Ergod. Th. & Dynam. Sys., (2025), **45**, 1503–1540
doi:10.1017/etds.2024.78

Piecewise contractions

SAKSHI JAIN† and CARLANGELO LIVERANI‡§

† *School of Mathematics, Monash University, 9 Rainforest Walk, Melbourne 3800, Australia*
(e-mail: sakshi.jain@monash.edu)
‡ *Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy*
§ *Department of Mathematics, William E. Kirwan Hall, 4176 Campus Drive, University of Maryland, College Park, MD 20742, USA*
(e-mail: liverani@mat.uniroma2.it)

(*Received* 3 *April* 2024 *and accepted in revised form* 9 *September* 2024)

Abstract. We study piecewise injective, but not necessarily globally injective, contracting maps on a compact subset of $\mathbb{R}^d$. We prove that, generically, the attractor and the set of discontinuities of such a map are disjoint, and hence the attractor consists of periodic orbits. In addition, we prove that piecewise injective contractions are generically topologically stable.

Key words: contracting maps, attractors, generic properties
2020 Mathematics Subject Classification: 37C20, 37C70 (Primary); 37B02, 37B25 (Secondary)

Contents

1. *Introduction*

Studying the dynamical properties of discontinuous hyperbolic dynamical systems is important for understanding many relevant systems (such as billiards and optimal control theory) but it is difficult if the system is multi-dimensional. A first step towards addressing this problem can be to divide the problem into two cases: the piecewise expanding case and the piecewise contracting case. Here we address the case of piecewise contracting systems. Because the attractor can differ drastically depending on the kind of piecewise contraction, a first approach is to identify a large class of piecewise contractions that exhibit similar dynamical properties.

In recent decades, the properties of the attractor of piecewise contractions have been studied under different settings (see [3, 7, 8, 21, 22]). To begin with, Bruin and Deane (see [3]) studied families of piecewise linear contractions on the complex plane $\mathbb{C}$ and proved that, for almost all parameters, each orbit is asymptotically periodic.

In the case of *one-dimensional* piecewise contractions, Nogueira and Pires, in [21], studied *globally* injective piecewise contractions on a half closed unit interval $[0, 1)$ with partition of continuity consisting of n elements and concluded that such maps have at most n periodic orbits: that is, the attractor can be a Cantor set or a collection of at most n periodic orbits or a union of a Cantor set and at most $n-1$ periodic orbits. In particular, when the attractor consists of exactly n periodic orbits, the map is asymptotically periodic (the limit set of every element in the domain is a periodic orbit). They also proved that every such map on n intervals is topologically conjugate to a piecewise linear contraction of n intervals whose slope in absolute value equals $1/2$. We would like to emphasise that this result does not imply that any two piecewise contractions close enough to each other are topologically conjugate to each other. Hence, this result is not a stability result for the class of piecewise contractions. In [12], piecewise increasing contractions on n intervals are considered and it is proved that the maximum number of periodic orbits is n. The authors prove that the collection parameters that give a piecewise contraction with non-asymptotically periodic orbits is a Lebesgue null measure set whose Hausdorff dimension is large or equal to n. In [22], Nogueira, Pires and Rosales proved that generically (under $\mathcal{C}^0$ topology) globally injective piecewise contractions of n intervals are asymptotically periodic and have at least one and at most n internal periodic orbits (such orbits persist under a sufficiently small $\mathcal{C}^0$-perturbation; refer to [22] for precise definition). In [24], the authors prove that almost all translations within a small neighbourhood of a λ-affine contraction are asymptotically periodic. In [4], Calderon, Catsigeras and Guiraud proved that the attractor of a piecewise injective contraction consists of finitely many periodic orbits and minimal Cantor sets. In [16, 20], the authors study symbolic coding associated to piecewise contractions on the unit interval and prove that they are related to the symbolic coding of rotations of the circle.

In *higher dimensions*, Catsigeras and Budelli (see [6]) proved that a finite dimensional piecewise contracting system with separation property (injective on the entire domain except for the discontinuity set) generically (under a topology that is finer than the topology we use for proving openness and coarser than the topology we use for proving density) exhibits at most a finite number of periodic orbits as its attractor. Here, we obtain similar, actually stronger, results without assuming the separation property.

In any (finite) dimension, in [8], the authors show that if the set of discontinuities and the attractor of a piecewise contraction are mutually disjoint, then the attractor consists of finitely many periodic orbits. This result is a by-product of our arguments as well. In [14], the authors study symbolic dynamics associated to piecewise contractions (referred to as quasi-contractions in that article) and categorize its association into different kinds of circle rotations.

In the month after we submitted this article, we noticed a new preprint [13] in which the authors provide a measure-theoretical criterion for asymptotic periodicity of a parametrized family of locally bi-Lipschitz piecewise contractions on a compact metric space.

Nevertheless, the occurrence of chaotic behaviour in such systems has been addressed in the literature, and [19] provides an example of a piecewise affine contracting map with positive entropy. The presence of a Cantor set in the attractor has also been studied rigorously. Examples of such maps for one dimension are given in [[8], Example 4.3] and [9], and for three dimensions are given in [[8], Example 5.1], where it is also proved that such a piecewise contraction turns out to have positive topological entropy. In [25], it is proved that, given a minimal interval exchange transformation with any number of discontinuities, there exists an injective piecewise contraction with Cantor attractors topologically semi-conjugate to it and, conversely, that piecewise contractions with Cantor attractors are topologically semi-conjugate to topologically transitive interval exchange transformations. Additionally, in [25] (respectively, in [7]), it is proved that the complexity (the complexity function of the itineraries of orbits; refer to [7, 25]) of a *globally injective* piecewise contraction (respectively, piecewise contraction with separation property) on the interval is eventually affine (which is eventually constant in the case of piecewise contractions with no Cantor attractors).

The global picture presented by the above articles is that the Cantor attractors are rare, but can exist in exceptional (but constructible) cases, and many such explicit examples have been rigorously studied.

Piecewise contractions have also been used to study some models of outer billiards (see [10, 11, 17]), where a billiard map is constructed such that it is a piecewise contraction, and so the properties of piecewise contractions are relevant in the study of such billiard maps.

Note that, in the above papers (and in this article), maps with only finitely many partition elements are considered.

The layout of this article is as follows.

Section 2 is dedicated to definitions, settings and the statement of results. In §3, we prove that the set of piecewise contractions with attractor disjoint from the set of discontinuities is open, under a rather coarse topology, and that if the maps are also piecewise injective (and hence not necessarily globally injective), then they are topologically stable. In §4, we prove that piecewise contractions with the attractor disjoint from the set of discontinuities are dense, under a rather fine topology, among the piecewise injective smooth contractions. Finally, we have three appendices collecting some needed technical facts.

2. *Settings and results*

Throughout this article, we work with $(X, d_0) \subset (\mathbb{R}^d, d_0)$, a compact subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$ and d_0 is the standard Euclidean metric on $\mathbb{R}^d$. Under these settings, we define a piecewise contraction as follows.

Definition 2.1. (Piecewise contraction) A map $f : X \to X$ is called a piecewise contraction if $\overline{f(X)} \subset \mathring{X}$ and there exist $m \in \mathbb{N}$ and a collection $\boldsymbol{P}(f) = \{P_i : P_i = \mathring{P}_i\}_{i=1}^m$ of subsets of X such that:

- $X = \bigcup_{i=1}^m \overline{P}_i$, where $P_i \cap P_j = \emptyset$ whenever $i \neq j$;
- $f|_{P_i}$ is a uniform contraction, that is, there exists $\lambda \in (0, 1)$ such that, for all $i \in \{1, 2, \ldots, m\}$ and for any $x, y \in P_i$,
$$d_0(f(x), f(y)) \leq \lambda d_0(x, y);$$
- there exists a partition $\{\tilde{P}_i\}$, $\tilde{P}_i \cap \tilde{P}_j = \emptyset$ for $i \neq j$, $P_i \subset \tilde{P}_i \subset \overline{P}_i$, $X = \bigcup_{i=1}^m \tilde{P}_i$ such that $f|_{\tilde{P}_i}$ is continuous.

Here λ is called the 'contraction coefficient' of *f*, $\boldsymbol{P}(f)$ is called the 'partition of continuity' and $\overline{P_i}$ and ∂P_i represent the closure and boundary, respectively, of a partition element P_i.

Remark 2.2. The third condition pertaining to the partition $\tilde{P}$ states that the values of *f* on the boundary of partition elements must be the limit of the values of *f* inside some elements, but no particular condition is imposed on which element.

For a piecewise contraction *f* with partition of continuity $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$, we define $\partial \boldsymbol{P}(f)$ as the boundary of the partition $\boldsymbol{P}(f)$ given by

$$\partial \boldsymbol{P}(f) = \bigcup_{i=1}^{m} \partial P_i.$$

We denote by $\Delta(f)$ the union of the set of discontinuities of *f* with ∂X. To avoid confusion in the choice of partition of continuity for a given piecewise contraction, we define the following.

Definition 2.3. (Maximal partition) A partition $\boldsymbol{P}(f)$ is called the maximal partition of a piecewise contraction *f* if $\partial \boldsymbol{P}(f) = \Delta(f)$.

Any partition of continuity of f is a refinement of the maximal partition. For a piecewise contraction with maximal partition $\boldsymbol{P}(f) = \{P_1^1, P_2^1, \ldots, P_m^1\}$, we define the partition $\boldsymbol{P}(f^n) = \{P_1^n, P_2^n, \ldots, P_{m_n}^n\}$, relative to the *n*th iterate f^n of *f*, where, for every $k \in \{1, 2, \ldots, m_n\}$,

$$P_k^n = P_{i_0}^1 \cap f^{-1}(P_{i_1}^1) \cap f^{-2}(P_{i_2}^1) \cap \cdots \cap f^{-(n-1)}(P_{i_{n-1}}^1), \tag{2.1}$$

and where $i_j \in \{1, 2, \ldots, m\}$ for every $j \in \{0, 1, \ldots, n-1\}$. Note that $\boldsymbol{P}(f)$ being the maximal partition of *f* does not imply that the partition $\boldsymbol{P}(f^n)$ is the maximal partition of f^n.

Remark 2.4. Throughout this article, for a given piecewise contraction f, the partition that we consider is the maximal partition $\boldsymbol{P}(f)$, whereas for its iterates f^n we consider the partition $\boldsymbol{P}(f^n)$, as defined in equation (2.1), which may not be the maximal partition of f^n.

One of the goals of this article is to understand the attractor of piecewise contractions. On that note, we recall the standard definition of an attractor.

Definition 2.5. (Attractor) For a piecewise contraction f with $\boldsymbol{P}(f) = \{P_i\}_{i=1}^m$, the attractor is defined as $\Lambda(f) = \bigcap_{n\in\mathbb{N}} \overline{f^n(X)}$.

When working with discontinuous maps, it is natural to talk of Markov maps (maps with Markov partitions), so we first give the following definition and then the definition of Markov maps in our settings.

Definition 2.6. (Stabilization of partition) For a piecewise contraction f, we say that the maximal partition $\boldsymbol{P}(f)$ stabilizes if there exists $N \in \mathbb{N}$ such that, for all $P \in \boldsymbol{P}(f^N)$, there exists $Q \in \boldsymbol{P}(f^N)$ such that $\overline{f^N(P)} \subset Q$. We call the least such number, N, the 'stabilization time' of $\boldsymbol{P}(f)$.

Definition 2.7. (Markov map) A piecewise contraction whose maximal partition stabilizes is called a Markov map.

Now we are able to state our first result (proved in §3).

THEOREM 2.8. *A piecewise contraction f with $\Lambda(f)$ as the attractor and $\Delta(f)$ as the union of the set of discontinuities and ∂X satisfies that $\Lambda(f) \cap \Delta(f) = \emptyset$ if and only if it is Markov. Moreover, the attractor of a Markov map consists of periodic orbits.*

Remark 2.9. Note that, given a Markov map f, N its stabilization time and $\boldsymbol{P}(f^N) = \{Q_1, \ldots, Q_l\}$ the associated partition, then f induces a dynamics $\mathbb{f}$ on $\Omega(f) := \{1, \ldots, l\}$ by the rule $f(Q_i) \subset Q_{\mathbb{f}(i)}$. See Lemma 3.2.

To state further results, we need to add some hypotheses on our system, and thus we give the following definition.

Definition 2.10. (Piecewise injective contraction) A piecewise contraction f with partition $\boldsymbol{P}(f) = \{P_i\}$, $i \in \{1, 2, \ldots, m\}$ is called a *piecewise injective contraction* if, for all $i \in \{1, 2, \ldots, m\}$, there exists $U_i = \mathring{U}_i \supset \overline{P_i}$ and an injective contraction $\tilde{f}_i : U_i \to \mathbb{R}^d$ such that $\tilde{f}_i|_{U_i} \supset P_i$ and $\tilde{f}_i|_{P_i} = f|_{P_i}$.

For any piecewise contractions f, g with partitions $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$ and $\boldsymbol{P}(g) = \{Q_1, Q_2, \ldots, Q_m\}$, respectively, we define

$$H(\boldsymbol{P}(f), \boldsymbol{P}(g)) = \{\psi \in \mathcal{C}^0(X, X) \ : \ \psi \text{ is an homeomorphism}$$
$$\text{for all } \ P \in \boldsymbol{P}(f), \ \text{there exists } Q \in \boldsymbol{P}(g) : \psi(P) = Q\}.$$

We define the distance ρ as follows (see Lemma 3.1 for the proof that ρ is a metric).

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

$$\rho(f,g) := \begin{cases} A & \text{if } H(\boldsymbol{P}(f), \boldsymbol{P}(g)) = \emptyset, \\ \inf_{\psi \in H(\boldsymbol{P}(f), \boldsymbol{P}(g))} \{\|\psi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - g \circ \psi\|_{\mathcal{C}^0(X,X)}\} & \text{otherwise,} \end{cases}$$

where $A = 2\,\mathrm{diam}(X)$ and id is the identity function, that is, $\mathrm{id}(x) = x$.

Evidently, $\rho(f, g) \leq A$ for any piecewise contractions f, g. Furthermore, notice that the metric (proved in Lemma 3.1) ρ is essentially a distance between two tuples $(f, \boldsymbol{P}(f))$ and $(g, \boldsymbol{P}(g))$ for any two piecewise contractions f, g, where $\boldsymbol{P}(f)$ and $\boldsymbol{P}(g)$ are the maximal partitions of f and g, respectively.

For an arbitrary $\sigma \in (0, 1)$, we define the distance d_1 as

$$d_1(f, g) = \sum_{n=1}^{\infty} \sigma^n \rho(f^n, g^n).$$

Under this metric, we have the following result (proved in §3).

THEOREM 2.11. *The set of Markov piecewise contractions is open in the set of piecewise contractions under the metric d_1. Moreover, any two Markov piecewise contractions f, g close enough (with respect to d_1), stabilize at the same time.*

To be able to state our stability result, we need to discuss the dynamics of the partition elements.

Definition 2.12. (Wandering set) For a piecewise contraction $f : X \to X$, a partition element $P \in \boldsymbol{P}(f)$ is called wandering if there exists $M \in \mathbb{N}$ such that $f^n(P) \cap P = \emptyset$ for all $n > M$. The set $\mathbb{W}(f) \subset \boldsymbol{P}(f)$, consisting of all wandering partition elements, is called the *wandering* elements set. We set $W(f) = \bigcup_{P \in \mathbb{W}(f)} P$. (Note that this set is not the usual wandering set, which is much bigger.)

Definition 2.13. (Non-wandering set) The complement $N\mathbb{W}(f)$ of the wandering elements set is called the non-wandering elements set: that is, $N\mathbb{W}(f) = \{P \in \boldsymbol{P}(f) : P \notin \mathbb{W}(f)\}$. We set $NW(f) = \bigcup_{P \in N\mathbb{W}(f)} P$.

Note that, for a Markov map, the set $N\mathbb{W}(f)$ corresponds to the non-wandering set of the dynamical system $(\Omega(f), \mathbb{f})$ defined in Remark 2.9. Accordingly, our definition of $NW(f)$ should not be confused with the usual non-wandering set of f, which, for a Markov map, consists of finitely many points (the set of periodic points; see Theorem 2.8). Hereby, we state our definition of topological stability.

Definition 2.14. (Topological stability) Let $\mathfrak{C}$ be contained in the set of piecewise contractions from $X \to X$ and let d be a metric defined on the set of piecewise contractions. We say that $(\mathfrak{C}, d)$ is *topologically stable* if, for every $f \in \mathfrak{C}$, there exists a $\delta > 0$ such that, for any piecewise contraction $g \in \mathfrak{C}$ with $d(f, g) < \delta$, f is semi-conjugate to g and g is semi-conjugate to f on the respective non-wandering sets: that is, there exist continuous functions $H_1 : NW(f) \to NW(g)$, $H_2 : NW(g) \to NW(f)$ such that $H_1 \circ f = g \circ H_1$, $f \circ H_2 = H_2 \circ g$ and $H_1(NW(f)) = NW(g)$, $H_2(NW(g)) = NW(f)$.

This definition of topological stability is somewhat different from the standard definition (see [[18], Definition 2.3.5]). In general, topological stability is defined for homeomorphisms such that one of them is semi-conjugate to the other (whereas we have semi-conjugacies for both sides), but on the entire space. Here it is necessary to define it only on the non-wandering set which, in fact, is the only subset of the space that contributes to the long-time dynamics. We can interpret our definition as stating that the long-time dynamics of two such functions are qualitatively the same.

THEOREM 2.15. *The set of Markov piecewise injective contractions (as defined in Definitions 2.7, 2.10) is topologically stable in the d_1 topology.*

The proof of the above theorem is given in §3. To state our result on density, we restrict ourselves to piecewise smooth contractions, which are defined as follows.

Definition 2.16. (Piecewise smooth contraction) A piecewise injective contraction f with partition $\boldsymbol{P}(f) = \{P_i\}$, $i \in \{1, 2, \ldots, m\}$ and injective extensions $\tilde{f}_i : U_i \to \mathbb{R}^d$ is called a *piecewise smooth contraction* if, for every $i \in \{1, 2, \ldots, m\}$:

- $\|\tilde{f}_i\|_{\mathcal{C}^3} < \infty$;
- $\|\tilde{f}_i^{-1}\|_{\mathcal{C}^1} < \infty$; and
- ∂P_i is contained in the union of finitely many $\mathcal{C}^2 (d-1)$-dimensional manifolds $\{M_j\}$, $\partial M_j \cap \overline{P}_i = \emptyset$. Such manifolds are pairwise transversal, and the intersection of any set of such manifolds consists of a finite collection of $\mathcal{C}^2$ manifolds.

