding to λ_s, and define

$$\phi(x;q) = \prod_{i=1}^{m} \sin(q_i\pi x_i/a_i). \tag{9}$$

Then it is readily verified that the function

$$u(t,x,y) = t^{(1-\alpha)/2}J_{(1-\alpha)/2}(\mu_s^{\frac{1}{2}}t)\phi(x;q)v_s(y)$$

is a nontrivial solution of the problem (2).

Conversely, suppose that the conditions (6) and (7) hold. Let us integrate the identity

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$$\begin{aligned} wLu - uMw = (wu_t - w_t u + \alpha uw/t)_t \\ + \partial_i(w\partial_i u - u\partial_i w) \\ - D_j[a_{jk}(wD_k u - uD_k w)] \end{aligned}$$

over $Q_s = X_s^* \times Y$, where X_s^* is the parallelepiped defined by $0 < s \leqq t < T$, $0 < x_i < a_i$ $(1 \leqq i \leqq m)$, and M is the adjoint operator of L given by

$$\begin{aligned} Mw = w_{tt} - \alpha(w/t)_t + \Delta w \\ - D_j(a_{jk}D_k w) + cw. \end{aligned}$$

By the divergence theorem, we have

$$\begin{aligned} &\int_{Q_s} [wLu - uMw]dtdxdy \\ (10) \qquad &= \int_{\partial Q_s} [(wu_t - w_t u + \alpha uw/t)v_t + (w\partial_i u - u\partial_i w)v_i \\ &\quad - a_{jk}(wD_k u - uD_k w)v_j^*]dS. \end{aligned}$$

Now let u be a solution of (2) and for any choice of λ_r, $\mu \neq 0$, and nonzero integers $p_1, \cdots, p_m$ satisfying (6) and (7), let

$$w(t,x,y) = t^{(1+\alpha)/2} J_{(1-\alpha)/2}(\mu^{\frac{1}{2}}t)\phi(x;p)v_r(y)$$

where ϕ is defined in (9) and v_r is an eigenfunction associated with λ_r. Since $Lu = 0$ and

$$Mw = -t^{(1+\alpha)/2} J_{(1-\alpha)/2}(\mu^{\frac{1}{2}}t)\phi(x;p)[D_j(a_{jk}D_k v_r) - cv_r + \lambda_r v_r] = 0,$$

the left hand side of (10) vanishes. Moreover, since $u = 0$ on ∂Q and $w = 0$ on $X^* \times \partial Y$ and $\partial X \times Y$, equation (10) becomes

$$(11) \qquad \int_{X \times Y} (wu_t - w_t u + \alpha uw/t) \Big|_{t=s}^{t=T} dxdy = 0.$$

We now let s approach zero. Since both w_t and w/t are bounded at $t = 0$, and u vanishes there, we obtain in the limit

$$(12) \qquad T^{(1+\alpha)/2} J_{(1-\alpha)/2}(\mu^{\frac{1}{2}}T) \int_{X \times Y} u_t(T,x,y)\phi(x;p)v_r(y)dxdy = 0.$$

In view of (6) and the completeness of the sets of eigenfunctions $\{\prod_{i=1}^m \sin(p_i\pi x_i/a_i)\}$ and $\{v_r\}$ in X and Y, respectively, (12) implies that $u_t(T,x,y) = 0$. With this additional information, we can now show that $u \equiv 0$ in Q.

Let us integrate the identity

$$0 = (2tu_t + u)Lu = [t(u_t^2 - (\partial_i u)^2 + a_{jk}D_j u D_k u + cu^2) + u(u_t + \tfrac{1}{2}\alpha u/t]_t$$

$$(13) \qquad + \partial_i[(2tu_t + u)\partial_i u] - D_j[a_{jk}(2tu_t + u)D_k u] + 2(\alpha-1)u_t^2 + \tfrac{1}{2}\alpha u^2/t^2$$

over Q_s and pass to the limit as $s \to 0$. Since $u = 0$ on ∂Q, $u_t = 0$ on $t = 0$ and $t = T$, all surface integrals arising from the integration vanish in the limit, so that we are left with the convergent integral

$$\int_Q [2(\alpha - 1)u_t^2 + \tfrac{1}{2}\alpha u^2/t^2]dtdxdy = 0.$$

Since $\alpha \leqq 0$, this yields the result that $u \equiv 0$ in Q.

3. The Neumann problem

We consider next the homogeneous Neumann problem

$$(14) \qquad Lu = 0 \text{ in } Q, \ \partial u/\partial n = 0 \text{ on } \partial Q,$$

where $\partial u/\partial n$ denotes the conormal derivative

$$\partial u/\partial n = a_{jk}D_k u v_j^*$$

on the part $X^* \times \partial Y$ of ∂Q.

THEOREM 3. *Let λ_r $(r = 1, 2, \cdots)$ be the nonzero eigenvalues of the problem*

$$(15) \qquad \begin{aligned} D_j(a_{jk}D_k v) - cv + \lambda v &= 0 \ \text{in } Y, \\ \partial v/\partial n &= 0 \ \text{on } \partial Y. \end{aligned}$$

Then every solution u of the problem (14) *vanishes identically* (*or* $u =$ const. *if* $c \equiv 0$) *for* $\alpha \geqq 0$ *if and only if*

$$(16) \qquad J_{(1+\alpha)/2}(\mu^{\frac{1}{2}}T) \neq 0$$

for any real number $\mu \neq 0$ *and integers* $p_1, \cdots, p_m$ *satisfying* (7).

