A NOTE ON THE STOCHASTIC RANK OF A BIPARTITE GRAPH

A.L. Dulmage and N.S. Mendelsohn

(received March 26, 1959)

1. Introduction, Definitions and Notation. A bipartite graph is a system consisting of two sets of vertices S and T and a set of edges K, each edge joining a vertex of S to a vertex of T. A set U of edges of K is said to be independent if no two edges of U have a vertex in common. The largest possible number of independent edges has been variously called the exterior dimension [3], term rank [4, 5, 7], etc. This number is the same as the smallest number of vertices in a set W such that each edge of K has at least one of its vertices in W. The edges of a finite bipartite graph can be represented as a set of cells in a matrix as follows. If $S = a_1, a_2, \ldots, a_n$ $T = b_1, b_2, \ldots b_m$, the edges of K are represented by some of the cells of an n by m matrix as follows: if K contains the edge joining a_i to b_j then the (i,j)th cell of the matrix represents this edge. It is convenient sometimes to represent the set K by a matrix A with real entries a_{ij} where $a_{ij} = 0$ if a_i is not joined to b_j in K and $a_{ij} > 0$ if a_i is joined to b_j in K. Any non-null graph K will have infinitely many matrix representations.

A non-null matrix A with non-negative entries is said to be doubly stochastic if every row sum and every column sum of A has the same value p. Such a matrix [2] is a linear combination of permutation matrices with positive coefficients; $A = \sum c_i P_i$ where $c_i > 0$, $\sum c_i = p$ and the matrices P_i are permutation matrices.

Let G_1 and G_2 be graphs with vertex and edge sets S_1, T_1, K_1 and S_2, T_2, K_2 respectively. G_1 is said to be embedded in G_2 if $S_1 \leq S_2$, $T_1 \leq T_2$, $K_1 \subset K_2$ and if when e is an edge of K_2 but

Can. Math. Bull., vol.2, no.3, Sept. 1959

https://doi.org/10.4153/CMB-1959-020-4 Published online by Cambridge University Press

not an edge of K_1 then at least one of the ends of e is not a vertex of S_1 or of T_1. A matrix representation of G_2 is always obtainable from a matrix representation A_1 of G_1 by bordering A_1 with extra rows or columns.

In [4], the authors have defined the stochastic rank σ of an n by n matrix A with non-negative entries as follows: if A can be embedded in a doubly stochastic matrix by bordering it with $n - \sigma$ rows and columns but cannot be embedded in a doubly stochastic matrix by bordering it with fewer than $n - \sigma$ rows and columns, A is said to have stochastic rank σ. If K is a graph of term rank ρ and a matrix representation of K has stochastic rank σ, it has been shown in [1] that $\sigma \leq \rho$. The stochastic rank σ_K of a graph K whose vertex sets contain the same number of elements, is defined to be the maximum of the stochastic ranks of all matrix representations of K. Theorem 6 of [4] states that $\sigma_K = \rho$ or $\sigma_K = \rho - 1$. In this paper, we obtain a graphical proof of this result together with a natural condition which distinguishes the two cases $\sigma_K = \rho$ and $\sigma_K = \rho - 1$. For this purpose the following concepts defined in [3] are needed. An edge e of K is inadmissible if e does not appear in any maximal set of independent edges of K, otherwise e is admissible. The set of all admissible edges of K is said to be the core of K. K is called a core-graph if every edge of K is admissible.

2. THEOREM. Let G be a bipartite graph whose vertex sets each contain n elements and whose term and stochastic ranks are ρ and σ respectively. Then $\sigma = \rho$ if G is a core-graph and $\sigma + 1 = \rho$ if G is not a core graph.

Proof. The edges of G in all cases will be represented as cells in an n by n matrix. Two cases are distinguished.

Case 1. $\rho = n$. Suppose G is a core-graph. If e_i is any edge of G, there is at least one set of n independent edges of which e_i is a member. Such a set of edges is represented by n cells of a matrix exactly one of which is in each row and column. Associate with each edge e_i such a set of cells S_i and let P_i be the permutation matrix whose entries are 1 in the cells of S_i and 0 elsewhere. Hence with each e_i of G we have associated the matrix P_i. (Different e_i could possibly be associated with the same P_i.) The matrix $A = \sum P_i$ is doubly stochastic and is a matrix representation of G. Hence $\sigma = n$.