For a piecewise smooth contraction f, we define the extended-metric

$$d_2(f, g) = \begin{cases} \sup\limits_{P \in \boldsymbol{P}(f)} \|f - g\|_{\mathcal{C}^2(P)} & \text{if } \boldsymbol{P}(f) = \boldsymbol{P}(g), \\ \infty & \text{otherwise.} \end{cases}$$

Note that $d_2(f, g) \geq \rho(f, g)$. The following density result is proved in §4.

THEOREM 2.17. *Piecewise smooth Markov contractions are d_2-dense in the space of piecewise smooth contractions.*

Remark 2.18. Theorems 2.8, 2.11 and 2.15 show that, for a piecewise contraction, to be Markov means being stable under a rather weak topology (d_1). Instead, Theorem 2.17 shows that to be Markov means being dense under a quite strong topology (d_2). These theorems collectively show that, for a piecewise contraction, being Markov is generic; hence, to have a Cantor set as an attractor is rare.

As already mentioned, a result on density is present in the literature (see [8]). However, it is proved under a much coarser metric as compared with d_2. More importantly, it assumes that the maps have the separation property, which implies that they are globally injective, whereas we assume only piecewise injectivity.

3. *Openness and topological stability*

Recall that, for any piecewise contraction f, $\boldsymbol{P}(f)$ stands for the maximal partition, so $\partial \boldsymbol{P}(f) = \Delta(f)$. In addition, for any $n > 1$, the elements of partition $\boldsymbol{P}(f^n)$ are given by equation (2.1), and $\#\boldsymbol{P}(f^n) = m_n$, $m_1 = m$.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Proof of Theorem 2.8. Let f satisfy $\Lambda(f) \cap \Delta(f) = \emptyset$. For all $n \in \mathbb{N}$, $\overline{f^{n+1}(X)} \subset \overline{f^n(f(X))} \subset \overline{f^n(X)}$ implies that $\{\overline{f^n(X)}\}$ is a nested sequence of non-empty compact sets. Further, Cantor's intersection theorem implies that $\Lambda(f) = \bigcap_{n\in\mathbb{N}} \overline{f^n(X)} \neq \emptyset$ and it is closed. Accordingly, there exists $\varepsilon > 0$ such that $d_0(\Lambda(f), \Delta(f)) > \varepsilon$. We claim that, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that, for $n \geq N_\varepsilon$, $\overline{f^n(X)} \subset B_{\varepsilon/2}(\Lambda(f))$. (For $r > 0$ and a set A, $B_r(A) = \{y :$ there exists $x \in A$ such that $d_0(x, y) < r\}$.) Indeed, if this was not the case, then there would exist a sequence $\{n_j\}$, $n_j \to \infty$, such that $\overline{f^{n_j}(X)} \cap B_{\varepsilon/2}(\Lambda(f))^c \neq \emptyset$. It follows that, for each $n \in \mathbb{N}$, there exists j such that $n_j \geq n$, and hence

$$\overline{f^n(X)} \cap B_{\varepsilon/2}(\Lambda(f))^c \supset \overline{f^{n_j}(X)} \cap B_{\varepsilon/2}(\Lambda(f))^c \neq \emptyset,$$

which, taking the intersection on n, yields a contradiction. Consequently, for every $P \in \boldsymbol{P}(f^{N_\varepsilon})$, there exists $Q \in \boldsymbol{P}(f^{N_\varepsilon})$ such that $\overline{f^{N_\varepsilon}(P)} \subset Q$; otherwise, there would exist $x \in f^{N_\varepsilon}(P) \cap \partial \boldsymbol{P}(f^{N_\varepsilon})$, that is, a $k \in \mathbb{N}$ such that

$$f^k(x) \in f^{k+N_\varepsilon}(P) \cap \partial \boldsymbol{P}(f) \subset \overline{f^{k+N_\varepsilon}(X)} \cap \Delta(f) \subset B_{\varepsilon/2}(\Lambda(f)) \cap \Delta(f) = \emptyset,$$

which is a contradiction.

Conversely, let f be a piecewise Markov contraction with stabilization time $N \in \mathbb{N}$. By definition, for every $P \in \boldsymbol{P}(f^N)$, there exists $Q \in \boldsymbol{P}(f^N)$ such that $\overline{f^N(P)} \subset Q$: that is, $\overline{f^N(P)} \cap \partial \boldsymbol{P}(f^N) = \emptyset$. Note that $\overline{f(X)} = \bigcup_{i=1}^m \overline{f(\tilde{P}_i)} \subset \bigcup_{i=1}^m \overline{f(P_i)}$, where $\tilde{P}_i$ is as given in Definition 2.1. Thus,

$$\Lambda(f) = \bigcap_{n\in\mathbb{N}} \overline{f^n(X)} \subset \bigcap_{n\in\mathbb{N}} \bigcup_{P\in\boldsymbol{P}(f^n)} \overline{f^n(P)} \subset \bigcup_{P\in\boldsymbol{P}(f^N)} \overline{f^N(P)} \subset \bigcup_{Q\in\boldsymbol{P}(f^N)} Q$$

implies that $\Lambda(f) \cap \partial \boldsymbol{P}(f) = \emptyset$.

To prove the second part of the theorem, let $x \in \Lambda(f)$ and let N be the stabilization time. By definition, for each $k \in \mathbb{N}$, there exists $y_k \in Q_k \in \boldsymbol{P}(f^N)$ such that $x = f^{kN}(y_k)$. In addition, there exists $P \in \boldsymbol{P}(f^N)$ such that $x \in P$. This implies that $\overline{f^{kN}(Q_k)} \subset P$ for all $k \in \mathbb{N}$. Let $l := \#\boldsymbol{P}(f^N)$. (For a discrete set M, $\#M$ denotes the cardinality of M.) Then there exists $k_1 \in \{0, \ldots, l\}$ such that $P = Q_{k_1}$ and hence $\overline{f^{k_1N}(P)} \subset P$. By the contraction mapping theorem, $f^{k_1N} : \overline{P} \to \overline{P}$ has a unique fixed point, say, $z \in \overline{P}$. Let $j \in \mathbb{N}$ be the smallest integer for which $f^j(z) = z$. Then $x \in \bigcap_{n\in\mathbb{N}} \overline{f^{nj}(P)} = \{z\}$. Hence, $\Lambda(f)$ consists of periodic orbits. □

To prove the result on openness and topological stability, we first prove that the functions ρ, d_1, defined in §2, are, in fact, metrics on the set of piecewise contractions. Note that, by definition of $H(\boldsymbol{P}(f), \boldsymbol{P}(g))$, if $\#\boldsymbol{P}(f) \neq \#\boldsymbol{P}(g)$, then $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) = \emptyset$.

LEMMA 3.1. *ρ is a metric.*

Proof. Let f, g, h be piecewise contractions.

If $\rho(f, g) = 0$, then there exists a sequence $\psi_n \in H(\boldsymbol{P}(f), \boldsymbol{P}(g))$, $\|\psi_n - \mathrm{id}\|_{\mathcal{C}^0(X,X)} \to 0$ and $\|f - g \circ \psi_n\|_{\mathcal{C}^0(X,X)} \to 0$ as $n \to \infty$, which implies that $\psi_n \to \mathrm{id}$ as $n \to \infty$ which further implies that $\boldsymbol{P}(f) = \boldsymbol{P}(g)$. Also, $\psi_n \to \mathrm{id}$ means that $f - g \circ \psi_n \to f - g$, and hence $f = g$.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Next, we check the symmetry of ρ. If $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) = \emptyset$, then $\rho(f, g) = \rho(g, f) = A$. If $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) \neq \emptyset$, then

$$\begin{aligned}\rho(f, g) &= \inf_{\psi \in H(\boldsymbol{P}(f), \boldsymbol{P}(g))} \{\|\psi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - g \circ \psi\|_{\mathcal{C}^0(X,X)}\} \\ &= \inf_{\psi^{-1} \in H(\boldsymbol{P}(g), \boldsymbol{P}(f))} \{\|\psi^{-1} - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f \circ \psi^{-1} - g\|_{\mathcal{C}^0(X,X)}\} \\ &= \rho(g, f).\end{aligned}$$

It remains to check the triangle inequality.

To show that $\rho(f, g) \leq \rho(f, h) + \rho(g, h)$, consider the following cases.

If $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) = \emptyset$, then $\rho(f, g) = A$, and $H(\boldsymbol{P}(f), \boldsymbol{P}(h)) = \emptyset$ or/and $H(\boldsymbol{P}(g), \boldsymbol{P}(h)) = \emptyset$: that is, either one of the two or both are empty sets. Therefore, $\rho(f, h) = A$ or $\rho(g, h) = A$ and so $\rho(f, g) = A \leq \rho(f, h) + \rho(h, g)$.

If $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) \neq \emptyset$, then there are the following two possibilities.

(1) If $H(\boldsymbol{P}(f), \boldsymbol{P}(h)) = \emptyset$ and $H(\boldsymbol{P}(h), \boldsymbol{P}(g)) = \emptyset$, then $\rho(f, h) + \rho(g, h) \geq A$ and

$$\begin{aligned}\rho(f, g) &= \inf_{\psi \in H(P(f), P(g))} \{\|\psi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - g \circ \psi\|_{\mathcal{C}^0(X,X)}\} \\ &\leq 2\ \mathrm{diam}(X).\end{aligned}$$

Since $A \geq 2\ \mathrm{diam}(X)$, we have the result.

(2) $H(\boldsymbol{P}(f), \boldsymbol{P}(g)) \neq \emptyset$, and $H(\boldsymbol{P}(g), \boldsymbol{P}(h)) \neq \emptyset$.

Given $\phi \in H(\boldsymbol{P}(g), \boldsymbol{P}(h))$ and $\varphi \in H(\boldsymbol{P}(f), \boldsymbol{P}(h))$, the homeomorphism $\psi = \phi^{-1} \circ \varphi \in H(\boldsymbol{P}(f), \boldsymbol{P}(g))$, and hence

$$\begin{aligned}\rho(f, g) &= \inf_{\psi \in H(\boldsymbol{P}(f), \boldsymbol{P}(g))} \{\|\psi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - g \circ \psi\|_{\mathcal{C}^0(X,X)}\} \\ &\leq \inf_{\varphi \in H(\boldsymbol{P}(f), \boldsymbol{P}(h))} \inf_{\phi \in H(\boldsymbol{P}(g), \boldsymbol{P}(h))} \{\|\phi^{-1} \circ \varphi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} \\ &\quad + \|f - h \circ \varphi\|_{\mathcal{C}^0(X,X)} + \|h \circ \varphi - g \circ \phi^{-1} \circ \varphi\|_{\mathcal{C}^0(X,X)}\} \\ &\leq \inf_{\varphi \in H(\boldsymbol{P}(f), \boldsymbol{P}(h))} \inf_{\phi \in H(\boldsymbol{P}(g), \boldsymbol{P}(h))} \{\|\phi^{-1} - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - h \circ \varphi\|_{\mathcal{C}^0(X,X)} \\ &\quad + \|\varphi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|h \circ \varphi - g \circ \phi^{-1} \circ \varphi\|_{\mathcal{C}^0(X,X)}\} \\ &= \inf_{\phi \in H(\boldsymbol{P}(g), \boldsymbol{P}(h))} \{\|\phi^{-1} - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|h \circ \varphi - g \circ \phi^{-1} \circ \varphi\|_{\mathcal{C}^0(X,X)}\} \\ &\quad + \inf_{\varphi \in H(\boldsymbol{P}(f), \boldsymbol{P}(h))} \{\|\varphi - \mathrm{id}\|_{\mathcal{C}^0(X,X)} + \|f - h \circ \varphi\|_{\mathcal{C}^0(X,X)}\} \\ &= \rho(f, h) + \rho(g, h).\end{aligned}$$

Hence, $\rho(\cdot, \cdot)$ is a metric. □

Lemma 3.1 implies that d_1 is also a metric since the series is convergent.

Proof of Theorem 2.11. Let f be Markov. We want to prove that there exists a neighbourhood of f consisting of only Markov contractions. Since f is Markov, there exists $N \in \mathbb{N}$ such that the maximal partition $\boldsymbol{P}(f)$ of f stabilizes with stabilization time N. Let

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

$g \neq f$ be a piecewise contraction such that $d_1(f, g) < \delta$ for $0 < \delta < \sigma^N A$ (recall that $A = \text{diam}(X)$).

We show that the partition of g stabilizes with the same stabilization time N.

Note that, for all $n \leq N$, $\rho(f^n, g^n) < A$ which implies that $H(\boldsymbol{P}(f^n), \boldsymbol{P}(g^n)) \neq \emptyset$. Let $P \in \boldsymbol{P}(g^N)$. Then there exist $\psi_N \in H(\boldsymbol{P}(f^N), \boldsymbol{P}(g^N))$ and $P' \in \boldsymbol{P}(f^N)$ such that $\psi_N(P') = P$. By stabilization, for P', there exists a unique $Q' \in \boldsymbol{P}(f^N)$ such that $\overline{f^N(P')} \subset Q'$. In addition, $\psi_N(Q') = Q$ for some $Q \in \boldsymbol{P}(g^N)$. Now, $\overline{f^N(P')} \subset Q'$ and Q' being open implies that $\varepsilon = \min_{P' \in \boldsymbol{P}(f^N)} d_0(\overline{f^N(P')}, \partial Q') > 0$. Choosing $0 < \delta < \varepsilon\sigma^N/3$, we claim that $\overline{g^N(P)} \subset Q$. Indeed, if there exists $x \in \overline{g^N(P)} \cap \partial Q$, then there exists a sequence $\{x_k\} \in g^N(P) \cap Q$ such that $x_k \to x$ as $k \to \infty$. Let $y_k \in P$ such that $g^N(y_k) = x_k$. Note that $\psi_N^{-1}(x) \in \partial Q'$. Now,

$$\begin{aligned}
\varepsilon &< d_0(f^N(\psi_N^{-1}(y_k)), \psi_N^{-1}(x)) \\
&\leq d_0(f^N(\psi_N^{-1}(y_k)), g^N(y_k)) + d_0(g^N(y_k), x) + d_0(x, \psi_N^{-1}(x)) \\
&\leq \|f^N \circ \psi_N^{-1} - g^N\|_{\mathcal{C}^0(X,X)} + \|\text{id} - \psi_N^{-1}\|_{\mathcal{C}^0(X,X)} + d_0(x_k, x) \\
&= \rho(f^N, g^N) + d_0(x_k, x) (\text{taking infimum over } \psi_N \in H(\boldsymbol{P}(f^N), \boldsymbol{P}(g^N)) \text{ on both sides}) \\
&< \delta\sigma^{-N} + d_0(x_k, x) \to \varepsilon\sigma^N\sigma^{-N}/3 \text{ as } k \to \infty,
\end{aligned}$$

and we get $\varepsilon < \varepsilon/3$, which is a contradiction. Hence, if f is Markov with N as the stabilization time, then, for

$$\delta < \min\left\{\sigma^N A, \frac{\sigma^N}{3} \inf_{P,Q \in \boldsymbol{P}(f^N)} d_0(\overline{f^N(P)}, \partial Q)\right\},$$

all piecewise contractions g, with $d_1(f, g) < \delta$, are Markov contractions and the stabilization time of g is also N. Thus, the collection of Markov contractions is open. □

To prove Theorem 2.15, we first need to prove the following lemma which, in itself, brings some important information about the dynamics of a Markov contraction.

LEMMA 3.2. *Let f be a Markov contraction with maximal partition $\boldsymbol{P}(f)$ and stabilization time N. Then, for every $P \in \boldsymbol{P}(f^N)$, there exists $Q \in \boldsymbol{P}(f^N)$ such that $f(P) \subset Q$.*

Proof. By Definition 2.6, there exists $P' \in \boldsymbol{P}(f^N)$ such that $f^N(P) \subset P'$. By the definition given in equation (2.1), there exist partition elements $\{P_i\}_{i=0}^{N-1}$ (not necessarily distinct) in $\boldsymbol{P}(f)$ such that

$$P = P_0 \cap f^{-1}P_1 \cap f^{-2}P_2 \cap \cdots \cap f^{-(N-1)}P_{N-1}.$$

Similarly, there exist partition elements $\{P_j'\}_{j=1}^{N-1}$ in $\boldsymbol{P}(f)$ such that

$$P' = P_0' \cap f^{-1}P_1' \cap f^{-2}P_2' \cap \cdots \cap f^{-(N-1)}P_{N-1}'.$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Let $x \in P$. Then $f^k(f(x)) \in P_{k+1}$ for every $k \in \{0, 1, \ldots, N-2\}$ and $f^N(x) = f^{N-1} f(x) \in P'$, which implies that $f(x) \in f^{-(N-1)}(P'_0)$. Consequently,

$$f(x) \in P_1 \cap f^{-1} P_2 \cap \cdots \cap f^{-(N-2)} P_{N-1} \cap f^{-(N-1)} P'_0 = Q \in \boldsymbol{P}(f^N).$$

Since $x \in P$ is arbitrary, $f(P) \subset Q \in \boldsymbol{P}(f^N)$. $\square$

Finally, to prove Theorem 2.15, we restrict to piecewise injective contractions and prove the stability result under the metric d_1. Recall that, for every piecewise injective contraction *f*, with maximal partition $\boldsymbol{P}(f) = \{P_1, \ldots, P_m\}$, there exists $U_i = \mathring{U}_i \supset \overline{P_i}$ and an injective continuous extension $\tilde{f} : U_i \to \mathbb{R}^d$ such that $\tilde{f}|_{P_i} = f|_{P_i}$. In this proof, we always consider these extensions which, to alleviate notation, we still denote by *f*.

Our strategy for the following proof is as follows. For piecewise injective Markov contractions *f* and *g* with maximal partitions $\boldsymbol{P}(f)$ and $\boldsymbol{P}(g)$, respectively, and for some $\delta > 0$, $d_1(f, g) < \delta$, we construct semi-conjugacies from $NW(f)$, the non-wandering set for *f*, to $NW(g)$, the non-wandering set for *g*, using a homeomorphism between the partitions given in the definition of the metric ρ.

