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ISOMORPHIC GROUP RINGS OVER DOMAINS

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ABSTRACT. Let R and S be rings, G and H abelian groups, and RG and SH the goup rings of G and H over R and S respectively. In this note we consider what relations must hold between G and H or between R and S if the group rings RG and SH are isomorphic. For example, it is shown that if R and S are integral domains of characteristic zero, G and H torsion abelian groups such that if G has an element of order p then p is not invertible in R, and RG and SH are isomorphic, then the rings R and S are isomorphic and the groups G and H are isomorphic.

Let R be a commutative ring, G an abelian group, and RG the group ring of G with coefficients in R. If $x \in RG$, then $x = \sum_{g \in G} r_g g$ with $r_g \in R$, $g \in G$ and $r_g = 0$ for all but a finite number of g. The homomorphism $\psi_R : RG \longrightarrow R$ defined by $\psi_R(x) = \Sigma r_g$ is called the augmentation homomorphism. For $x \in RG$ we will often denote $\psi_R(x)$ by $c(x)$ and call this quantity the content of x.

If A is either a commutative ring or an abelian group and p is a prime, let $A_p = \{x \in A | x^{p^n} = 1$ for some integer $n\}$. A_p is the set of p torsion elements of A. Here 1 denotes the identity of A. In the group ring RG, let $V_{R,p} = V_p = \{x \in RG | x \in (RG)_p$ and $c(x) = 1\}$. V_p is called the normalized p torsion of RG.

If $x \in (RG)_p$, x is a p torsion element in RG and so $\psi_R(x) = c(x)$ is a p torsion element in R. There is, then, and element $\bar{x} \in V_p$ with $x = c(x)\bar{x}$. This representation of x shows that $(RG)_p$ is the direct product of R_p and V_p.

We let $\operatorname{Supp} G$ denote the set of all primes p for which G_p is a nontrivial group, and let R^* represent the unit group of the ring R. May ([3], p.493 and 497) has determined sufficient conditions on R to guarantee that G_p is a direct summand of V_p. We list his result in lemma 1.

LEMMA 1. *Let R be an indecomposable ring of characteristic* 0 *and G be an abelian group. Suppose that* $\operatorname{Supp} G \cap R^* = \emptyset$. *If $p \in \operatorname{Supp} G$, then G_p is a direct summand of V_p. If, in addition, R is an integral domain, then $G_q = V_q$ for every prime q.*

If G is an abelian group, we let $T(G)$ denote the torsion subgroup of G. ζ_n will represent a primitive n^{th} root of unity chosen so the $\zeta_{mn}^m = \zeta_n$ for all m and n. Let $\rho_n(x)$ denote the n^{th} cyclotomic polynomial. If r is an element of the ring R and $\rho_n(r) = 0$, we will call r a primitive n^{th} root of unity.

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THEOREM 2. *Let R be an integral domain of characteristic* 0, *S a ring, and G and H abelian groups with* $\operatorname{Supp} G \cap R^* = \emptyset$. *Suppose that* $RG \simeq SH$. *Then $T(H)$ is isomorphic to a direct summand of $T(G)$.*

PROOF. Let $\varphi : RG \longrightarrow SH$ be the given isomorphism and let $p \in \operatorname{Supp} G$. By Lemma 1, RG_p is the direct product of R_p and G_p. Also RG ([3], p. 489) contains no nontrivial idempotents and so the same must be true of $\varphi(RG) = SH$ and thus, S has no nontrivial idempotents. In particular S is an indecomposable ring of characteristic 0. $p = \varphi(p)$ is neither a unit nor a zero divisor of SH, since p has similar properties in RG. Hence $\operatorname{Supp} G \cap S^* = \emptyset$.

Let $q \in \operatorname{Supp} H$. Then there is an element $h \in H$, of order q, and an element $u \in RG$ such that $\varphi(u) = h$. Since u is a torsion element $u = \alpha g$ with $g \in T(G)$ and $\alpha \in T(R^*)$. $u^q = 1$ implies that $g^q = 1$ and $\alpha^q = 1$. If $g = 1$, then $\alpha^q = 1$, $\alpha \neq 1$ in the domain R implies α satisfies $\rho_q(x) = 0$, i.e. $\alpha^{q-1} + \alpha^{q-2} + \cdots + 1 = 0$. But then $\varphi(u) = \varphi(\alpha) = h$ satisfies $h^{q-1} + h^{q-2} + \cdots + 1 = 0$ which contradicts the linear independence of $1, h, h^2, \ldots, h^{q-1}$ over S. Thus $g \neq 1$ and $g \in \operatorname{Supp} G$. We can now conclude that $\operatorname{Supp} H \cap S^* = \emptyset$. From Lemma 1, V_p is the direct summand of H_p and T_p for some subgroup T_p of V_p, and so $(SH)_p$ is the direct product of S_p, T_p and H_p. Since $\varphi((RG)_p) = (SH)_p$ we have that $R_p \times G_p \simeq S_p \times T_p \times H_p$ for any $p \in \operatorname{Supp} G$. Because R is an integral domain, R_p is either isomorphic to a cyclic group of order p^k for some $k \geqq 0$, or is isomorphic to $Z(p^\infty)$. In either case we claim S_p contains a direct summand isomorphic to R_p.