If G is not a core-graph it contains an inadmissible edge e. Any permutation matrix P which contains a 1 in the cell representing e also contains a 1 in at least one cell not representing an edge of G. Hence G cannot be represented by a doubly stochastic n by n matrix. Hence $\sigma < n = \rho$. Also since G has term rank $\rho = n$ we may assume that the vertex sets may be so ordered that the cells along the main diagonal of an n by n matrix all represent edges of G. G is now embedded in a larger graph G_1 whose new edges are represented by cells in an (n + 1)th row and (n + 1)th column as follows. The cell (n + 1, n + 1) represents a new edge. If (i, j) represents an inadmissible edge of G, let (n + 1, i) and (j, n + 1) represent edges of G_1. The graph G_1 is of term rank n + 1 and is a core-graph. For if e is an admissible edge of G the cells which represent n independent edges of G which include e, together with (n + 1, n + 1) represent n + 1 independent edges of G_1. On the other hand for the inadmissible edge of G represented by (i, j) the set of cells (i, j), (n + 1, i), (j, n + 1) together wilh all cells (r, r), $r \neq i$, $r \neq j$, $r \neq n + 1$ form a set of (n + 1) cells representing independent edges of G_1. Hence by the first part of case 1, G_1 can be represented by a doubly stochastic (n + 1) by (n + 1) matrix. Hence $\sigma = \rho - 1$.

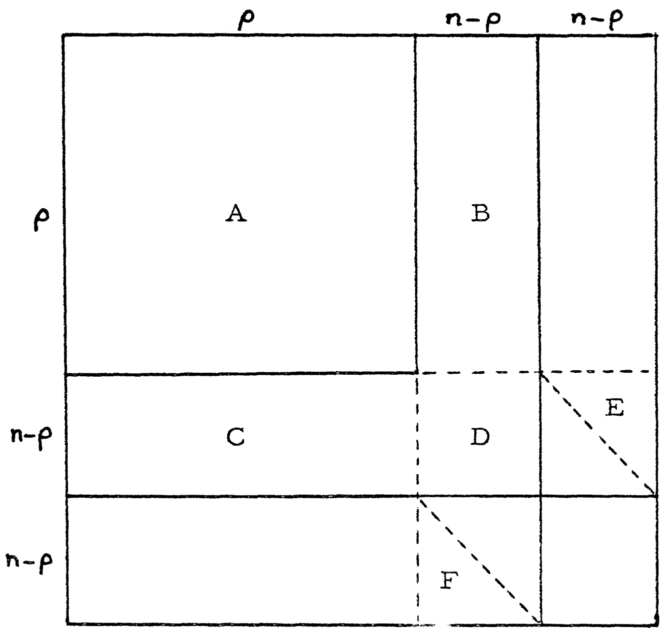

Figure 1

https://doi.org/10.4153/CMB-1959-020-4 Published online by Cambridge University Press

Case 2. $\rho < n$. Again we assume that the vertex sets are so ordered that the first ρ diagonal elements of an n x n matrix represent edges of G as in figure 1. If the matrix is partitioned into four parts A, B, C, D, in which A consists of the first ρ rows and columns and B, C, D as in the diagram then the region D represents no edges of G. Augment the matrix by the addition of $n - \rho$ rows and columns. Embed G in a graph G_1 whose additional edges are represented only by the main diagonal cells of the square regions E and F abutting D as in the diagram. Then G_1 is a graph whose vertex sets each contain $2n - \rho$ elements and whose term rank is $2n - \rho$. Furthermore, each admissible edge of G and each added edge is an admissible edge of G_1 and each inadmissible edge of G (if such exists) is inadmissible in G_1. Hence, G_1 is a core-graph if and only if the same is true of G. Case 2 has now been reduced to case 1.

REFERENCES

1. D. Konig, Theorie der endlichen und unendlichen Graphen, (New York, 1950).

2. A.L. Dulmage and I. Halperin, On a theorem of Frobenius--Konig and J. von Neumann's game of hide and seek, Trans. Roy. Soc. Can. Ser. III, 49 (1955), 23-29.

3. A.L. Dulmage and N.S. Mendelsohn, Coverings of bipartite graphs, Can. J. Math. 10 (1958), 517-534.

4. ...., The term and stochastic ranks of a matrix, Can. J. Math. 11 (1959), 269-279.

5. O. Ore, Graphs and matching theorems, Duke Math. J. 22 (1955), 625-639.

6. H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Can. J. Math. 9 (1957), 371-377.

7. ...., The term rank of a matrix, Can. J. Math. 10 (1957), 57-65.

University of Manitoba

https://doi.org/10.4153/CMB-1959-020-4 Published online by Cambridge University Press