Proof of Theorem 2.15. Let $N \in \mathbb{N}$ be the stabilization time of $\boldsymbol{P}(f)$. By Theorem 2.11, there exists $\delta > 0$ such that, if $d_1(f, g) < \delta$, then $\boldsymbol{P}(g)$ also stabilizes at time *N*. By Theorem 2.8, the attractors of *f* and *g* consist of eventually periodic orbits. Let $P \in \boldsymbol{P}(f^N)$ be a periodic element of the partition. Then there exists $n_0 \in \mathbb{N}$ such that $\overline{f^{n_0}(P)} \subset P$. Then $d_1(f, g) < \delta$ implies that there exists $Q \in \boldsymbol{P}(g^N)$ such that $\overline{g^{n_0}(Q)} \subset Q$. By the contraction mapping theorem, for $f^{n_0} : \overline{P} \to \overline{P}$, $g^{n_0} : \overline{Q} \to \overline{Q}$, there exist x_f and x_g, the unique fixed points of f^{n_0} and g^{n_0}, respectively, in *P* and *Q*.

Using Lemma 3.2 inductively, let P_i, Q_i be the partition elements in $\boldsymbol{P}(f^N)$, $\boldsymbol{P}(g^N)$, respectively, such that $f^i(\overline{P}) \subset P_i$, $g^i(\overline{Q}) \subset Q_i$. Let $\widehat{P} = \bigcup_{i=1}^{n_0} \overline{P}_i$, $\widehat{Q} = \bigcup_{i=1}^{n_0} \overline{Q}_i$. Then, for each $i \in \mathbb{N}$, $\widehat{P}_i = f^i(\widehat{P}) \subset \widehat{P}$, $\widehat{Q}_i = g^i(\widehat{Q}) \subset \widehat{Q}$.

Note that, for δ small enough, we have $H(\boldsymbol{P}(f^n), \boldsymbol{P}(g^n)) \neq \emptyset$ for every $n \leq n_0$: that is, there exists $\psi \in H(\boldsymbol{P}(f^{n_0}), \boldsymbol{P}(g^{n_0}))$ such that $\|\psi - \mathrm{id}\|_{C^0(X,X)} < \delta$ and $\|f^n - g^n \circ \psi\|_{C^0(X,X)} < \delta$, and thus $\psi(\widehat{P}) = \widehat{Q}$.

Next, define $\widehat{g} = \psi \circ f \circ \psi^{-1}$. Since, by Definition 2.10, f^{-1} is well defined on $\widehat{P}$, we have that $\widehat{g}$ is invertible on $\widehat{Q}$ and $\widehat{g}^{-1}(\partial \widehat{Q}) = \psi \circ f^{-1}(\partial \widehat{P})$.

Next, for $\varepsilon > 0$ small enough, let $\widehat{P}_{1,\varepsilon}$ be the ε-neighbourhood of $f(\widehat{P})$ and let $\widehat{P}_{c,\varepsilon}$ be the ε-neighbourhood of $\widehat{P}^{\complement}$, the complement of $\widehat{P}$. Similarly, let $\widehat{Q}_{c,\varepsilon} = \psi \circ f^{-1}(\widehat{P}_{c,\varepsilon})$ and $\widehat{Q}_{1,\varepsilon} = \psi \circ f^{-1}(\widehat{P}_{1,\varepsilon})$. By the Markov property, we have $\overline{\widehat{P}}_{1,\varepsilon} \cap \overline{\widehat{P}}_{c,\varepsilon} = \emptyset$, and thus $\overline{\widehat{Q}}_{1,\varepsilon} \cap \overline{\widehat{Q}}_{c,\varepsilon} = \emptyset$. Hence, by Urysohn's lemma, there exists a function $\theta \in \mathcal{C}^0(X, [0, 1])$ such that $\theta|_{\overline{\widehat{Q}}_{1,\varepsilon}} = 1$ and $\theta|_{\overline{\widehat{Q}}_{c,\varepsilon}} = 0$. Finally, define the continuous functions

$$\tilde{g}(x) = \theta(x) g(x) + (1 - \theta(x)) \widehat{g}(x),$$
$$h_0(x) = \tilde{g} \circ \psi \circ f^{-1}(x) \quad \text{for all } x \in \overline{\widehat{P} \setminus \widehat{P}_1}.$$

LEMMA 3.3. *Provided $\delta > 0$ is small enough, we have $h_0(\widehat{P} \setminus \widehat{P}_1) = \widehat{Q} \setminus \widehat{Q}_1$, $h_0(\partial \widehat{P}) = \partial \widehat{Q}$, $h_0(\partial \widehat{P}_1) = \partial \widehat{Q}_1$.*

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Proof. First, using the properties of the homeomorphism ψ, we have, for each $x \in \overline{\widehat{P} \setminus \widehat{P}_1}$,

$$|h_0(x) - x| = |\theta(x)[g \circ \psi \circ f^{-1}(x) - x] + (1 - \theta(x))[\psi(x) - x]| \leq 2\delta. \tag{3.1}$$

If $x \in \widehat{P}_{c,\varepsilon} \cap \widehat{P}$, then $\psi \circ f^{-1}(x) \in \widehat{Q}_{c,\varepsilon}$, and thus $h_0(x) = \widehat{g} \circ \psi \circ f^{-1}(x) = \psi(x)$. Whereas, if $x \in \widehat{P}_{1,\varepsilon}$, then $\psi \circ f^{-1}(x) \in \widehat{Q}_{1,\varepsilon}$, and thus $h_0(x) = g \circ \psi \circ f^{-1}(x)$. In addition,

$$h_0(\partial \widehat{P}) = \psi(\partial \widehat{P}) = \partial \widehat{Q},$$
$$h_0(\partial \widehat{P}_1) = g \circ \psi \circ f^{-1}(\partial \widehat{P}_1) = g \circ \psi(\partial \widehat{P}) = g(\partial \widehat{Q}) = \partial \widehat{Q}_1.$$

Also, if δ is small enough, then h_0 is invertible on $(\widehat{P}_{c,\varepsilon} \cup \widehat{P}_{1,\varepsilon}) \cap \widehat{P}$. Thus, to prove surjectivity, it suffices to prove that each $p \in (\widehat{Q} \setminus \widehat{Q}_1) \setminus h_0((\widehat{P}_{c,\varepsilon} \cup \widehat{P}_{1,\varepsilon}) \cap \widehat{P})$ belongs to $h_0(\widehat{P} \setminus \widehat{P}_1)$. Let $B = \{z \in \mathbb{R}^n \ : \ \|z\| \leq 3\delta\}$, $x = p + z$ and $h_0(x - p) = x - h_0(x)$. Then $h_0(x) = p$ is equivalent to

$$z = h_0(z).$$

Since equation (3.1) implies that $h_0(B) \subset B$, by Brouwer's fixed-point theorem it follows that there exists at least one $z \in B$ such that $h_0(z + p) = p$ and, for $\delta \leq \varepsilon/6$, $z + p \in \widehat{P} \setminus \widehat{P}_1$. □

For the sake of convenience, let $\widehat{P}_0 = \widehat{P}$ and $\widehat{Q}_0 = \widehat{Q}$. For every $i \in \mathbb{N}$, we define $h_i : \overline{\widehat{P}_i \setminus \widehat{P}_{i+1}} \to \overline{\widehat{Q}_i \setminus \widehat{Q}_{i+1}}$ as $h_i(x) = g \circ h_{i-1} \circ f^{-1}(x)$. Thus, we define the semi-conjugacy $H : \overline{\widehat{P}} \to \overline{\widehat{Q}}$ as

$$H(x) = \begin{cases} h_i(x), & x \in \overline{\widehat{P}_i \setminus \widehat{P}_{i+1}}, \\ x_g, & x = x_f. \end{cases}$$

H is continuous because, for $x \in \partial \widehat{P}_i$, $H(x) = h_{i+1}(x) = g \circ h_i \circ f^{-1}(x) = h_i(x)$ and, for any sequence $(x_i) \in \widehat{P}_i \setminus \widehat{P}_{i+1}$ with $x_i \to x_f$ as $i \to \infty$, $(Hx_i) \in \widehat{Q}_i \setminus \widehat{Q}_{i+1}$, so $H(x_i) \to x_g = H(x_f)$. Indeed, H is surjective, for $p \in \overline{\widehat{Q}}$, and there exists $i \in \mathbb{N} \cup \{0\}$ such that $p \in \overline{\widehat{Q}_i \setminus \widehat{Q}_{i+1}}$. We use induction on i. If $i = 0$, then, using Lemma 3.3, there exists $z \in \overline{\widehat{Q}_i \setminus \widehat{Q}_{i+1}}$ such that $h_0(z) = p$. Instead, if $i \neq 0$, then $p_1 = g^{-1}(p) \in \overline{\widehat{Q}_{i-1} \setminus \widehat{Q}_i}$. Inductively, as h_{i-1} is surjective, there exists $q_1 \in \overline{\widehat{P}_{i-1} \setminus \widehat{P}_i}$ such that $h_{i-1}(q_1) = p_1$. Now, by definition, $q = f(q_1) \in \overline{\widehat{P}_i \setminus \widehat{P}_{i+1}}$, and finally $g \circ h_i \circ f^{-1}(q) = p$.

Furthermore, H is the wanted semi-conjugacy between f, g, on $\widehat{P}$ and $\widehat{Q}$ because, for $x \in \widehat{P}$,

$$H_{\widehat{P}} \circ f(x) = h_{i+1} \circ f(x) = g \circ h_i \circ f^{-1} \circ f(x) = g \circ h_i(x) = g \circ H(x).$$

To obtain a semi-conjugacy on the whole of $NW(f)$, we repeat the same steps for every periodic partition element $P \in \boldsymbol{P}(f^N)$, $Q \in \boldsymbol{P}(g^N)$ with the same $\psi \in H(\boldsymbol{P}(f^N), \boldsymbol{P}(g^N))$. Calling $\{\widehat{P}^k\}$ the collection of the union of elements associated to a periodic orbit, we have $\bigcup_k \widehat{P}^k = NW(f)$. Pasting these functions together, we obtain a function $\Theta : NW(f) \to NW(g)$ with $\Theta(x) = H_{\widehat{P}^k}(x)$ for $x \in \widehat{P}^k \subset NW(f)$. Then Θ is continuous on $NW(f)$ as $\Theta|_{\partial \widehat{P}^k} = \psi|_{\partial \widehat{P}^k}$ for every k.

To construct the semi-conjugacy from the other side, we use the fact that ψ is a homeomorphism and repeat the same construction using ψ^{-1} instead of ψ and switching the roles of f and g. □

4. *Density*

We prove Theorem 2.17 in two steps. First, we show that Markov maps are dense in a special class of systems (piecewise strongly contracting) and then we will show that such a class is itself dense in the collection of piecewise smooth contractions.

Definition 4.1. (Piecewise strongly contracting) A piecewise smooth contraction f with contraction coefficient λ and maximal partition $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$ is said to be piecewise strongly contracting if there exists $p \in \mathbb{N}$ such that $\lambda^p m_p < 1/2$, where $m_p = \#\boldsymbol{P}(f^p)$.

We prove the following results.

PROPOSITION 4.2. *Markov maps are d_2-dense in the collection of piecewise strong contractions.*

PROPOSITION 4.3. *Piecewise strong contractions are d_2-dense in the collection of piecewise smooth contractions.*

These two propositions readily imply our main result.

Proof of Theorem 2.17. Let f be a piecewise smooth contraction. By Proposition 4.3, for each $\varepsilon > 0$, there exists a piecewise strong contraction f_1 such that $d_2(f, f_1) < \varepsilon/2$. In addition, by Proposition 4.2, there exists a piecewise smooth Markov contraction f_2 such that $d_2(f_1, f_2) < \varepsilon/2$, and hence the result. □

In the rest of the paper, we prove Propositions 4.2 and 4.3.

The basic idea of the proof is to introduce iterated function systems (IFSs) associated with the map. The attractor of the IFS is greater than the one of the map (see §4.1 for the relationship between the two sets), and hence if we can prove that the attraction of the IFS is disjoint from the discontinuities of the map, so will be the attractor of the map. The advantage is that, in this way, the study of the boundaries of the elements of $\boldsymbol{P}(f^n)$ is reduced to the study of the pre-images of the discontinuities of f under the IFS. Hence, we can iterate smooth maps rather than discontinuous ones.

This advantage is first exploited in §4.2, where we prove Proposition 4.2 using an argument that is, essentially, a quantitative version of Sard's theorem.

To prove Proposition 4.3, the rough idea is to use a transversality theorem (see Appendix B) to show that if a lot of pre-images intersect, then, generically, their intersection should have smaller and smaller dimensions until no further intersection is generically possible. Unfortunately, if we apply a transversality theorem to a composition of maps of the IFS, we get a perturbation of the composition and not of the single maps. How to perturb the single maps in such a way that the composition has the wanted properties is not obvious.

Our solution to this problem is to make sure that if we perturb the maps in a small neighbourhood B, and we consider arbitrary compositions of the perturbed maps, then all the images of B along the composition never intersect B. Hence, if we restrict the composition to B, all the maps, except the first, will behave as their unperturbed version. To ensure this, it suffices to prove that such compositions have no fixed points near the singularity manifolds (such an implication is proved in Lemma 4.10). To this end, in Propositions 4.11 and 4.17, we show that one can control the location of the fixed points of the compositions of the map of the IFS by an arbitrarily small perturbation.

After this, we can finally set up an inductive scheme to ensure that the pre-images of the discontinuity manifolds keep intersecting transversally. This is the content of Proposition 4.19 from which Proposition 4.3 readily follows.

4.1. *IFSs associated to the map and their properties.* We start by recalling the definition of an IFS relevant to our argument and exploring some of its properties.

Definition 4.4. (IFS) The set $\Phi = \{\phi_1, \phi_2, \ldots, \phi_m\}$, $m \geq 2$, is an IFS if each map $\phi_i : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz contraction. (In the following, we will consider only $\mathcal{C}^3$ maps ϕ_i.)

Let f be a piecewise smooth contraction with maximal partition $\boldsymbol{P}(f)$. By analogy with [**23**], we define an IFS associated to f as follows.

By Definition 2.16, $\tilde{f}|_{U_i}$ is $\mathcal{C}^3$ for every i and $D\tilde{f}|_{U_i} \leq \lambda < 1$. Using the $\mathcal{C}^r$ version of Kirszbraun–Valentine theorem A.2, we obtain a $\mathcal{C}^3$ extension $\phi_i : \mathbb{R}^d \to \mathbb{R}^d$ of $\tilde{f}|_{U_i}$, and hence of $f|_{P_i}$, so that $\|D\phi_i\| \leq \lambda < 1$ for all $i \in \{1, 2, \ldots, m\}$. We denote a $\mathcal{C}^3$ IFS associated to f by

$$\Phi_f = \{\phi_1, \phi_2, \ldots, \phi_m\}.$$

Remark 4.5. Unfortunately, it is not obvious if one can obtain an extension in which ϕ_i are invertible. This would simplify the following arguments as one would not have to struggle to restrict the discussion to the sets U_i (e.g., see (4.11)). However, since $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ finite to one maps are generic by Tougeron's theorem (see [[**15**], Theorem 2.6, pp. 169]), we can assume, by an arbitrarily small perturbation of f, that the ϕ_i are finite to one.

For $m, n \in \mathbb{N}$, let $\Sigma_n^m = \{1, 2, \ldots, m\}^n$ and $\Sigma^m = \{1, 2, \ldots, m\}^{\mathbb{N}}$ be the standard symbolic spaces. We endow Σ^m with the metric d_γ for some $\gamma > 1$:

$$d_\gamma(\sigma, \sigma') = \sum_{i=1}^{\infty} \frac{|\sigma_i - \sigma_i'|}{\gamma^i}. \tag{4.1}$$

In addition, let $\tau : \Sigma^m \to \Sigma^m$ be the left subshift: $\tau(\sigma_1, \sigma_2, \sigma_3, \ldots) = (\sigma_2, \sigma_3, \ldots)$.

Set $K = \max\{\|x\| : x \in X\}$ ($\|\cdot\|$ is the general Euclidean norm). Let $M = \sup_i \|\phi_i(0)\|$. Then, for each $y \in \mathbb{R}^d$,

$$\|\phi_i(y)\| \leq \|\phi_i(y) - \phi_i(0)\| + M \leq \lambda\|y\| + M.$$

Thus, setting

$$Y = \{y \in \mathbb{R}^d \ : \ \|y\| \le \max\{K, (1-\lambda)^{-1}M\}\}, \tag{4.2}$$

we have $\phi_i(Y) \subset Y$, for all $i \in \{1, \ldots, m\}$, $X \subset Y$ and Y is a d-dimensional manifold with boundary.