Proof of claim: Suppose R contains a primitive p^{th} root of unity ζ_p. Then $\zeta_p \in R_p$ and ζ_p satisfies $\rho_p(\zeta_p) = 0$. Hence $\psi_S\varphi(\zeta_p)$ also satisfies $\rho_p(x) = 0$. Thus $\psi_S\varphi$ is injective on $\langle \zeta_p \rangle$ and so on R_p. In particular, S_p contains a subgroup $A = \psi_S\varphi(R_p)$ isomorphic to R_p. We must check that A is a direct summand of S_p.

If $R_p \simeq Z(p^\infty)$, then A, being a divisible subgroup, is a direct summand of S_p. So suppose now R_p is a finite cyclic group of order p^k. Let $t \in S_p$ and suppose $t^{p^j} \in A - \{1\}$ with j-minimal. Then t^{p^j} is a solution of $\rho_{p^l}(x) = 0$ for some l, and so t is a solution of $\rho_{p^{l+j}}(x) = 0$. Since t is then a p^{l+j}th root of unity, we have that $l + j \leqq k$. Let ζ_{p^k} generate R_p and $a = c(\varphi(\zeta_{p^k}))$ generate A. Since t^{p^j} is a solution of $\rho_{p^l}(x) = 0$ we can write $t^{p^j} = a^{(p^{k-l})s}$ with $(s,p) = 1$. So $t^{P^j} = (a^{s(p^{k-l-j})})^{p^j}$. This says that A is a pure subgroup of S_p, which is also bounded. From ([2], p. 18), A is a direct summand of S_p. This completes the proof of the claim.

Write S_p as ${}_pA \times {}_pB$ with ${}_pA \simeq R_p$. Then

$$(*) \qquad R_p \times G_p \simeq {}_pA \times {}_pB \times T_p \times H_p$$

If R_p is finite, Walker's theorem ([4], p. 900) permits us to cancel the R_p and A_p from (*) giving $G_p \simeq {}_pB \times T_p \times H_p$, while if $R_p \simeq Z(p^\infty)$ we can cancel R_p and ${}_pA$ from (*) since R_p is a divisible group. In either case we have that $G_p \simeq {}_pB \times T_p \times H_p$ and H_p is isomorphic to a direct summand of G_p. Since $T(G) \simeq \oplus_p G_p$ and $T(H) \simeq \oplus_p H_p$ the theorem is now established.

COROLLARY 3. *Let R and S be integral domains of char 0 and G and H abelian groups such that $RG \simeq SH$. Suppose that* $\operatorname{Supp} G \cap R^* = \emptyset$. *Then* $T(G) \simeq T(H)$.

PROOF. Let $p \in \operatorname{Supp} G$. From Lemma 1, $(RG)_p = R_p \times G_p$. As in the proof of Theorem 2 $\operatorname{Supp} H \cap S^* = \varphi$ and so again by Lemma 1 $(SH)_p = S_p \times H_p$. Since S is an integral domain, S_p is either isomorphic to a cyclic group of order p^k for some $k \geqq 0$, or to $Z(p^\infty)$. Neither of these groups has any nontrivial direct summands. But the theorem shows that R_p is a direct summand of S_p. Hence $S_p \simeq R_p$ or $R_p \simeq \{1\}$ and S_p is not the trivial group. In the latter case, S_p would then contain a p^{th} root of unity while R does not, contradicting a conclusion in the proof of the theorem. Hence $R_p \simeq S_p$ and by Walker's theorem $H_p \simeq G_p$. □

In general we cannot say that R and S must be isomorphic even if $T(G) \simeq T(H)$. We can take, for example, any nonisomorphic torsion free abelian groups A_1 and A_2 and a torsion group B. Let $C = A_1 \oplus A_2 \oplus B$. Then $ZC \simeq Z(A_1)(A_2 \oplus B) \simeq Z(A_2)(A_1 \otimes B)$. If $R = Z(A_1)$, $S = Z(A_2)$, $G = A_2 \oplus B$ and $H = A_1 \oplus B$, then the integral domains R and S are not isomorphic even though $ZG \simeq SH$ and the hypotheses of Corollary 3 are met. However, even though $G/T(G)$ is not isomorphic to $H/T(H)$, we still have $R(G/T(G)) \simeq S(H/T(H))$. We check this, in some generality, in the following