PROOF. The condition (16) is actually necessary for any value of the parameter α. In fact, if there exist a nonzero eigenvalue λ_s of (15), a real number $\mu_s \neq 0$, and integers $q_1, \cdots, q_m$ satisfying (7) such that

$$(17) \qquad J_{(1+\alpha)/2}(\mu_s^{\frac{1}{2}}T) = 0$$

then the function

$$(18) \qquad u(t, x, y) = t^{(1-\alpha)/2}J_{(\alpha-1)/2}(\mu_s^{\frac{1}{2}}t)\psi(x; q)v_s(y)$$

constitutes a nontrivial solution of the problem (14) for any value of α. Here

$$(19) \qquad \psi(x;q) = \prod_{i=1}^{m} \cos(q_i \pi x_i / a_i)$$

and v_s is an eigenfunction corresponding to λ_s. Indeed, by (15) it is easily shown that (18) satisfies $Lu = 0$, $\partial u/\partial n = 0$ on $X^* \times \partial Y$, and $\partial_i u = 0$ on $x_i = 0, x_i = a$ $(1 \leqq i \leqq m)$. Moreover, since

$$u_t = -\mu_s^{\frac{1}{2}} t^{(1-\alpha)/2} J_{(1+\alpha)/2}(\mu_s^{\frac{1}{2}} t)\psi(x;q)v_s(y) = O(t),$$

it follows that $u_t(0,x,y) = 0$ and by (17) $u_t(T,x,y) = 0$. Thus (18) is a nontrivial solution of the problem (14).

On the other hand, let $\alpha \geqq 0$ and assume that the condition (16) holds. Let λ_r be a nonzero eigenvalue of (15) with the corresponding eigenfunction v_r. For any choice of real number $\mu \neq 0$ and integers $p_1, \cdots, p_m$ satisfying (7) and (16), let

$$(20) \qquad w(t,x,y) = t^{(1+\alpha)/2} J_{(\alpha-1)/2}(\mu^{\frac{1}{2}} t)\psi(x;p)v_r(y)$$

where ψ is given by (19). By direct differentiation, it is readily verified that $Mw = 0$ in Q, $\partial w/\partial n = 0$ on ∂Q except on $t = 0$ and $t = T$. Hence, if u is a solution of (14), substitution of (20) for w in (10) leads again to the integral (11). Since

$$\begin{aligned} w_t = {} & [\alpha t^{(\alpha-1)/2} J_{(\alpha-1)/2}(\mu^{\frac{1}{2}} t) \\ & - \mu^{\frac{1}{2}} t^{(1+\alpha)/2} J_{(\alpha+1)/2}(\mu^{\frac{1}{2}} t)]\psi(x;p)v_r(y) \end{aligned}$$

it follows that

$$\begin{aligned} -w_t + \alpha w/t &= \mu^{\frac{1}{2}} t^{(1+\alpha)/2} J_{(\alpha+1)/2}(\mu^{\frac{1}{2}} t)\psi(x;p)v_r(y) \\ &= O(t^{\alpha+1}). \end{aligned}$$

Therefore, as s is allowed to approach zero in (11), we obtain in the limit

$$\mu^{\frac{1}{2}} T^{(1+\alpha)/2} J_{(\alpha+1)/2}(\mu^{\frac{1}{2}} T) \int_{X \times Y} u(T,x,y)\psi(x;p)v_r(y)dxdy = 0,$$

By the hypothesis (16) and the completeness of the sets of eigenfunctions $\{\prod_{i=1}^{m} \cos(p_i \pi x_i/a_i)\}$ and $\{v_r\}$ in X and Y, respectively, we conclude that $u(T,x,y) = 0$ if $c > 0$ and $u(T,x,y) = \text{const.}$ if $c \equiv 0$. Notice that in the case $c \equiv 0$, the problem (15) has the eigenfunction $v = 1$ corresponding to the eigenvalue $\lambda = 0$.

Let us consider the case $c > 0$. It remains to be shown that $u \equiv 0$ in Q. For this purpose, we note that the identity (13) no longer applies. We integrate instead the identity

$$\begin{aligned} [2t^{\alpha+1}u_t + (\alpha+1)t^\alpha u]Lu = {} & [t^{\alpha+1}(u_t^2 - (\partial_i u)^2 + a_{jk}D_j u D_k u + cu^2) \\ & + (\alpha+1)t^\alpha u u_t]_t + \partial_i[t^\alpha(2tu_t + (\alpha+1)u)\partial_i u] \end{aligned}$$

$$- D_j[t^\alpha a_{jk}(2tu_t + (\alpha + 1)u)D_k u] - 2t^\alpha u_t^2$$

over Q and apply the divergence theorem. Because u is a solution of (14) and $u = 0$ at $t = T$, it is clear that all surface integrals arising from the integration vanish. Thus we have

$$-2 \int_Q t^\alpha u_t^2 \, dt dx dy = 0$$

from which the conclusion that $u \equiv 0$ in Q follows.

If $c \equiv 0$, then the above argument gives $u =$ const. in Q.

4. Concluding remarks

By using the same technique, it is possible to prove uniqueness theorems for equation (1) subject to mixed boundary conditions of the type considered in [2] with respect to the variables x, y and with either the condition $u = 0$ or $u_t = 0$ on $t = 0$ and $t = T$.

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