Next, define $\Theta_f : \Sigma^m \to Y$ as

$$\Theta_f(\sigma) = \bigcap_{n\in\mathbb{N}} \phi_{\sigma_1} \circ \phi_{\sigma_2} \circ \cdots \circ \phi_{\sigma_n}(Y), \tag{4.3}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots,) \in \Sigma^m$. The sets $\{\phi_{\sigma_1} \circ \phi_{\sigma_2} \circ \cdots \phi_{\sigma_n}(Y)\}_{n\in\mathbb{N}}$ form a nested sequence of compact subsets of Y. In addition, $\operatorname{diam}(\phi_{\sigma_1} \circ \phi_{\sigma_2} \circ \cdots \circ \phi_{\sigma_n}(Y)) \to 0$ as $n \to \infty$, so, by Cantor's intersection theorem, $\Theta_f(\sigma)$ is a single element in Y for every $\sigma \in \Sigma$, which implies that Θ_f is well defined. We define the attractor of the IFS Φ_f as

$$\Lambda(\Phi_f) = \Theta_f(\Sigma^m). \tag{4.4}$$

LEMMA 4.6. *The function $\Theta_f : \Sigma^m \to Y$ is continuous. In turn, $\Lambda(\Phi_f)$ is compact.*

Proof. For given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\lambda^k \operatorname{diam}(Y) < \varepsilon$. Let $\delta = \gamma^{-k}$, where γ is the one in (4.1). For $\sigma, \sigma' \in \Sigma^m$, $d_\gamma(\sigma, \sigma') < \delta = \gamma^{-k}$ implies that, for all $i < k$, $\sigma_i = \sigma'_i$, and hence, for any $x, y \in Y$,

$$\begin{aligned} &d_0(\Theta_f(\sigma), \Theta_f(\sigma')) \\ &\quad \le \sup_{x,y\in Y} d_0(\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_k} \circ \phi_{\sigma_{k+1}} \cdots \circ \phi_{\sigma_n}(x), \\ &\qquad\qquad \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_k} \circ \phi_{\sigma'_{k+1}} \cdots \circ \phi_{\sigma'_n}(y)) \\ &\quad < \lambda^k \sup_{x,y\in Y} d_0(\phi_{\sigma_{k+1}} \circ \cdots \circ \phi_{\sigma_n}(x), \phi_{\sigma'_{k+1}} \cdots \circ \phi_{\sigma'_n}(y)) \\ &\quad < \lambda^k \operatorname{diam}(Y) < \varepsilon, \end{aligned}$$

and hence the continuity with respect to σ. Since Σ^m is compact, the attractor $\Lambda(\Phi_f)$ is compact as it is the continuous image of a compact set. □

Remark 4.7. Let $\Phi_f = \{\phi_1, \ldots, \phi_m\}$ be an IFS associated to a piecewise contraction f. For $p \in \mathbb{N}$, let $\boldsymbol{P}(f^p) = \{P_1^p, P_2^p, \ldots, P_{m_p}^p\}$ be the partition of f^p given as in equation (2.1). Define the corresponding IFS associated to f^p as

$$\Phi_{f^p} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m_p}\},$$

where, for all $i \in \{1, 2, \ldots, m_p\}$, there exists unique $\sigma^i = (\sigma_1^i, \ldots, \sigma_p^i) \in \Sigma_p^m$ (uniquely determined by the partition element $P_i^p \in \boldsymbol{P}(f^p)$) such that

$$\varphi_i = \phi_{\sigma_1^i} \circ \phi_{\sigma_2^i} \circ \cdots \circ \phi_{\sigma_p^i}.$$

The attractor of Φ_{f^p} is $\Lambda(\Phi_{f^p}) = \Theta_{f^p}(\Sigma^{m_p})$. To avoid confusion, we denote the elements in Σ^m by σ and the elements in Σ^{m_p} by ω.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

LEMMA 4.8. *For a piecewise smooth contraction f with IFS Φ_f and for $p \in \mathbb{N}$, the following relationship holds.*

$$\Lambda(f) \subset \Lambda(f^p) \subset \Lambda(\Phi_{f^p}) \subset \Lambda(\Phi_f).$$

Proof. For $p \in \mathbb{N}$, let $\boldsymbol{P}(f^p) = \{P_1^p, P_2^p, \ldots, P_{m_p}^p\}$ and let $m_1 = m$. We start by proving the first inclusion, that is $\Lambda(f) \subset \Lambda(f^p)$. Let $x \in \Lambda(f) = \bigcap_{n\in\mathbb{N}} \overline{f^n(X)}$, that is, for every $n \in \mathbb{N}$, $x \in \overline{f^n(X)}$. Accordingly,

$$x \in \bigcap_{n\in\mathbb{N}} \overline{(f^p)^n(X)} = \Lambda(f^p).$$

Thus, $\Lambda(f) \subset \Lambda(f^p)$. For the third inclusion, that is, $\Lambda(\Phi_{f^p}) \subset \Lambda(\Phi_f)$, let $\Phi_{f^p} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m_p}\}$. Then, for every $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma^{m_p}$ there exists $\sigma^\omega = (\sigma^{\omega_1}, \sigma^{\omega_2}, \ldots)$, where $\sigma^{\omega_i} = (\sigma_1^{\omega_i}, \ldots, \sigma_p^{\omega_i}) \in \Sigma_p^m$, such that

$$\Theta_{f^p}(\omega) = \bigcap_{n\in\mathbb{N}} \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}(Y) = \bigcap_{n\in\mathbb{N}} \phi_{\sigma_1^{\omega_1}} \circ \cdots \circ \phi_{\sigma_p^{\omega_1}} \circ \cdots \circ \phi_{\sigma_p^{\omega_n}}(Y).$$

Thus, $\Lambda(\Phi_{f^p}) = \bigcup_{\omega\in\Sigma^{m_p}} \Theta_{f^p}(\omega) = \bigcup_{\{\sigma^\omega\,:\,\omega\in\Sigma^{m_p}\}} \Theta_f(\sigma^\omega) \subset \bigcup_{\sigma\in\Sigma^m} \Theta_f(\sigma) = \Lambda(\Phi_f)$. Finally, for the second inclusion, let $x \in \Lambda(f^p) = \bigcap_{n\in\mathbb{N}} \overline{f^{pn}(X)}$. Then, for all $n \in \mathbb{N}$, there exists $y_n \in X$ such that $d_0(x, f^{pn}(y_n)) < 1/n$. By definition of Φ_{f^p}, there exists $\omega^n = (\omega_1^n, \omega_2^n, \ldots, \omega_n^n, \ldots) \in \Sigma^{m_p}$ such that $f^{pn}(y_n) = \varphi_{\omega_1^n} \circ \cdots \circ \varphi_{\omega_n^n}(y_n)$, where $\varphi_{\omega_i^n} \in \Phi_{f^p}$. This implies that, for all $n \in \mathbb{N}$, $d_0(x, \varphi_{\omega_1^n} \circ \cdots \circ \varphi_{\omega_n^n}(y_n)) < 1/n$. By compactness of Σ^{m_p}, there exists a subsequence $\{n_k\}$ and $\omega \in \Sigma^{m_p}$ such that $\omega^{n_k} \to \omega$. Since, by Lemma 4.6, Θ_{f^p} is continuous,

$$d_0(x, \Theta_{f^p}(\omega)) = \lim_{k\to\infty} d_0(x, \Theta_{f^p}(\omega^{n_k})) = \lim_{k\to\infty} d_0(x, f^{pn_k}(y_{n_k})) = 0.$$

Hence, by definition of the attractor, $x \in \Lambda(\Phi_{f^p})$. □

4.2. *A simple perturbation and the proof of Proposition 4.2.* For $\delta \in \mathbb{R}^d$ with $|\delta| > 0$ sufficiently small, and a piecewise contraction f with IFS $\Phi_f = \{\phi_1, \phi_2, \ldots, \phi_m\}$, we define perturbations f^δ, Φ_{f^δ} as

$$f^\delta(x) = f(x) + \delta, \ \phi_i^\delta = \phi_i + \delta. \tag{4.5}$$

Provided $|\delta|$ is small enough, the perturbation f^δ satisfies $\overline{f^\delta(X)} \subset \mathring{X}$, and hence f^δ is a piecewise smooth contraction with corresponding IFS $\Phi_{f^\delta} = \{\phi_1^\delta, \phi_2^\delta, \ldots, \phi_m^\delta\}$. One can easily check that $d_2(f, f^\delta) = |\delta|$. By definition (see (4.3)), $\Theta_{f^\delta} : \Sigma^m \to Y$ reads

$$\Theta_{f^\delta}(\sigma) = \bigcap_{n\in\mathbb{N}} \phi_{\sigma_1}^\delta \circ \phi_{\sigma_2}^\delta \circ \cdots \circ \phi_{\sigma_n}^\delta(Y)$$

and the respective attractor is $\Lambda(\Phi_{f^\delta}) = \bigcup_{\sigma\in\Sigma^m} \Theta_{f^\delta}(\sigma)$.

Observe that, for any $p \in \mathbb{N}$, the corresponding IFS associated to $(f^\delta)^p$ is given by $\Phi_{(f^\delta)^p} = \{\varphi_1^\delta, \varphi_2^\delta, \ldots, \varphi_{m_p}^\delta\}$, where, for every $i \in \{1, 2, \ldots, m_p\}$, there exists $\sigma_i \in \Sigma_p^m$ such that $\varphi_i^\delta = \phi_{\sigma_1^i}^\delta \circ \phi_{\sigma_2^i}^\delta \circ \cdots \circ \phi_{\sigma_p^i}^\delta$.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

LEMMA 4.9. *The map $\Theta_{f^\delta}(\sigma) \mapsto \bigcap_{n\in\mathbb{N}} \phi^\delta_{\sigma_1} \circ \phi^\delta_{\sigma_2} \circ \cdots \circ \phi^\delta_{\sigma_n}(Y)$ is uniformly Lipschitz continuous in δ: that is, there exists $a > 0$ such that, for all $\sigma \in \Sigma^m$, $d_0(\Theta_{f^\delta}(\sigma), \Theta_{f^{\delta'}}(\sigma)) \le a d_0(\delta, \delta')$.*

Proof. Let $\delta, \delta' > 0$, $n \in \mathbb{N}$, $x \in Y$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$. Then

$$\begin{aligned} d_0(\Theta_{f^\delta}(\sigma), \Theta_{f^{\delta'}}(\sigma)) &\le d_0(\Theta_{f^\delta}(\sigma), \phi^\delta_{\sigma_1} \circ \cdots \circ \phi^\delta_{\sigma_n}(x)) \\ &\quad + d_0(\Theta_{f^{\delta'}}(\sigma), \phi^{\delta'}_{\sigma_1} \circ \cdots \circ \phi^{\delta'}_{\sigma_n}(x)) \\ &\quad + d_0(\phi^\delta_{\sigma_1} \circ \cdots \circ \phi^\delta_{\sigma_n}(x), \phi^{\delta'}_{\sigma_1} \circ \cdots \circ \phi^{\delta'}_{\sigma_n}(x)) \\ &\le \lambda^n d_0(\Theta_{f^\delta}(\tau^n\sigma), x) + \lambda^n d_0(\Theta_{f^{\delta'}}(\tau^n\sigma), x) \\ &\quad + d_0(\delta, \delta') + d_0(\phi_{\sigma_1} \circ \phi^\delta_{\sigma_2} \circ \cdots \circ \phi^\delta_{\sigma_n}(x), \phi_{\sigma_1} \circ \phi^{\delta'}_{\sigma_2} \circ \cdots \circ \phi^{\delta'}_{\sigma_n}(x)) \\ &\le 2\lambda^n \operatorname{diam}(Y) + d_0(\delta, \delta') \\ &\quad + \lambda d_0(\phi^\delta_{\sigma_2} \circ \cdots \circ \phi^\delta_{\sigma_n}(x), \phi^{\delta'}_{\sigma_2} \circ \cdots \circ \phi^{\delta'}_{\sigma_n}(x)). \end{aligned}$$

Iterating the above argument yields

$$\begin{aligned} d_0(\Theta_{f^\delta}(\sigma), \Theta_{f^{\delta'}}(\sigma)) &\le \lim_{n\to\infty} \{2\lambda^n \operatorname{diam}(Y) + d_0(\delta, \delta')(1 + \lambda + \lambda^2 + \cdots + \lambda^n)\} \\ &= (1-\lambda)^{-1} d_0(\delta, \delta'), \end{aligned}$$

and letting $a = 1/(1-\lambda)$ concludes the proof. □

Proof of Proposition 4.2. Let $p \in \mathbb{N}$ be such that $m_p \lambda^p \le \frac{1}{2}$. Lemma 4.8 asserts that $\Lambda(f^\delta) \subset \Lambda(\Phi_{(f^\delta)^p})$. Hence, by Theorem 2.8, it suffices to prove that, for every $\varepsilon > 0$ small enough, there exists $\delta \in B_\varepsilon(0)$ such that the attractor $\Lambda(\Phi_{(f^\delta)^p})$ is disjoint from $\partial \boldsymbol{P}(f^\delta) = \partial \boldsymbol{P}(f)$.

Suppose, to the contrary, that, for every $\delta \in B_\varepsilon(0)$, $\Lambda(\Phi_{(f^\delta)^p}) \cap \partial \boldsymbol{P}(f) \ne \emptyset$. Accordingly, there exists $\omega(\delta) \in \Sigma^{m_p}$ for which $\Theta_{(f^\delta)^p}(\omega(\delta)) \in \partial \boldsymbol{P}(f)$. By definition, $\partial \boldsymbol{P}(f) = \bigcup_{P \in \boldsymbol{P}(f)} \partial P$. Therefore, there exists $P_i \in \boldsymbol{P}(f)$ and $A \subset B_\varepsilon(0)$ with $\mu_d(A) \ge \mu_d(B_\varepsilon(0))/m = C_d \varepsilon^d / m$ (μ_d is the d-dimensional Lebesgue measure) such that, for all $\delta \in A$, $\Theta_{(f^\delta)^p}(\omega(\delta)) \in \partial P_i$.

Moreover, for each $k \in \mathbb{N}$, there exist $\omega^* = (\omega^*_1, \omega^*_2, \ldots, \omega^*_k) \in \Sigma^{m_p}_k$ such that the set defined as

$$A_k(\omega^*) = \{\delta \in A : \omega(\delta)_j = \omega^*_j,\ j \le k\}$$

is non-empty and $\mu_d(A_k(\omega^*)) \ge \mu_d(A)/m_p^k \ge C_d \varepsilon^d m^{-1} {m_p}^{-k}$. Accordingly, for $\omega(\delta) \in A_k(\omega^*)$ and the IFS associated to $\Phi_{(f^\delta)^p} = \{\varphi^\delta_1, \varphi^\delta_2, \ldots, \varphi^\delta_{m_p}\}$,

$$\partial P_i \ni \Theta_{(f^\delta)^p}(\omega(\delta)) = \varphi^\delta_{\omega^*_1} \circ \Theta_{(f^\delta)^p}(\tau\omega(\delta)) = \varphi^\delta_{\omega^*_1} \circ \varphi^\delta_{\omega^*_2} \circ \cdots \circ \varphi^\delta_{\omega^*_k} \circ \Theta_{(f^\delta)^p}(\tau^k \omega(\delta)),$$

where τ is the left shift as defined above and $\varphi^\delta_{\omega^*_j} = \phi^\delta_{\sigma^p_{j,1}} \circ \phi^\delta_{\sigma^p_{j,2}} \circ \cdots \circ \phi^\delta_{\sigma^p_{j,p}}$, for $\omega^*_j = (\sigma^p_{j,1}, \sigma^p_{j,2}, \ldots, \sigma^p_{j,p}) \in \Sigma^m_p$ with $\phi^\delta_{\sigma^p_{j,s}} \in \Phi_{f^\delta}$.

Next, for some $\bar{x} \in X$, define $\theta : A_k(\omega^*) \to X$ as $\theta(\delta) = \varphi^\delta_{\omega^*_1} \circ \varphi^\delta_{\omega^*_2} \circ \cdots \circ \varphi^\delta_{\omega^*_k}(\bar{x})$. Then

$$d_0(\Theta_{(f^\delta)^p}(\omega(\delta)), \theta(\delta)) \le \lambda^{pk} d_0(\Theta_{(f^\delta)^p}(\tau^k \omega(\delta)), \bar{x}) \le \operatorname{diam}(Y)\lambda^{pk} =: B_* \lambda^{pk}.$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Thus, $\theta(\delta)$ belongs to a $B_*\lambda^{pk}$ neighbourhood of ∂P_i. Since ∂P_i is contained in the union of finitely many $\mathcal{C}^2$ manifolds, the Lebesgue measure of a $B_*\lambda^{pk}$ neighbourhood of ∂P_i is bounded above by $C\lambda^{pk}\mu_{d-1}(\partial P_i)$ for a fixed constant $C > 0$. Accordingly,

$$\mu_d(\theta(A_k(\omega^*))) \leq C\lambda^{pk}\mu_{d-1}(\partial P_i). \tag{4.6}$$

On the other hand,

$$\mu_d(\theta(A_k(\omega^*))) = \int_{A_k(\omega^*)} |\det(D\theta(\delta))|\, d\delta,$$

where, by definition of $\theta(\delta)$, $D\theta(\delta) = \mathbb{1} + D\varphi_{\sigma_1^*} + D\varphi_{\sigma_1^*}D\varphi_{\sigma_2^*} + \cdots + D\varphi_{\sigma_1^*}\cdots D\varphi_{\sigma_{k-1}^*}$. Note that

$$\|D\varphi_{\sigma_1^*} + D\varphi_{\sigma_1^*}D\varphi_{\sigma_2^*} + \cdots + D\varphi_{\sigma_1^*}\cdots D\varphi_{\sigma_{k-1}^*}\| \leq \frac{\lambda^p}{1-\lambda^p},$$

where $\|\cdot\|$ is the standard operator norm defined as $\|L\| = \sup_{\|v\|=1}\|Lv\|$ for any linear operator $L : \mathbb{R}^d \to \mathbb{R}^d$. It follows that, for all $v \in \mathbb{R}^d$,

$$\|D\theta(\delta)v\| \geq \|v\| - \frac{\lambda^p}{1-\lambda^p}\|v\| \geq \frac{1}{2}\|v\|,$$

since $\lambda^p \leq 1/2m^p$ and $m^p \geq 2$. Hence, the eigenvalues of $D\theta(\delta)$ are larger, in modulus, than $\frac{1}{2}$. Accordingly, $|\det(D\theta(\delta))| \geq 2^{-d}$ and

$$\mu_d(\theta(A_k(\omega^*))) \geq 2^{-d}\mu_d(A_k(\omega^*)) \geq C_d 2^{-d}\varepsilon^d m^{-1} {m_p}^{-k}, \tag{4.7}$$

which, for k large enough, is in contradiction with (4.6), and this concludes the proof. □

4.3. *Fixed points in a generic position.* Fix $N \in \mathbb{N}$. For all $q \leq N+1$ and $\sigma = (\sigma_1, \ldots, \sigma_q) \in \Sigma_q^m(\Phi)$, let $x_\sigma(\Phi)$ be the unique fixed point of $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_q}$: that is,

$$\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_q}(x_\sigma(\Phi)) = x_\sigma(\Phi). \tag{4.8}$$

The goal of this section is to define a perturbation that puts the above fixed points in a generic position. We start with the following trivial but useful fact concerning the location of such fixed points.