THEOREM 4. *Let R and S be integral domains of char 0, and G and H abelian groups such that $RG \simeq SH$. Suppose that* $\operatorname{Supp} G \cap R^* = \emptyset$ *and $T(G)$ is a direct summand of G, then $R(G/T(G)) \simeq S(H/T(H))$.*

PROOF. Let $\varphi : RG \longrightarrow SH$ be the given isomorphism. As before $\varphi((RG)_p) = (SH)_p$, and $(RG)_p = R_p \times G_p$, $(SH)_p = S_p \times H_p$ with $R_p \simeq S_p$ by the proof of Corollary 3. Also, we have $T((RG)^*) = T(R^*)T(G)$ and we may define the map $\pi : T(R^*)T(G) \longrightarrow T(G)$ given by $\pi(rg) = g$ with $r \in T(R^*)$, $g \in T(G)$. Let $h \in T(H)$, then $\varphi^{-1}(h) = r_h g_h$ with $r_h \in T(R^*) g_h \in T(G)$. Define $\psi : T(H) \longrightarrow T(G)$ by $\psi(h) = g_h$. ψ is a homomorphism since it is the composite of φ^{-1} restricted to $T(H)$ and π. We check that ψ is an onto isomorphism.

Suppose $h \in T(H)$ and $\psi(h) = 1$. Then $\psi^{-1}(h) = r_h$ with $r_h \in T(R^*)$. Suppose h is of order n, then $r_h \in R$, with R an integral domain, is an n^{th} root of unity, and so r_h satisfies the equation $\rho_n(x) = 0$. But then h satisfies $\rho_n(x) = 0$ which contradicts the linear independence of $1, h, h^2, \ldots, h^{n-1}$ over S. Hence $n = 1$ and ψ is injective. To check ψ is onto, it is sufficient to check that $\psi(H_p) = G_p$ for each prime p. Fix $p \in \operatorname{Supp} G$. Let $A = \varphi^{-1}(H_p)$. Since $\varphi(R_p \times G_p) = S_p \times H_p$ we have that

$$\frac{R_p \cdot G_p}{A} \simeq \frac{S_p \cdot H_p}{H_p} \simeq S_p$$

if $h \in H_p$ with $h \neq 1$, then $\varphi^{-1}(h)$ cannot be a root of unity and thus satisfy a cyclotomic equation, since h does not. So $A \cap R_p = \{e\}$. Then

$$\frac{A \cdot R_p}{A} \simeq \frac{R_p}{R_p \cap A} \simeq R_p$$

Since $R_p \simeq S_p$, and this group which must be either a cyclic group of order p^k for some k, or $Z(p^\infty)$, does not contain a proper subgroup isomorphic to itself, we can conclude that $A \cdot R_p = R_p \cdot G_p$ because AR_p/A is a subgroup of R_pG_p/A. Thus $\pi(A) = G_p$ and $\psi(H_p) = G_p$. This shows ψ to be a surjective isomorphism.

Because $T(G)$ is a direct summand of G, we can find a torsion-free subgroup U of G with $G = U \cdot T(G)$.

Let $\tau : RG \longrightarrow RG$ be the R map defined by $\tau(u) = u$ if $u \in U$ $\tau(g) = \varphi^{-1}(\psi^{-1}(g))$ if $g \in T(G)$.

Since ψ is a surjective isomorphism, τ is well defined. It is straightforward to check that τ is an automorphism of RG. Then $\hat{\varphi} = \varphi\tau$ is an isomorphism from RG onto SH such that $\hat{\varphi}(T(G)) = T(H)$. Let I_1 be the ideal of RF generated by $\{1 - g|g \in T(G)\}$ and I_2 the ideal of SH generated by $\{1 - h|h \in T(H)\}$. $\hat{\varphi}(I_1) = I_2$ and thus

$$R(G/T(G)) \simeq RG/I_1 \simeq SH/I_2 \simeq S(H/T(H))$$

which establishes the result. □

Corollary 5. *Let R and S be integral domains of characteristic 0, and G and H torsion abelian groups such that $RG \simeq SH$. Suppose that* $\operatorname{Supp} G \cap R^* = \emptyset$. *Then $G \simeq H$ and $R \simeq S$.*

Proof. The groups are isomorphic by Corollary 3 and the domains are isomorphic by Theorem 4. □

Using the techniques of the previous results we can extend Theorem 7.2 of [1].