LEMMA 4.10. *Given an IFS Φ, if for some $y \in \mathbb{R}^d$, $\delta > 0$, $p \in \mathbb{N}$ and $\sigma \in \Sigma_p^m$,*

$$\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(B_\delta(y)) \cap B_\delta(y) \neq \emptyset,$$

then the unique fixed point of $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}$ belongs to $B_{c_*\delta}(y)$, $c_* = 2/(1-\lambda) > 2$.

Proof. The fact that $\|\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(y) - y\| \leq 2\delta$ implies that, for each $x \in B_{c_*\delta}(y)$,

$$\begin{aligned}\|\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(x) - y\| &\leq \|\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(x) - \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(y)\| + 2\delta \\ &\leq (\lambda^p c_* + 2)\delta \leq c_*\delta.\end{aligned}$$

Hence, $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}(B_{c_*\delta}(y)) \subset B_{c_*\delta}(y)$. The lemma follows by the contraction mapping theorem. □

Let $\mathcal{A}_p = \{\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p} : \sigma \in \Sigma_p^m\}$ and $A_p(\Phi) := \{x_\sigma(\Phi) : \sigma \in \Sigma_p^m(\Phi)\}$ and let $\#A_N(\Phi) = k_\star$. We can now explain what we mean by having the fixed points in a generic position.

PROPOSITION 4.11. *For each $\varepsilon > 0$, there exists an ε-perturbation $\Phi^0 = \{\phi_k^0\}$ of Φ such that, for all $q \le p \le N$, $\sigma \in \Sigma_q^m(\Phi)$, $\omega \in \Sigma_p^m(\Phi)$, $\omega \ne \sigma$, if $\sigma_{p+1} \ne \omega_q$, then $x_\sigma(\Phi^0) \ne x_\omega(\Phi^0)$, whereas if $\sigma_{p+1} = \omega_q$ and $x_\sigma(\Phi^0) = x_\omega(\Phi^0)$, then $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_q}$ and $\phi_{\omega_1} \circ \cdots \circ \phi_{\omega_p}$ are both some power of a $\Theta \in \bigcup_{s=1}^p \mathcal{A}_s$.*

Proof. We proceed by induction on p. If $\sigma_1, \omega_1 \in \{1, \ldots, m\}$ and $x_{\sigma_1}(\Phi) = x_{\omega_1}(\Phi)$, then we can simply make the perturbation $\tilde{\phi}_{\sigma_1}(x) = \phi_{\sigma_1}(x) + \eta$ for some $\|\eta\| < \varepsilon/2$. This proves the statement for $p = 1$. To simplify the notation, we keep calling Φ also the perturbed IFS. We suppose that the statement is true for p, after a perturbation of size at most $(1 - 2^{-p})\varepsilon$, and we prove it for $p + 1$.

Since $A(p+1) = \{x_\sigma(\Phi) : \sigma \in \Sigma_{p+1,*}^m(\Phi)\}$ is a finite discrete set,

$$\delta_{p+1}^* = \min\left\{1, \inf_{\substack{x,y \in A(p+1) \\ x \ne y}} \|x - y\|\right\} > 0.$$

Let $\delta \le c_*^{-2}\delta_{p+1}^*/2$ (recall that $c_* = 2/(1-\lambda)$). Suppose that $z := x_\sigma(\Phi) = x_\omega(\Phi)$ and $\omega_q = \sigma_{p+1}$, where $\sigma \in \Sigma_{p+1}^m(\Phi)$, $\omega \in \Sigma_q^m(\Phi)$, $q \le p+1$ and $\sigma \ne \omega$. Let $j \in \{0, \ldots, q-1\}$ be the largest integer such that $\sigma_{p+1-j} = \omega_{q-j}$. If $j = q-1$, then it must be that $q \le p$; otherwise, we would have $\omega = \sigma$. Hence,

$$\phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q}(z) = z = \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1}}(z) = \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1-q}} \circ \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q}(z).$$

That is, $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1-q}}(z) = z$. It follows, by the inductive hypothesis, that there exist $k \in \mathbb{N}$ and $\Theta \in \mathcal{A}_s$, $s \le q$, such that

$$\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1-q}} = \Theta^k,$$
$$\phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q} = \Theta^j,$$

so $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1}} = \Theta^{k+j}$, as claimed. It remains to consider the case $j < q-1$. Let $\psi := \phi_{\omega_{q-j}} \circ \cdots \circ \phi_{\omega_q}$. Then $\psi = \phi_{\sigma_{p+1-j}} \circ \cdots \circ \phi_{\sigma_{p+1}}$ and

$$[\psi \circ \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_{q-j-1}}](\psi(z)) = \psi(z) = [\psi \circ \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p-j}}](\psi(z)),$$

where, by construction, $\omega_{q-j-1} \ne \sigma_{p-j}$. By renaming the indices, we are thus reduced to the case $\omega_q \ne \sigma_{p+1}$. The following lemma is useful for analysing this case.

SUB-LEMMA 4.12. *If, for $j \in \{1, \ldots, p\}$,*

$$\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(B_\delta(z)) \cap B_\delta(z) \ne \emptyset,$$

then $\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(z) = z$ and $\sigma_j = \sigma_{p+1}$, and the same for ω.

Proof. Lemma 4.10 implies that there exists $z_1 \in B_{c_*\delta}(z)$ such that $\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(z_1) = z_1$. In addition,

$$\|z - \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_j}(z_1)\| = \|\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1}}(z) - \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1}}(z_1)\| \le \lambda c_* \delta.$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Thus, $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_j}(B_{c_*\delta}(z)) \cap B_{c_*\delta}(z) \neq \emptyset$, and hence Lemma 4.10 implies that there exists $z_2 \in B_{c_*^2\delta}(z)$ such that $\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_j}(z_2) = z_2$. The definition of δ implies that $z_1 = z_2$, which, in turn, implies that $z_1 = z$. Hence,

$$\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_j}(z) = z,$$
$$\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(z) = z,$$

and, by the inductive hypothesis, this is possible only if $\sigma_j = \sigma_{p+1}$. The argument for ω is identical. □

If there exists $j \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, q\}$ such that

$$\begin{aligned} &\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(B_\delta(z)) \cap B_\delta(z) \neq \emptyset, \\ &\phi_{\omega_{k+1}} \circ \cdots \circ \phi_{\omega_q}(B_\delta(z)) \cap B_\delta(z) \neq \emptyset, \end{aligned} \tag{4.9}$$

then, by Sub-Lemma 4.12, $\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(z) = z = \phi_{\omega_{k+1}} \circ \cdots \circ \phi_{\omega_q}(z)$, which contradicts our inductive hypothesis. Thus, if the first inequality is satisfied for some j, the second cannot be satisfied for any k, and vice versa. Suppose that there does not exist k for which the second inequality of (4.9) is satisfied (the other possibility being completely analogous).

Define the perturbation

$$\tilde{\phi}_k = \begin{cases} \phi_k \circ h_{z,\delta} & \text{if } k = \omega_q, \\ \phi_k & \text{otherwise,} \end{cases}$$

where, by analogy with (4.18), for $v \in \mathbb{R}^d$, $\|v\| = 1$,

$$h_{z,\delta}(x) = \begin{cases} x & \text{for all } x \notin B_{c_*^{-1}\delta}(z), \\ x + c_*^{-3}\delta^3 g(1 - c_*\delta^{-1}\|x - z\|)v & \text{otherwise.} \end{cases}$$

Note that $\|h_{z,\delta} - \mathrm{id}\|_{\mathcal{C}^2} \leq 2^{-p-1}\varepsilon_0$. Since $h_{z,\delta}(B_{c_*^{-1}\delta}(z)) \subset B_{c_*^{-1}\delta}(z)$, it follows that the effect of the perturbation is always confined to $B_\delta(z)$ and its images. Moreover, by Sub-Lemma 4.12, if there exists j such that $\phi_{\sigma_{j+1}} \circ \cdots \circ \phi_{\sigma_{p+1}}(B_\delta(z)) \cap B_\delta(z) \neq \emptyset$, we have seen that $\sigma_j = \sigma_{p+1} \neq \omega_q$, so next we apply a map, ϕ_{σ_j}, that has not been modified. On the other hand, if $x \in B_\delta(z)$ and for some k we have $\omega_k = \omega_q$, we have $\phi_{\omega_{k+1}} \circ \cdots \circ \phi_{\omega_q}(x) \notin B_\delta(z)$. Next, we apply ϕ_{ω_k} outside the region where it has been modified. It follows that, for each $x \in B_\delta(z)$,

$$\tilde{\phi}_{\omega_1} \circ \cdots \circ \tilde{\phi}_{\omega_q}(x) = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q} \circ h_{z,\delta}(x),$$
$$\tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_{p+1}}(x) = \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{p+1}}(x).$$

Calling $z(\delta)$ the unique fixed point of $\phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q} \circ h_{z,\delta}$,

$$z'(\delta) = (\mathbb{1} - D\phi_{\omega_1} \cdots D\phi_{\omega_q} Dh_{z,v})^{-1} D\phi_{\omega_1} \cdots D\phi_{\omega_q} \partial_\delta h_{z,\delta}.$$

Since $z(0) = z$ and

$$\partial_\delta h_{z,\delta} = 3c_*^{-3}\delta^2 g(1 - c_*\delta^{-1}\|z(\delta) - z\|)v + c_*^{-2}\delta\|z(\delta) - z\|g'(1 - c_*\delta^{-1}\|z(\delta) - z\|)v,$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

we have that it is not possible that $z(\delta) = z$ for all $\delta \le c_*^{-2}\delta^*_{p+1}/2$. Thus, we can make a perturbation for which the two fixed points are different, and, for δ small, they cannot be equal to the other fixed points.

For any other couple of elements $\sigma \in \Sigma^m_{p+1}(\Phi)$, $\omega \in \Sigma^m_q(\Phi)$, we can repeat the same process and obtain the perturbation with two different fixed points, as above. Note that, as the size of the perturbation is $\delta < c_*^{-1}\delta^*_{p+1}/2$, the distance between the newly obtained fixed points in $\bigcup_{q\le p} \mathcal{A}_q$ stays positive as the perturbation does not move the fixed points more than $\delta^*_{p+1}/2$. □

From now on, we assume that Φ satisfies Proposition 4.11.

4.4. *Pre-images of the boundary manifolds and how to avoid them.* Next, we need some notation and a few lemmata to describe the structure of the pre-images of the discontinuity manifolds conveniently. This allows us to develop the tools to prove Proposition 4.3.

Let f be a piecewise smooth contraction with $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$ and $\Phi_f = \{\phi_1, \phi_2, \ldots, \phi_m\}$. Recall that, by hypothesis, $\partial \boldsymbol{P}(f)$ is contained in the finite union of $\mathcal{C}^2$ manifolds, which we will call *boundary manifolds*. Let l_0 be the number of boundary manifolds in $\partial \boldsymbol{P}(f)$. Recall also that, for every $i \in \{1, 2, \ldots, m\}$, U_i is the open neighbourhood of $P_i \in \boldsymbol{P}(f)$ such that $\tilde{f}|_{U_i}$ is injective and hence invertible. Accordingly, by the construction of Φ_f, we have that $\phi_i|_{U_i}$ has a well-defined inverse for all $i \in \{1, 2, \ldots, m\}$.

Let $\epsilon_0 = \min\{d_H(P_i, U_i^c) : i \in \{1, \ldots, m\}\}$, where the complement is taken in $\mathbb{R}^d$. For each $\epsilon \le \epsilon_0/2$, we can consider the ϵ-neighbourhood V_i of P_i and the $\epsilon/2$ neighbourhood V_i^-.

Choosing ϵ small enough, we can describe the boundary manifolds by embeddings $\psi_i \in \mathcal{C}^2(D_i^+, \mathbb{R}^d)$, $i \in \{1, 2, \ldots, l_0\}$, such that $\overline{\psi_i(D_i^+)} \subset U_p$ for some $p \in \{1, \ldots, m\}$, and there exists an open set $D_i \subset D_i^+ \subset \mathbb{R}^{d-1}$ such that $\psi_i(D_i) \cap V_p \ne \emptyset$ and $\partial\psi_i(D_i) \cap \overline{V}_p = \emptyset$ (this is possible by Definition 2.16). For each IFS Φ and $\sigma \in \Sigma_n^m$, recalling the definition (4.2) of Y, we let

$$\begin{aligned} D_\sigma(\Phi) &= \{x \in Y \ : \ x \in V^-_{\sigma_n}, \phi_{\sigma_{k+1}} \circ \cdots \circ \phi_{\sigma_n}(x) \in V^-_{\sigma_k}, k \in \{1, \ldots, n-1\}\}, \\ D^+_\sigma(\Phi) &= \{x \in Y : x \in V_{\sigma_n}, \phi_{\sigma_{k+1}} \circ \cdots \circ \phi_{\sigma_n}(x) \in V_{\sigma_k}, k \in \{1, \ldots, n-1\}\}. \end{aligned} \tag{4.10}$$

We call a sequence σ *admissible* if $D_\sigma(\Phi) \ne \emptyset$. We define the set $\Sigma^m_{n,i}$ of the i-admissible sequences as

$$\begin{aligned} \Sigma^m_{n,i}(\Phi) &= \{\sigma \in \Sigma^m_n \ : \ \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_n}(D_\sigma(\Phi)) \cap \psi_i(D_i) \ne \emptyset\}, \\ D_{\sigma,i}(\Phi) &= \{x \in D_\sigma(\Phi) \ : \ \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_n}(x) \in \psi_i(D_i)\}. \end{aligned} \tag{4.11}$$

Remark 4.13. Note that, for each $N \in \mathbb{N}$, there is a $\delta > 0$ such that, for each $i \in \{1, \ldots, l_0\}$, $n \le N$, admissible word $\sigma \in \Sigma^m_{n,i}$, point $x \in D^+_\sigma(\Phi_f)$ and small enough perturbations $\tilde{\Phi} = \{\tilde{\phi}_i\}$ of Φ, we have, for each $j \le n$, $\tilde{\phi}_{\sigma_j} \circ \cdots \circ \tilde{\phi}_{\sigma_n}(B_\delta(x)) \subset \tilde{\phi}_{\sigma_j}(U_{\sigma_j})$, so that the inverse function $\tilde{\phi}^{-1}_{\sigma_n} \circ \cdots \circ \tilde{\phi}^{-1}_{\sigma_1}$ is well defined on $\tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_n}(B_\delta(x))$.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Remark 4.14. By the definition (2.1) of the partition $\boldsymbol{P}(f^n)$, it follows that

$$\partial \boldsymbol{P}(f^n) \subset \bigcup_{i=1}^{l_0} \bigcup_{r=0}^{n} \bigcup_{\sigma\in\Sigma^m_{r,i}} D_{\sigma,i}(\Phi_f),$$

where $\Sigma^m_{0,i}(\Phi) = \{0\}$ and $\phi_0 = \mathrm{id}$, so, for $\sigma \in \Sigma^m_{0,i}(\Phi)$, we have $D_{\sigma,i}(\Phi_f) = \psi_i(D_i)$.

Unfortunately, the sets $D_{\sigma,i}(\Phi_f)$ may have a rather complex topological structure, whereas we would like to cover $\partial\boldsymbol{P}(f^n)$ with a finite set of $(d-1)$-dimensional manifolds described by a single chart. This is our next task.

In the following, we will write $\Sigma^m_{n,i}$ only if it does not create confusion. In addition, we set

$$\Sigma^m_{n,*} = \bigcup_{i=1}^{l_0} \Sigma^m_{n,i}. \tag{4.12}$$

Note that, if $\sigma \in \Sigma^m_{n,*}$ and $x \in D_{\sigma,i}(\Phi)$, then there exists a unique $y \in D_i$ such that $\phi_\sigma(x) := \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_n}(x) = \psi_i(y)$, so the following is well defined: $\psi_i^{-1} \circ \phi_\sigma(D_{\sigma,i}(\Phi)) = W_{\sigma,i}$. Note that $W_{\sigma,i}$ is a compact set. For all $y \in D_i$, let $A(y)$ be the set of $\sigma \in \Sigma^m_{p,i}$, $p \le n$, such that $y \in W_{\sigma,i}$. As noted in Remark 4.13, there exists a $\delta(y) > 0$ such that $\psi_i(B_{\delta(y)}(y)) \subset \bigcap_{\sigma\in A(y)} \phi_\sigma(D_{\sigma,i}(\Phi))$. For a fixed $N \in \mathbb{N}$, we have that $\{B_{\delta(y)}(y) \ : \ y \in \overline{D_i}\}$ is a Besicovitch cover and, by the Besicovitch covering theorem, we can obtain a subcover in which each point can belong to at most c_d balls (for some c_d depending only on the dimension d). We can then extract a finite subcover

$$\mathcal{W}_i^N = \{B_{\delta(y_k)}(y_k)\} \tag{4.13}$$

of $\overline{D_i}$. We set, for $\sigma \in \Sigma^m_{p,i}$ with $p \in \{0, \ldots, N\}$, $M^N_{\sigma,i}(\Phi) = \{\phi^{-1}_{\sigma_p} \circ \cdots \circ \phi^{-1}_{\sigma_1} \circ \psi_i(B_{\delta(y_k)}(y_k))\}$, $M^N_{0,i} = \{\psi_i(D_i)\}$, which is the wanted collection of $(d-1)$-dimensional manifolds. Note that they are not necessarily disjoint. However, they have the wanted property, as the following remark states.

Remark 4.15. By the definition given by equation (2.1) of the partition $\boldsymbol{P}(f^n)$, it follows that, for each $N \ge n$,

$$\partial \boldsymbol{P}(f^n) \subset \bigcup_{i=1}^{l_0} \bigcup_{r=0}^{n} \bigcup_{\sigma\in\Sigma^m_{r,i}} \bigcup_{M\in M^N_{\sigma,i}(\Phi_f)} M,$$

where $\Sigma^m_{0,i} = \{0\}$ and $\phi_0 = \mathrm{id}$, so, for $\sigma \in \Sigma^m_{0,i}$, we have $M_{\sigma,i}(\Phi_f) = \psi_i(D_i)$.

Also, for all IFS Φ and $N \in \mathbb{N} \cup \{0\}$, we define

$$D^N(\Phi) = \bigcup_{n=0}^{N} \bigcup_{i=1}^{l_0} \bigcup_{\sigma\in\Sigma^m_{n,i}} \bigcup_{M\in M^N_{\sigma,i}(\Phi)} M. \tag{4.14}$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

In addition, for $\delta > 0$, we define the closure of the δ-neighbourhood of $D^N(\Phi_f)$ as

$$D_\delta^N(\Phi) = \bigcup_{x \in D^N(\Phi_f)} \overline{B_\delta(x)}. \tag{4.15}$$

Remark 4.16. The basic idea of the proof is to make a perturbation such that the images of $\partial \boldsymbol{P}(f^n)$ do not self-intersect too many times. This can be done easily for a single map ϕ_i. However, we are dealing with compositions in which the same map can appear many times. So we have to avoid the possibility that the perturbation at one time interferes with itself at a later time. This can be achieved if there are no fixed points close to the pre-images of the singularities. This is our next task.

PROPOSITION 4.17. *Let $\Phi = \{\phi_1, \phi_2, \ldots, \phi_m\}$ be an IFS with contraction coefficient λ. For each $N \in \mathbb{N}$ and $\varepsilon > 0$ small enough, there exists an IFS $\tilde{\Phi} = \{\tilde{\phi}_1, \ldots, \tilde{\phi}_m\}$ and $\delta_\varepsilon \in (0, \varepsilon)$ such that $\|\phi_i - \tilde{\phi}_i\|_{\mathcal{C}^2} \leq \varepsilon$, and, for any $p \leq N$, $\sigma = (\sigma_1, \ldots, \sigma_p) \in \Sigma_{p,*}^m(\Phi)$, we have that $\tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_p}|_{D_\sigma(\tilde{\Phi})}$ is invertible. Moreover, $x \in D_{\delta_\varepsilon}^N(\tilde{\Phi})$ implies that $\tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_p}(x) \neq x$. Finally, there exists $c_* > 2$ such that, for any $\delta \in (0, c_*^{-1}\delta_\varepsilon/2)$ and $y \in D_{\delta_\varepsilon/2}^N(\tilde{\Phi})$,*

$$\tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_p}(B_\delta(y)) \cap B_\delta(y) = \emptyset.$$

Proof. The last statement of the proposition is an immediate consequence of the first part and Lemma 4.10. As for the first part, note that if, for each $\sigma \in \Sigma_{p,*}^m$, $p \leq N$, the fixed points of $\phi_\sigma := \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}$ do not belong to $D^N(\tilde{\Phi})$, then the proposition holds with δ_ε small enough. Thus, it suffices to prove the latter fact.

By Remark 4.13, it follows that there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$ and ε-perturbation $\tilde{\Phi}$ of Φ, for all $\sigma \in \Sigma_{p,*}^m(\Phi)$, $p \leq N$, the inverse map of ϕ_σ is well defined in $D_\sigma(\tilde{\Phi})$. In addition, by Lemma C.3, $\Sigma_{p,*}^m(\Phi) = \sigma_{p,*}^m(\tilde{\Phi})$. From now on we assume that $\varepsilon \leq \varepsilon_0$.

We can then apply Proposition 4.11 to obtain the IFS Φ^0, a $\varepsilon/4$ perturbation of ϕ, where the fixed points differ unless they are associated with sequences composed by the repetition of the same word. Next, we want to proceed by induction on the sequences in $\Sigma_{N,*}^m(\Phi)$.

To this end, it is necessary to have an order structure on $\Sigma_{N,*}^m(\Phi)$. We introduce the following order: $0 \prec \sigma$ for all $\sigma \neq 0$, and if $p > q$ and $\sigma \in \Sigma_{p,*}^m(\Phi)$, $\sigma' \in \Sigma_{q,*}^m(\Phi)$, then $\sigma' \preceq \sigma$. If $p = q$, then the σ are ordered lexicographically. This is a total ordering, and hence we can arrange them as sequences $\{\sigma^i\}_{i \in \mathbb{N}}$ with $\sigma^j \prec \sigma^i$ if and only if $i > j$. Next, define $\ell(j)$ to be the length of the word σ^j: that is $\sigma^j \in \Sigma_{\ell(j),*}^m$. Recall the definition of fixed points (4.8). It is convenient to set $\sigma^0 = 0$ and $x_0(\phi) = \emptyset$. Also, let $\Lambda_0 = \max\{\|(D_x\phi_i)^{-1}\|_{\mathcal{C}^0(V_i)} \ : \ \phi_i \in \Phi^0\}$.

The idea is to define a sequence of perturbations Φ^k, $\|\Phi^{k+1} - \Phi^k\|_{\mathcal{C}^2} = \varepsilon_k \leq \varepsilon 2^{-k-1}$, such that, for all $j \leq k$,

$$x_{\sigma^j}(\Phi^k) \notin D^N(\Phi^k). \tag{4.16}$$

Note that the above implies that there exists $\Lambda > 2$ such that

$$\Lambda \geq \max\{\|(D_x\phi_i)^{-1}\|_{\mathcal{C}^0(U_i)} \ : \ \phi_i \in \Phi^k\}$$

for all $k \in \mathbb{N}$. In particular, the above implies that

$$D^N(\Phi^{k+1}) \subset D^N_{2\Lambda^N \varepsilon_k}(\Phi^k). \tag{4.17}$$

Using the notation introduced just before Proposition 4.11, let

$$\bar{\delta}_k = \min\bigg\{1, \frac{1}{2c_*} \inf_{\substack{x,y \in A_N(\Phi^k) \\ x \neq y}} \|x - y\|\bigg\} > 0.$$

We proceed by induction on σ. For σ^0, the statement in the induction is trivially true. We assume that it is true for σ^k and we prove the statement for σ^{k+1}. Let

$$\delta_k = \min\{\min\{d_0(x_{\sigma^j}(\Phi^k), D^N(\Phi^k)) \ : \ j \leq k\}, \bar{\delta}_k/4\}.$$

We consider ε_k-perturbations of Φ^k with $\varepsilon_k \leq \Lambda^{-N}\delta_k/4$. For $\sigma \in \Sigma_p^m$, we use the notation $\phi_\sigma = \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_p}$. If $x_{\sigma^{k+1}}(\Phi^k) \notin D^N(\Phi^k)$, then we set $\Phi^{k+1} = \Phi^k$ and the induction step is satisfied. Otherwise, as before, for some $a > 2$, and any $\delta \in (0, 1/\sqrt{a})$, $v \in V$ and $\bar{x} \in \mathbb{R}^d$, we define $h_{\bar{x},\delta,v} : \mathbb{R}^d \to \mathbb{R}^d$ as

$$h_{\bar{x},\delta,v}(x) = \begin{cases} x & \text{for all } x \notin B_\delta(\bar{x}), \\ x + \delta^3 g(1 - \delta^{-1}\|x - \bar{x}\|)v & \text{otherwise,} \end{cases} \tag{4.18}$$

where $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ is a monotone function such that $g(y) = 0$ for all $y \leq 0$, $g(y) = 1$ for all $y \geq 1/2$ and $\|g'\|_\infty < a$. Let $p = \ell(\sigma^{k+1})$. For each $\delta > 0$ and $v \in \mathbb{R}^d$, $\|v\| \leq 1$, we consider the perturbations $\Phi_{\delta,v} = \{\phi_{i,\delta,v}\}$ defined by

$$\phi_{i,\delta,v}(x) = \begin{cases} \phi_i(x) & \text{if } i \neq \sigma_p^{k+1}, \\ \phi_{\sigma_p^{k+1}} \circ h_{x_{\sigma^{k+1}}(\Phi^k),\delta,v}(x) & \text{if } i = \sigma_p^{k+1}, \end{cases}$$

where $\Phi^k = \{\phi_i\}$. Note that $\|\phi_i - \phi_i \circ h_{\bar{x},\delta,v}\|_{\mathcal{C}^2} \leq C_g\delta$ and $\|\phi_i \circ h_{\bar{x},\delta,v}\|_{\mathcal{C}^3} \leq C_g$ for some constant $C_g > 1$. Thus, these are ε_k perturbations provided $\delta \leq C_g^{-1}\varepsilon_k \leq C_g^{-1}\Lambda^{-N}\delta_k/4$.

LEMMA 4.18. *There exist $C_* > 0$ and $\delta_* \in (0, \min\{C_g^{-1}\varepsilon_k, \Lambda^{-2N}\})$ such that, for all $\delta \leq \delta_*$, $\|v\| \leq 1$ and each $j \leq k+1$,*

$$\frac{3}{2}\delta^2\|v\|\Lambda^{-N} \leq \|x_{\sigma^j}(\Phi^k) - x_{\sigma^j}(\Phi_{\delta,v})\| \leq \frac{\delta}{2}.$$

Moreover, $\partial_v x_{\sigma^j}(\Phi_{\delta,v})$ is invertible and

$$\|(\partial_v x_{\sigma^j}(\Phi_{\delta,v}))^{-1}\| \leq C_*\delta^{-3}\Lambda^N.$$

Proof. Let $q = \ell(\sigma^j)$. If $\sigma_s^j \neq \sigma_p^{k+1}$, for all $s \leq q$, then $x_{\sigma^j}(\Phi^k) = x_{\sigma^j}(\Phi_{\delta,v})$. Otherwise, let $\bar{s}$ be the largest such that, for all $s \leq \bar{s}$, $\sigma_s^j \neq \sigma_p^{k+1}$,

$$\begin{aligned}&\phi_{\sigma_{\bar{s}}^j,\delta,v} \circ \cdots \circ \phi_{\sigma_q^j,\delta,v}(x_{\sigma^j}(\Phi_{\delta,v})) \\ &\quad = \phi_{\sigma_{\bar{s}}^j,\delta,v} \circ \cdots \circ \phi_{\sigma_q^j,\delta,v} \circ \phi_{\sigma_1^j,\delta,v} \circ \cdots \circ \phi_{\sigma_{\bar{s}-1}^j,\delta,v} \\ &\qquad \times (\phi_{\sigma_{\bar{s}}^j,\delta,v} \circ \cdots \circ \phi_{\sigma_q^j,\delta,v}(x_{\sigma^j}(\Phi_{\delta,v}))).\end{aligned}$$

Note that, $(\sigma_{\bar{s}}^j, \ldots, \sigma_q^j, \sigma_1^j, \ldots, \sigma_{\bar{s}-1}^j,) = \sigma^{j_1}$ for some $j_1 < k+1$. Moreover, $\sigma_q^j = \sigma_p^{k+1}$ and

$$x_{\sigma^{j_1}}(\Phi_{\delta,v}) = \phi_{\sigma_{\bar{s}}^j,\delta,v} \circ \cdots \circ \phi_{\sigma_q^j,\delta,v}(x_{\sigma^j}(\Phi_{\delta,v})).$$

By hypothesis,

$$\|x_{\sigma^j}(\Phi_{\delta,v}) - x_{\sigma^j}(\Phi^k)\| \leq \Lambda^p \|x_{\sigma^{j_1}}(\Phi_{\delta,v}) - x_{\sigma^{j_1}}(\Phi^k)\|.$$

We can thus consider only the case in which $\sigma_q^j = \sigma_p^{k+1}$. Let $1 \leq \bar{s} < q$ be the largest integer, if it exists, such that $\sigma_s^j = \sigma_p^{k+1}$ for all $s \leq \bar{s}$. Then for $y \in B_\delta(x_{\sigma^{k+1}}(\Phi^k))$,

$$x_{\sigma^j}(\Phi_{\delta,v}) = \phi_{\sigma_1^j,\delta,v} \circ \cdots \circ \phi_{\sigma_{\bar{s}}^j,\delta,v} \circ \phi_{\sigma_{\bar{s}+1}^j} \circ \cdots \circ \phi_{\sigma_q^j}(h_{x_{\sigma^{k+1}}(\Phi^k),\delta,v}(y)).$$

Since, by construction, $h_{x_{\sigma^{k+1}}}(B_\delta(x_{\sigma^{k+1}}(\Phi^k))) \subset B_\delta(x_{\sigma^{k+1}}(\Phi^k))$, we have that applying $\phi_{\sigma_{\bar{s}}^j,\delta,v}$ differs from applying $\phi_{\sigma_{\bar{s}}^j}$ only if

$$\phi_{\sigma_{\bar{s}+1}^j} \circ \cdots \circ \phi_{\sigma_q^j}(B_\delta(x_{\sigma^{k+1}}(\Phi^k))) \bigcap B_\delta(x_{\sigma^{k+1}}(\Phi^k)) \neq \emptyset.$$

But then Lemma 4.10 implies that

$$\|x_{(\sigma_{s+1}^j,\ldots,\sigma_q^j)}(\Phi^k) - x_{\sigma^{k+1}}(\Phi^k)\| \leq c_* \delta.$$

Then our choice of δ and the induction hypothesis implies that $x_{\sigma^{k+1}}(\Phi^k) = x_{(\sigma_{s+1}^j,\ldots,\sigma_q^j)}(\Phi^k) \notin D^N(\Phi^k)$, which is contrary to our current assumption. It follows that, provided $x_{\sigma^j}(\Phi_{\delta,v}) \in B_\delta(x_{\sigma^{k+1}}(\Phi^k))$,

$$x_{\sigma^j}(\Phi_{\delta,v}) = \phi_{\sigma^j}(h_{x_{\sigma^{k+1}}(\Phi^k),\delta,v}(x_{\sigma^j}(\Phi_{\delta,v}))).$$

To simplify notation, let $z(\delta, v) = x_{\sigma^j}(\Phi_{\delta,v})$, $h_{\delta,v} = h_{x_{\sigma^{k+1}}(\Phi^k),\delta,v}$ and $\phi_{\delta,v} = \phi_{\sigma^j} \circ h_{\delta,v}$. We can study $z(\delta, v)$ by applying the implicit function theory, which yields

$$\frac{d}{d\delta} z(\delta, v) = (\mathbb{1} - D_{z(\delta,v)}\phi_{\delta,v})^{-1} D_{h_{\delta,v}(z(\delta,v))}\phi_{\delta,v}\partial_\delta h_{\delta,v}(z(\delta, v)).$$

If $z(\delta, v) \in B_{\delta/2}(x_{\sigma^{k+1}}(\Phi^k))$, then $h_{\delta,v}(z(\delta, v)) = z(\delta, v) + \delta^3 v$, $D_{z(\delta,v)}h_{\delta,v} = \mathbb{1}$ and $\partial_\delta h_{\delta,v}(z(\delta, v)) = 3\delta^2 v$. Thus, setting $A := D_{z(\delta,v)+\delta^3 v}\phi_{\delta,v}$ we obtain

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

$$\frac{d}{d\delta}z(\delta, v) = 3\delta^2(\mathbb{1} - A)^{-1}Av.$$

Since the maximal eigenvalue of $(\mathbb{1} - A)^{-1}A$ is bounded by $(1 - \lambda)^{-1}\lambda$, there exists a δ_* such that, for all $\delta \leq \delta_*$, $z(\delta, v) \in B_{\delta/2}(x_{\sigma^{k+1}}(\Phi^k))$. Moreover,

$$\|(\mathbb{1} - A)^{-1}Av\| \geq \Lambda^{-N}/2\|v\|,$$

and thus $z(\delta, v) \notin B_{(3/2)\delta^2\|v\|\Lambda^{-N}}(x_{\sigma^{k+1}}(\Phi^k))$. Finally, for each $\delta \leq \delta_0$,

$$\partial_v z(\delta, v) = \delta^3(\mathbb{1} - A)^{-1}A,$$

from which the last statement of the lemma follows. □

By Lemma 4.18, equation (4.17) and our choice of ε_k, $x_{\sigma^j}(\Phi_{\delta,v}) \notin D^N_{3\delta_k/4}(\Phi^k)$ and $D^N(\Phi^{k+1}) \subset D^N_{\delta_k/4}(\Phi^k)$ for all $j \leq k$. Thus, $x_{\sigma^j}(\Phi_{\delta,v}) \notin D^N(\Phi^{k+1})$ for all $j \leq k$. We are left with $x_{\sigma^{k+1}}(\Phi^{k+1})$, recalling that $x_{\sigma^{k+1}}(\Phi^k) \in D^N(\Phi^k)$. Let $\omega \in \Sigma^m_{q,i}$, $q \leq N$ and $M \in M^N_{\omega,i}(\Phi^k)$ such that $x_{\sigma^{k+1}}(\Phi^k) \in M$. Then

$$\phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q}(x_{\sigma^{k+1}}(\Phi^k)) \in \overline{\psi_i(D_i)}.$$

First, suppose that $\omega_q \neq \sigma^{k+1}_p$. It follows that, for $y \in B_{\delta/2}(x_{\sigma^{k+1}}(\Phi^k))$,

$$\phi_{\omega_1,\delta,v} \circ \cdots \circ \phi_{\omega_q,\delta,v}(y) \neq \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q}(y)$$

only if, for some $s < q$, $\omega_s = \sigma^{k+1}_p$ and

$$\phi_{\omega_{s+1}} \circ \cdots \circ \phi_{\omega_q}(B_\delta(x_{\sigma^{k+1}}(\Phi^k))) \cap B_\delta(x_{\sigma^{k+1}}(\Phi^k)) \neq \emptyset, \tag{4.19}$$

but this is ruled out by our choice of δ_k and Proposition 4.11. The above discussion shows that $M \cap B_{\delta/2}(x_{\sigma^{k+1}}(\Phi^k))$ is an element of $M^N_{\omega,i}(\Phi_{\delta,v})$ as well. Hence, it suffices to ensure that $x_{\sigma^{k+1}}(\Phi_{\delta,v}) \notin M$. Since Lemma 4.18 shows that varying v the fixed point visits an open ball, and since M has zero measure, it follows that there exists an open set of v which yields the wanted property.