Theorem 6. *Let R be an integral domain of characteristic 0, S a ring, and G and H torsion abelian groups. Suppose that* $\operatorname{Supp} G \cap R^* = \emptyset$, *and that if $p \in \operatorname{Supp} G$, R does not contain a p^2 root of unity. Then $RG \simeq SH$, if and only if there exist subgroups K, L of G with*

(i) *$G = KL$ (internal direct sum)*

(ii) *$L \simeq H$*

(iii) *$S \simeq RK$*

Proof. If such subgroups exist,

$$RG \simeq (RK)L \simeq SL \simeq SH.$$

Conversely, suppose $\varphi : RG \longrightarrow SH$ is the given isomorphism. If $p \in \operatorname{Supp} G$, by Lemma 1, $(RG)_p = R_p \times G_p$. Suppose $u \in RG$ is a p^{th} root of unity. Then $u^p = 1$ and u satisfies $\rho_p(x) = 0$. Write $u = rg$ with $r \in R_p g \in G_p$. Then $r^p = 1$ and $g^p = 1$. If $g \neq 1$, then rg satisfies $\rho_p(x) = 0$. This says that g satisfies $\eta(x) = \rho_p(rx) = 0$ which contradicts the linear independence of $1, g, g^2, \ldots, g^{p-1}$ over R. Hence $u = r$ and u is a p^{th} root of unity in R. We now can conclude that all solutions of $\rho_p(x) = 0$ are in R and there are either 0 or $p - 1$ of them, the latter case when R has a p^{th} root of unity. Because φ is an isomorphism, there are either 0 or $p-1$ solutions of $\rho_p(x)=0$ in SH, and they are similarly all in S.

Let $1 \neq h \in H_p$ and write $\varphi^{-1}(h) = r_h g_h$ with $r_h \in R_p$ $g_h \in G_p$. If $h^{p^n} = 1$, then $r_h^{p^n} = 1$ which implies $R_h^p = 1$ since R does not contain a p^2 root of unity. Since r_h is either 1 or a p^{th} root of unity, $\varphi(r_h) \in S_p$. Let π be the projection map from $R_p \times G_p \longrightarrow G_p$, and $L_p = \pi\varphi^{-1}(H_p)$.

If $v \in H_p$ is such that $\pi\varphi^{-1}(v) = 1$. Then $\varphi^{-1}(v) = r_v$ with $r_v \in R_p$. But then either $r_v = 1$ or r_v satisfies $\rho_p(x) = 0$. This latter case contradicts the linear independence of $1, v, v^2, \ldots, v^{p-1}$ over S. Hence $L_p \simeq H_p$ and $L = \oplus L_p$ is isomorphic to $H = \oplus H_p$.

Let $\tau_1 : H \longrightarrow (SH)^*$ be the homomorphism defined by $\tau_1(h) = \varphi(r_h)h$ for $h \in H_p$ and $\tau : SH \longrightarrow SH$ the S-linear map extending τ_1. It is easy to check that τ is an automorphism of SH. Let $\hat{\varphi} = \tau\varphi$. Then $\hat{\varphi}$ is an isomorphism of RG onto SH and $\hat{\varphi}(L) = H$.

Let I_1 be the ideal of RG generated by $\{1 - l | l \in L\}$ and I_2 the ideal of H generated by $\{1 - h | h \in H\}$. $\hat{\varphi}(I_1) = I_2$ and so $R(G/L) \simeq RG/I_1 \simeq SH/I_2 \simeq S(H/H) \simeq S$.

As in the proof of Theorem 2, S is indecomposible and so by Lemma 1, if $p \in \operatorname{Supp} G$, there is a subgroup T_p of V_p (in SH) such that $V_p = T_p \times H_p$. Then $(SH)_p = S_p \times T_p \times H_p$. Let $T = {}_p \oplus T_p$ and $K = \{g \in G | \hat{\varphi}(g) \in S^* \times T\}$. K is a subgroup of G and $K \cap L = \{1\}$. To complete the proof we need only check that $KL = G$. We show that $G_p \subset KL$. Let $g \in G_p$. Then $\hat{\varphi}(g) = \omega_p h_p$ with $\omega_p \in S_p \times H_p$, $h_p \in H_p$.

Let $l \in L$ be such that $\hat{\varphi}(l) = h_p$ then $g = (gl^{-1})l$ and $\hat{\varphi}(gl^{-1}) = \hat{\varphi}(g)\hat{\varphi}(l^{-1}) = \omega_p h_p h_p^{-1} = \omega_p$. thus $gl^{-1} \in K$. This completes the proof. □

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