It remains to analyse the case $\omega_q = \sigma^{k+1}_p$. In this case, for $y \in B_{\delta/4}(x_{\sigma^{k+1}}(\Phi^k))$,

$$\phi_{\omega_1,\delta,v} \circ \cdots \circ \phi_{\omega_q,\delta,v}(y) = \phi_{\omega_1,\delta,v} \circ \cdots \circ \phi_{\omega_q}(y + \delta^3 v) \neq \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_q}(y + \delta^3 v)$$

only if equation (4.19) is satisfied, which, by our choice of δ_k, is possible only if $x_\omega(\Phi^k) = x_\sigma(\Phi^k)$. But then Proposition 4.11 implies that there exist $\sigma^r \preceq \sigma^{k+1}$ such that $\phi_\omega = \phi^{m_1}_{\sigma^r}$ and $\phi_\sigma = \phi^{m_2}_{\sigma^r}$, which would mean that

$$\overline{\psi_i(D_i)} \ni \phi^{m_1}_{\sigma^r}(x_{\sigma^{k+1}}(\Phi^k)) = x_{\sigma^{k+1}}(\Phi^k).$$

Again, Lemma 4.18 allows to find an open set of v for which $x_{\sigma^{k+1}}(\Phi^k) \notin \overline{\psi_i(D_i)}$. The last possibility is that

$$\phi_{\omega_1,\delta,v} \circ \cdots \circ \phi_{\omega_q,\delta,v}(y) = \phi_{\omega_1,\delta,v} \circ \cdots \circ \phi_{\omega_q}(y + \delta^3 v).$$

This implies that the perturbed manifold M is displaced by, at most, $2\Lambda^N\delta^3\|v\|$ whereas Lemma 4.18 implies that the fixed point moves by at least $(3/2)\delta^2\|v\|\Lambda^{-N} \geq 2\Lambda^N\delta^3\|v\|$.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Hence, again we have an open set of v, which produces perturbations with the wanted property. As a last observation, note that if there are other manifolds $M \in D^N(\Phi^k)$ such that $x_{\sigma^{k+1}}(\Phi^k) \in M$, then we can repeat the same argument and we have just a smaller open set of v that does the job. This concludes the overall induction and hence the proof of Proposition 4.17. □

4.5. *Perturbations with low complexity and the proof of Proposition 4.3.* Thanks to Proposition 4.17, we can finally construct the wanted perturbation $\tilde{f}$.

Let f be a piecewise smooth contraction with the maximal partition $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$. Let $l_0 \in \mathbb{N}$ be the number of manifolds in $\partial \boldsymbol{P}(f)$. Define $l_1 = \max\{c_d l_0, d\}$. (Recall that, by construction, c_d is the maximal number of manifolds in $M^N_{\sigma,i}$, $N \in \mathbb{N}$ and $\sigma \in \Sigma^m_{0,i}$, that can contain a point in $\psi_i(D_i^+)$.)

Given two manifolds defined by maps ψ_1, ψ_2, we write $\psi_1 \pitchfork \psi_2$ if the manifolds are transversal (see Definition B.1 for the definition of transversality). On the contrary, if the two manifolds have an open (in the relative topology) intersection, we call them *compatible* and write $\psi_1 \curlywedge \psi_2$. If two manifolds are not compatible, then we write $\psi_1 \not\curlywedge \psi_2$.

PROPOSITION 4.19. *Let $f : X \to X$ be a piecewise smooth contraction with maximal partition $\boldsymbol{P}(f)$. Then, for any $N \in \mathbb{N}$ and $\varepsilon > 0$ small enough, there exists a piecewise smooth contraction $\tilde{f}$, with $d_2(f, \tilde{f}) < \varepsilon$ such that no more than $2^{dm^{d-1}l_1}$ partition elements of $\boldsymbol{P}(\tilde{f}^N)$ can meet at one point.*

Proof. Before starting the proof, we need to introduce some language.

Consider an IFS $\Phi_f = \{\phi_1, \phi_2, \ldots, \phi_m\}$ associated to f with contraction coefficient λ. Let $\varepsilon_0 \in (0, \varepsilon/4)$ be small enough. Then, by Proposition 4.17, there exist $\delta_* \leq \delta_\varepsilon \in (0, \varepsilon_1)$ and an ε_0-perturbation (here, and in the following, by *perturbation* we mean a function that is $\mathcal{C}^2$ close and with a uniformly bounded $\mathcal{C}^3$ norm) of Φ_f (which, abusing the notation, we still call Φ_f) such that, for every $p \leq N$, $\sigma = (\sigma_1, \ldots, \sigma_p) \in \Sigma^m_{p,*}$ and $\xi \in D^N_{\delta_*}(\Phi_f) \cap V_p$,

$$\phi_{\sigma_1} \circ \phi_{\sigma_2} \circ \cdots \circ \phi_{\sigma_p}(B_{\delta_*}(\xi)) \cap B_{\delta_*}(\xi) = \emptyset. \tag{4.20}$$

Note that there exists $\epsilon_0 \leq \varepsilon_0$ such that (4.20) persists for ϵ_0-perturbations of Φ.

By compactness, for $\delta \in (0, \min\{\delta_*/2, \delta_N\})$, where $\delta_N > 0$ is such that it satisfies the condition of Remark 4.13 for each ϵ_0-perturbation, there exists a finite open cover $\{B_{\delta/2}(z_i)\}_{i=1}^t$ of $\overline{D^N_\delta(\Phi_f)}$ (which, by definition, contains $\partial \boldsymbol{P}(f^N)$) such that, for each i, $z_i \in P_j$ and $B_\delta(z_i) \subset V_j$, for some j. (See the discussion at the beginning of §4.4 for the definition of V_j.)

Let $Y = \overline{Y} \subset \mathbb{R}^d$ be compact, such that $\overline{\phi(Y)} \subset Y$ for all $\phi \in \Phi$.

By convention, we set $\mathcal{Z}^1_0(\Phi) = \{Y\}$ and $\mathcal{Z}^1_1(\Phi) := \{\psi_\omega\ ;\ \omega \in \{1, \ldots, l_*\}\}$ to be the collection of the manifolds $\psi_i(A)$ for $A \in \mathcal{W}^N_i$, as defined in (4.13). Also, we call $\mathcal{Z}^1_k(\Phi) := \{\psi^k_\omega\ :\ \omega \in \{1, \ldots, l_*\}^k\}$ the manifolds consisting of the intersection of the k manifolds $\{\psi_{\omega_i}\ :\ i \in \{1, \ldots, k\}; i \neq j \Longrightarrow \psi_{\omega_i} \not\curlywedge \psi_{\omega_j}\}$. (These are indeed manifolds; see Definition 2.16. To simplify notation, we use ψ_ω both for the manifold and for the map that defines it.) By construction, $\mathcal{Z}^1_k(\Phi) = \emptyset$ for $k > l_0$. In addition, the maximal dimension of the manifolds in $\mathcal{Z}_k^1(\Phi)$, for $k > 1$, is $d-2$ (since the boundary manifolds are pairwise transversal; see Definition 2.16). Note that $\mathcal{Z}_1^1(\Phi_f)$ is a collection that covers the boundary manifolds; for simplicity, we call the elements of $\mathcal{Z}_1^1(\Phi_f)$, from now on, original boundary manifolds. For each $s \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$, let $\mathcal{Z}_0^s = \{Y\}$, $U_0 = Y$ and (to alleviate notation, from now on we write $\phi_{\sigma_i}^{-1}$ to mean the inverse of $\phi_{\sigma_i}|_{U_{\sigma_i}}$, while the domain of $\phi_{\sigma_i}^{-1} \circ \psi$ consists of the points where the composition is well defined)

$$\begin{aligned}\mathcal{Z}_k^{s+1}(\Phi) = \bigg\{ &\phi_{\sigma_1}^{-1} \circ \psi_{\omega_1} \cap \cdots \cap \phi_{\sigma_n}^{-1} \circ \psi_{\omega_n} \;:\; n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}, \sum_{i=1}^n k_i = k, \\ &\psi_{\omega_i} \in \mathcal{Z}_{k_i}^s(\Phi), \sigma_i \in \Sigma_{1,*}^m(\Phi); i \neq j, \sigma_i = \sigma_j \Longrightarrow \psi_{\omega_i} \not\pitchfork \psi_{\omega_j} \bigg\}, \\ \mathcal{Z}_*^s(\Phi) = &\bigcup_{k \in \mathbb{N} \cup \{0\}} \mathcal{Z}_k^s(\Phi).\end{aligned}$$

Note that $\mathcal{Z}_*^s$ contains the admissible pre-images of the boundary manifolds under composition of at most s maps in Φ and all their intersections. In particular, the sets in $\mathcal{Z}_1^s$ cover $\partial\mathcal{P}(f^s)$, and we call them boundary manifolds. In addition, if a set belongs to $\mathcal{Z}_k^s(\Phi)$, then, by definition, it is determined by the intersection of the pre-images of k original boundary manifolds. Also, we remark that $\mathcal{Z}_k^{s+1} \supset \mathcal{Z}_k^s$ since $\phi_0 = \mathrm{id}$ (see Remark 4.14 and Definition (4.12)). Next, let N_s be the maximal number of manifolds that can intersect in $\mathcal{Z}_*^s$: that is, $\mathcal{Z}_k^s = \emptyset$ for $k > N_s$. We have seen that $N_1 \leq l_1$. Moreover, each original boundary manifold can have at most m^s different pre-images obtained by the compositions of s maps. This implies that, at each point, we have at our disposal at most $\sum_{s'=0}^{s-1} m^{s'} l_1$ different manifolds from $\bigcup_i M_{0,i}^N$ to intersect. If $m = 1$, then $N_s \leq s l_1$; if $m \geq 2$, then $N_s \leq (m^s - 1)/(m-1) l_1 < 2m^{s-1} l_1$. Accordingly, $N_s \leq s m^{s-1} l_1$. (We remark that, by definition, the pre-images are taken via invertible maps, and hence the manifolds cannot self-intersect.)

Our goal is to produce a sequence of perturbations Φ^s of $\Phi_f =: \Phi^0$ such that Φ^s is a $2^{-s}\varepsilon$ perturbation of Φ^{s-1} with the following property.

($\star$) *The set $\mathcal{Z}_k^s(\Phi^s)$ consists of manifolds of dimension strictly smaller than $d - j$ for* all $k > j m^{j-1} l_1$, *whereas $\mathcal{Z}_1^s(\Phi^s)$ consists of $d-1$ dimensional manifolds. This property persists for small perturbations of Φ^s.*

Note that the above implies that $\mathcal{Z}_k^s(\Phi^s) = \emptyset$ for each $s \in \mathbb{N}$ and $k > d m^{d-1} l_1$. Accordingly, at most $d m^{d-1} l_1$ pre-images of the original boundary manifolds under composition of at most s elements of Φ^s can have non-empty intersections. In turn, defining $f_s(x) = \phi_i(x)$ for $x \in P_i$ and $\phi_i \in \Phi^s$, we obtain a perturbation of f smaller than $\sum_{j=1}^s 2^{-j}\varepsilon \leq \varepsilon$ such that $\mathcal{P}(f_s^s)$ has at most $2^{d m^{d-1} l_1}$ elements meeting at point.

Indeed, suppose p elements of $\mathcal{P}(f_s^s)$ meet at a point x. The boundaries of such elements in a neighbourhood small enough of x consist of codimension one manifolds belonging to $\mathcal{Z}_1^s(\Phi^s)$, and they have to intersect at x. Suppose that the total number of such boundary manifolds is q. Then x must belong to a manifold in $\mathcal{Z}_q^s(\Phi^s)$, and hence it must be $q \le dm^{d-1}l_1$. Note that we can uniquely define a partition element by specifying on which side it lies with respect to all its boundary manifolds. Since there are at most 2^q possibilities, it must be $p \le 2^q$. It follows that $p \le 2^{dm^{d-1}l_1}$. The lemma then follows by choosing $s = N$.

It remains to prove property ($\star$). We proceed by induction. If $s = 1$ and $k \in \{1, \ldots, l_1\}$, then the manifolds in $\mathcal{Z}_k^1$ are indeed of codimension at least one, and the manifolds in $\mathcal{Z}_1^1$ are of codimension one, whereas, if $k > l_1$, then $\mathcal{Z}_k^1 = \emptyset$, so $\Phi_f = \Phi^0$ satisfies our hypothesis. We assume that the hypothesis is verified for some s and prove it for $s + 1$.

Let $\epsilon_s \le \min\{2^{-s-1}\varepsilon, \epsilon_0\}$ be such that all the ϵ_s-perturbations of Φ^s still satisfy ($\star$). This implies that, provided Φ^{s+1} is a ϵ_s perturbation of Φ^s, $\mathcal{Z}_*^{s'}(\Phi^{s+1})$ has the wanted property for all $s' \le s$.

Accordingly, we must analyse only sets of the type $\phi_{\sigma_1}^{-1} \circ \psi_1 \cap \cdots \cap \phi_{\sigma_n}^{-1} \circ \psi_n$, where $\psi_i \in \mathcal{Z}_{k_i}^s(\Phi^s)$ and $\phi_{\sigma_i} \in \Phi^s \cup \{\mathrm{id}\}$, $\sigma_i \in \{0, \ldots, m\}$. (By an innocuous abuse of notation here we use ψ_i to refer to generic elements.) By definition, such sets are elements of $\mathcal{Z}_k^{s+1}(\Phi^s)$, with $k = \sum_{i=1}^n k_i$. Note that the $\psi_i \in \mathcal{Z}_0^s(\Phi^s)$ do not contribute to the intersection. We can thus assume, without loss of generality, that $k_i > 0$. Note that if $n = 1$, then the manifolds belong to $\bigcup_{i=0}^m \phi_i^{-1}(\mathcal{Z}_*^s(\Phi^s)) \subset \mathcal{Z}_*^{s+1}(\Phi^s)$ which have automatically the wanted property, and so has any ϵ_s-perturbation. We consider thus only the case $n \ge 2$. In addition, if $\phi_{\sigma_i} = \phi_{\sigma_j}$, $i \ne j$, then $\phi_{\sigma_i}^{-1} \circ \psi_i \cap \phi_{\sigma_j}^{-1} \circ \psi_j = \phi_{\sigma_i}^{-1} \circ (\psi_j \cap \psi_j)$, and since, by definition, $\psi_i \not\pitchfork \psi_j$, $\psi_j \cap \psi_j \in \mathcal{Z}_{k_i+k_j}^s(\Phi^s)$. Hence, we can substitute to the intersection of the manifolds $\phi_{\sigma_i}^{-1} \circ \psi_i \cap \phi_{\sigma_j}^{-1} \circ \psi_j$ the manifold $\phi_{\sigma_i}^{-1} \circ (\psi_j \cap \psi_j)$. We can thus assume, without loss of generality, that $i \ne j$ implies that $\phi_{\sigma_i} \ne \phi_{\sigma_j}$.

We define the map $F : \mathbb{R}^d \to \mathbb{R}^{nd}$ by $F(x) := (\phi_{\sigma_1}(x), \ldots, \phi_{\sigma_n}(x))$ and the stratified sub-variety $C = \{(\psi_1(x_0), \ldots \psi_n(x_n)) \ : \ x_i \in \overline{D}_i\}$, where $D_i \subset \mathbb{R}^{d_i}$ is the domain of the map ψ_i. By Lemma B.4, for a constant c to be chosen later, there exists a $\frac{1}{2}c\epsilon_s$-perturbation $\hat{F} = (\hat{\phi}_{\sigma_1}(x), \ldots, \hat{\phi}_{\sigma_n}(x))$ of F, transversal to C. If $\phi_{\sigma_i} \ne \mathrm{id}$ for all i, then we set $\tilde{F} = \hat{F}$. If, for some i, $\phi_{\sigma_i} = \mathrm{id}$, then $\hat{\phi}_{\sigma_i}$ is a small perturbation of identity, and hence it is invertible with $\mathcal{C}^3$ inverse. (Indeed, if $\|h - \mathrm{id}\|_{\mathcal{C}^1} = \alpha < 1$, then h is a diffeomorphism. In fact, if $h(x) = h(y)$,

$$0 = \int_0^1 \frac{d}{dt} h(ty + (1-t)x)\, dt = y - x + \int (Dh - \mathbb{1})(x - y)\, dt,$$

which implies that $\|x - y\| \le \alpha \|x - y\|$: that is $x = y$. Thus, h is globally invertible, and the claim follows by the inverse function theorem.) By possibly relabelling, we can assume that $i = 1$. Then we define

$$\tilde{F}(x) = (\tilde{\phi}_{\sigma_1}, \ldots, \tilde{\phi}_{\sigma_n}) = (x, \hat{\phi}_{\sigma_2} \circ \hat{\phi}_{\sigma_1}^{-1}, \ldots, \hat{\phi}_{\sigma_n} \circ \hat{\phi}_{\sigma_1}^{-1}).$$

$\tilde{F}$ is still transversal to C and, if c is small enough, by Lemmata C.1 and C.2, it is a $c\epsilon_s$-perturbation of F. Let $d_i = d - j_i$ be the dimension of the manifold ψ_i. Then

$k_i \le j_i m^{j_i-1} l_1$. Lemma B.4 implies that the sets $\bigcap_{i=1}^n \phi_{\sigma_i}^{-1} \circ \psi_i$ are manifolds with dimension (actually, they are stratified sub-varieties, but we can restrict them to manifolds without loss of generality)

$$\sum_{i=1}^n (d - j_i) - (n-1)d = d - \sum_{i=1}^n j_i =: d - \bar{j} \le d - \max\{j_i\} - 1.$$

Note that

$$k := \sum_{i=1}^n k_i \le \sum_{i=1}^n j_i m^{j_i-1} l_1 \le \sum_{i=1}^n j_i m^{\bar{j}-1} l_1 = \bar{j} m^{\bar{j}-1} l_1.$$

Accordingly, if $k > jm^{j-1} l_1$, then $\bar{j} \ge j+1$ and the manifold has a dimension strictly smaller than $d - j$, as required. (Of course, if $k > dm^{d-1} l_1$, then the intersection is empty.)

We would then like to define a perturbed IFS $\tilde{\Phi}^s$ as

$$\tilde{\phi}_k = \begin{cases} \tilde{\phi}_{\sigma_i} & \text{if } k = \sigma_i, \\ \phi_k & \text{otherwise.} \end{cases}$$

Unfortunately, this may perturb the new manifolds ψ_i as well, since they are now defined via pre-images of $\tilde{\Phi}^s$. This is the last problem we need to take care of. To this end, we make the perturbation only locally starting from the ball $B_\delta(z_1)$. Once we check that the perturbation is as required in this ball, we will consider the other balls, making the new perturbations small enough not to upset the property obtained in $B_{\delta/2}(z_1)$.

Let $g \in \mathcal{C}^\infty(\mathbb{R}^d, [0, 1])$ be such that

$$g(z) = \begin{cases} 1, & z \in B_{\delta/2}(z_1), \\ 0, & z \notin B_{3\delta/4}(z_1), \end{cases}$$

and $\|g\|_{\mathcal{C}^r} \le C\delta^{-r}$ for $r \in \{0, 1, 2, 3\}$ for some $C > 0$. Define, for each $i \in \{1, \ldots, m\}$,

$$\phi_{i,1}(x) = \phi_i(x) + g(x)(\widetilde{\phi}_i(x) - \phi_i(x)).$$

Provided we choose c small enough,

$$\|\phi_{i,1} - \phi_i\|_{\mathcal{C}^2} = \|g\|_{\mathcal{C}^2} \|\widetilde{\phi}_i - \phi_i\|_{\mathcal{C}^2} < C\delta^{-2} C_\# c \epsilon_s < \epsilon_s/4.$$

Then we define the perturbation $\Phi^{s,1} = \{\phi_{i,1}\}$. Note that the $\Phi^{s,1}$ equals Φ^s outside the ball $B_\delta(z_1)$ and agrees with $\tilde{\Phi}^s$ inside the ball $B_{\delta/2}(z_1)$.

Recall that we perturbed the system in order to control the intersection of the manifolds $\phi_{\sigma_1}^{-1} \circ \psi_1 \cap \cdots \cap \phi_{\sigma_n}^{-1} \circ \psi_n$. By construction, each ψ_i is the intersection of manifolds $\phi^{-1}_{\sigma_s^{i,j}} \circ \cdots \phi^{-1}_{\sigma_1^{i,j}} \circ \bar{\psi}_{i,j}$ with $\bar{\psi}_{i,j} \in \mathcal{Z}_1^1(\Phi^s)$ and $\sigma^{i,j} \in \{0, \ldots, m\}^s$. We are thus interested in $A := \phi^{-1}_{\sigma_i,1} \circ \phi^{-1}_{\sigma_s^{i,j},1} \circ \cdots \circ \phi^{-1}_{\sigma_1^{i,j},1} \circ \bar{\psi}_{i,j} \cap B_\delta(z_1)$, which are the perturbation of $\phi_{\sigma_i}^{-1} \circ \psi_i$. Let

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

$$h_i(x) = \begin{cases} x & \text{if } x \notin B_\delta(z_1), \\ \phi_{\sigma^i}^{-1} \circ \phi_{\sigma^i,1} & \text{otherwise.} \end{cases}$$

By choosing ϵ_s small enough, we have $\|h_i - \mathrm{id}\|_{\mathcal{C}^2} \le \delta/4$; in particular, h_i is a diffeomorphism. It follows that $h_i(B_\delta(z_1)) \subset B_\delta(z_1)$. Let $x \in A$. Then Proposition 4.17 implies that

$$\begin{aligned} \phi_{\sigma_1^{i,j},1} \circ \cdots \phi_{\sigma_s^{i,j},1} \circ \phi_{\sigma_i,1}(x) &= \phi_{\sigma_1^{i,j},1} \circ \cdots \phi_{\sigma_s^{i,j},1} \circ \phi_{\sigma_i} \circ h_i(x) \\ &= \phi_{\sigma_1^{i,j}} \circ \cdots \circ \phi_{\sigma_s^{i,j}} \circ \phi_{\sigma_i} \circ h_i(x) \\ &= \phi_{\sigma_1^{i,j}} \circ \cdots \circ \phi_{\sigma_s^{i,j}} \circ \phi_{\sigma^i,1}(x), \end{aligned}$$

and hence the part of ψ_i contained in $\phi_{\sigma^i,1}(B_\delta(z_1))$ is unchanged. This implies that, inside the ball $B_{\delta/2}(z_1)$, the IFS $\Phi^{s,1}$ has the wanted property for the manifold $\phi_{\sigma_1,1}^{-1} \circ \psi_1 \cap \cdots \cap \phi_{\sigma_n,1}^{-1} \circ \psi_n$. Moreover, by the openness of the transversality property, there exists $\epsilon_{s,1} \le \epsilon_s/4$ such that the wanted property persists in $B_{\delta/2}(z_1)$ for each $\epsilon_{s,1}$ perturbation. We can now consider all the other pre-images and do the same procedure with $\epsilon_{s,j} \le 4^{-j-1}\epsilon_s$ for the jth intersection manifold. In this way, we can construct an IFS $\Phi^{s,q} =: \tilde{\Phi}^{s,1}$ for some $q \in \mathbb{N}$ that is a $\epsilon_s/3$ perturbation of Φ^s and has the wanted property in $B_{\delta/2}(z_1)$. We can then repeat the same procedure in the ball $B_\delta(z_2)$ to obtain an IFS $\tilde{\Phi}^{s,2}$ that is a $\epsilon_s/9$ perturbation of $\tilde{\Phi}^{s,1}$, small enough not to upset what we have achieved in $B_{\delta/2}(z_1)$. Iterating such a construction, we finally obtain $\Phi^{s+1} = \tilde{\Phi}^{s,t}$, which has the wanted property on all the space since $\{B_{\delta/2}(z_i)\}_{i=1}^t$ is a covering of $D^N(\Phi_f^s)$ and is a $\sum_{i=1}^t \epsilon_s 3^{-k} \le \epsilon_s \le 2^{-s-1}\varepsilon$ perturbation of Φ^s. This concludes the induction argument. □

Finally, we can prove Proposition 4.3

Proof of Proposition 4.3. Let f be a piecewise smooth contraction with contraction coefficient $\lambda < 1$ and maximal partition $\boldsymbol{P}(f) = \{P_1, P_2, \ldots, P_m\}$. If $\lambda < 1/2m$, then the proof is complete; otherwise, we have the following.

Let the IFS associated to f be $\Phi_f = \{\phi_1, \phi_2, \ldots, \phi_m\}$ and let $l_1 = \max\{c_d l_0, d\}$, where l_0 is the number of boundary manifolds in $\partial\boldsymbol{P}(f)$, d is the dimension of the space and c_d is the maximum number of original boundary manifold overlaps (as defined in (4.13)). Let $N \in \mathbb{N}$ be the least number such that $\lambda^N 2^{dm^{d-1}l_1} < 1/4$. By Lemma 4.19, for $\varepsilon > 0$ small enough, there exists a piecewise contraction $\tilde{f}$ such that $d_2(f, \tilde{f}) < \varepsilon$ and no more than $2^{dm^{d-1}l_1}$ elements of the partition $\partial\boldsymbol{P}(\tilde{f}^N)$ have a non-empty intersection of their closure.

Accordingly, for each $x \in X$, there is a $\delta(x)$ such that $B_{\delta(x)}(x)$ intersects at most $2^{dm^{d-1}l_1}$ elements of $\boldsymbol{P}(\tilde{f}^N)$. Since X is compact, we can extract a finite cover $\{B_{\delta(x_j)/2}(x_j)\}$. Set $\delta = \frac{1}{2}\min\{\delta(x_j)\}$ and let $k \in \mathbb{N}$ be such that, for any partition element $P \in \boldsymbol{P}(\tilde{f}^{kN})$, $\mathrm{diam}(\tilde{f}^{kN}(P)) < \delta/2$; hence $\tilde{f}^{kN}(P) \subset B_{\delta(x_j)}(x_j)$ for some j. Therefore, it can intersect at most $2^{dm^{d-1}l_1}$ elements of $\boldsymbol{P}(\tilde{f}^N)$.

To conclude, let L be the number of elements of $\boldsymbol{P}(\tilde{f}^{kN})$. Then $\#\boldsymbol{P}(\tilde{f}^{2kN}) \le L 2^{dm^{d-1}l_1}$ and $\#\boldsymbol{P}(\tilde{f}^{jkN}) \le L(2^{dm^{d-1}l_1})^j$ for $j \in \mathbb{N}$. Since $\lambda^{kN}(2^{dm^{d-1}l_1}) < 1/4$, there exists $j_* \in \mathbb{N}$ such that $L(2^{dm^{d-1}l_1})^{j_*}\lambda^{j_* kN} < 1/2$. Hence $\tilde{f}^{j_* kN}$ is strongly contracting. □

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Acknowledgements. We thank Henk Bruin, Oliver Butterley, Giovanni Canestrari, Roberto Castorrini, Stefano Galatolo, Arnaldo Nogueira, Benito Pires and Antonio Rapagnetta for helpful discussions, comments and suggestions. We would also like to thank the anonymous referee for pointing out the inconsistencies and typographical errors in the previous version. This work was supported by the PRIN Grants 'Regular and stochastic behaviour in dynamical systems' (PRIN 2017S35EHN), 'Stochastic properties of dynamical systems' (PRIN 2022NTKXCX) and by the MIUR Excellence Department Project MathTOV awarded to the Department of Mathematics, University of Rome Tor Vergata. S.J. was additionally funded by ARC (DP220100492) and acknowledges Monash University and C.L. acknowledges membership to the GNFM/INDAM.

A. *Appendix. Extension theorem*

Here we discuss the extension theorems needed in the paper. Recall the classical extension theorem for Lipschitz functions.

THEOREM A.1. ((Kirszbraun–Valentine theorem) [26]) *Let $f : S(\subset \mathbb{R}^d) \to \mathbb{R}^d$ be a Lipschitz continuous function. Then f can be extended to any set $T \subset \mathbb{R}^d$ to a Lipschitz continuous function with the same Lipschitz constant.*

The above can be easily extended to $\mathcal{C}^r$ functions.

THEOREM A.2. ($\mathcal{C}^r$ version of Kirszbraun–Valentine theorem) *Let $S \subset \mathbb{R}^d$ be a compact set and let $f : S(\subset \mathbb{R}^d) \to \mathbb{R}^d$ be a $\mathcal{C}^r$ function, for $r \in \mathbb{N}$, such that* $\mathrm{Lip}(f) = \lambda < 1$ *and $f^{-1}|_S$ is $\mathcal{C}^1$. Then f can be extended to $\mathbb{R}^d$ to a $\mathcal{C}^r$ function f_* such that* $\mathrm{Lip}(f_*) = \mathrm{Lip}(f)$ *and $f_*^{-1}|_S = f^{-1}|_S$.*

Proof. Note that $\|f|_S\|_{\mathcal{C}^r}$ being finite and S being compact implies that there exists an open neighbourhood U of S such that f is $\mathcal{C}^r$ in U. By the inverse function theorem, f is invertible in U with $\|f^{-1}|_U\|_{\mathcal{C}^1} < \infty$. By the Kirszbraun–Valentine theorem A.1, there exists $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\tilde{f}|_S = f$ and $\mathrm{Lip}(\tilde{f}) = \mathrm{Lip}(f) = \lambda$. Then $\tilde{f}|_U \circ f^{-1}|_U = \mathrm{id}$, so we can define $\tilde{f}^{-1}|_U = f^{-1}$. Now let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{C}^\infty$ function compactly supported on $\overline{U}$ and $\int \phi = 1$ and define the convolution

$$\tilde{f} \star \phi(x) = \int \tilde{f}(x-y)\phi(y)\,dy.$$

For $\delta > 0$, let V be a δ-neighbourhood of U. Define a $\mathcal{C}^\infty$ function $g : \mathbb{R}^d \to \mathbb{R}$ as

$$g(x) = \begin{cases} 1, & x \in U, \\ 0, & x \notin V \end{cases}$$

such that $\|g\|_{\mathcal{C}^r} < c_r$. Finally, define $f_* : \mathbb{R}^d \to \mathbb{R}^d$ as $f_*(x) = g(x)f(x) + (1-g(x))\tilde{f} \star \phi(x)$. Let $x \in U$. Then

$$f_*(x) = g(x)f(x) + (1-g(x))\tilde{f} \star \phi(x) = f(x) + 0 = f(x).$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Therefore, f_* is an extension of f. To check whether it is Lipschitz, let $x_1, x_2 \in \mathbb{R}^d$, then

$$\begin{aligned}
&\|f_*(x_1) - f_*(x_2)\|_\infty \\
&\quad = \|g(x_1)f(x_1) + (1-g(x_1))\tilde{f} \star \phi(x_1) - g(x_2)f(x_2) - (1-g(x_2))\tilde{f} \star \phi(x_2)\|_\infty \\
&\quad \le \|g(x_1)(f(x_1) - \tilde{f} \star \phi(x_1)) - g(x_2)(f(x_2) - \tilde{f} \star \phi(x_2))\|_\infty \\
&\qquad + \|\tilde{f} \star \phi(x_1) - \tilde{f} \star \phi(x_2)\|_\infty \\
&\quad \le 0 + \|\tilde{f} \star \phi(x_1) - \tilde{f} \star \phi(x_2)\|_\infty \le \mathrm{Lip}(f) d_0(x_1, x_2).
\end{aligned}$$

Hence, f_* is Lipschtiz with $\mathrm{Lip}(f_*) = \mathrm{Lip}(f)$. Now, using Lemma 5.2 in [5],

$$\begin{aligned}
\|f_*\|_{\mathcal{C}^r} &= \|gf + (1-g)\tilde{f} \star \phi\|_{\mathcal{C}^r} \\
&\le \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r} + (1 - \|g\|_{\mathcal{C}^r})\bigg\| \int \tilde{f}(x-y)\phi(y)\, dy \bigg\|_{\mathcal{C}^r} \\
&\le \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r} + (1 - \|g\|_{\mathcal{C}^r}) \int \|f\|_{\mathcal{C}^r}\|\phi(y)\|_{\mathcal{C}^r}\, dy \\
&< \infty.
\end{aligned}$$

Therefore, $f_* \in \mathcal{C}^r$. □

B. *Appendix. Transversality*

For the convenience of the reader, we state the transversality theorem as used in the main text. We refer to [2] for details. The theorem in [[2], Ch. 6, §29.E] is stated for smooth maps and manifolds, but it can easily be reduced to the following version by using the $\mathcal{C}^r$ version of Sard's theorem. Also, the author discusses in detail the extension of the theorem to stratified sub-varieties, which is the case we are interested in and for which we state the theorem. One can also find the $\mathcal{C}^r$ version of the transversality theorem in [1], but there the reader needs to be mindful of the specific properties they ask on manifolds.

Definition B.1. (Transversal mapping) For every manifold A, B and submanifold $C \subset B$, a $\mathcal{C}^r$ mapping $f : A \to B$ is said to be transversal to $C (f \pitchfork C)$ at a point a if either $f(a) \notin C$ or the tangent plane to C at $f(a)$, if $f(a) \in \partial C$, and the image of the tangent plane to A at a are transversal: that is,

$$D_a f(T_a A) \oplus T_{f(a)} C = T_{f(a)} B.$$

f is said to be transversal to C if f is transversal to C at a for every $a \in A$.

Definition B.2. For every manifold A, B and stratified sub-variety $C \subset B$, a $\mathcal{C}^r$ mapping $f : A \to B$ is said to be transversal to C if it is transversal to C and all its substrata.

THEOREM B.3. (Transversality theorem-$\mathcal{C}^r$ version [[2], Ch. 6, §29.E]) *Let A be a compact manifold and let C be a compact stratified sub-variety of a manifold B. Then the $\mathcal{C}^r$ mappings $f : A \to B$ with $f \pitchfork C$ form an open everywhere dense set in the space of all $\mathcal{C}^r$ mappings $A \to B$.*

We apply the above theorem to the following situation (Y and the functions ϕ_i are as defined in the §4.1).

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

LEMMA B.4. *Let $Y = \{x : \|x\| \leq R\} \subset \mathbb{R}^d$, for some $R > 0$, let $B \subset Y$ be open and let $\phi_i \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$, $i \in \{1, \ldots, m\}$ be such that they are invertible when restricted to B. Also, let $W_i \subset B$, $i \in \{1, \ldots, m\}$, be d_i-dimensional compact manifolds with boundaries. The maps $F : \mathbb{R}^d \to \mathbb{R}^{md}$, defined by $F(x) := (\phi_1(x), \ldots, \phi_m(x))$, which are transversal to the stratified sub-variety $C = W_1 \times \cdots \times W_m$ form an open and dense set. Moreover, if $F|_B \pitchfork C$, then the manifold $(\bigcup_{i=1}^m \phi_i^{-1}(W_i \setminus \partial W_i)) \bigcap B$ is empty if there exists k such that $\sum_{i=1}^m d_i - (k-1)d < 0$; otherwise, it has dimension at most $\sum_{i=1}^m d_i - (m-1)d$.*

Proof. Our strategy is to transform the present setting to the setting suitable for the application of Theorem B.3 and then proceed further.

The fact that $W_1 \times \cdots \times W_k$ is a stratified sub-variety can be checked directly. A minor problem is that neither Y nor $\mathbb{R}^d$ are compact manifolds. To overcome this problem, we define the function $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $g(y) = 1$ for $y \leq 1$ and $g(y) = 0$ for $y \geq a$, with $|g'(x)| \leq 2a^{-1}$ ($a > 2$ to be chosen large enough), and we define $\hat{F} : \mathbb{R}^d \to \mathbb{R}^{md}$ as

$$\hat{F}(x) = g(R^{-1}\|x\|)F(x).$$

Note that, for $x \in Y$, $\hat{F} = F$, whereas, for $\|x\| > aR$, $\hat{F}(x) = 0$. Hence, $\hat{F}$ can be seen as a smooth function on the torus $\mathbb{R}^m/\mathbb{Z}_{aR}$ (which indeed is a compact manifold). We can thus apply Theorem B.3 to obtain the first part of the Lemma.

To prove the second part, note that if $z \in B$ is such that $\hat{F}(y) = F(y) \in C \setminus \partial C$, then

$$D_y F(\mathbb{R}^d) + T_{F(y)}C = \mathbb{R}^{md}. \tag{B.1}$$

If there exists a $y \in \mathbb{R}^d$ such that $F(y) \in C \setminus \partial C$, then $\phi_i(y) \in W_i$, and hence $y = \phi_i^{-1}(W_i)$: that is, $\bigcap_{i=1}^m \phi_i^{-1}(W_i) \supset \{y\} \neq \emptyset$. Next, let $\bar{d} = \sum_{i=1}^m d_i$. If $d + \bar{d} < md$, then (B.1) cannot be satisfied. It follows that if $\bar{d} < (m-1)d$, then $\bigcap_{i=1}^m \phi_i^{-1}(W_i) = \emptyset$.

If $\bar{d} \geq (m-1)d$, we study equation (B.1): for an arbitrary $(\tilde{\beta}_1, \ldots, \tilde{\beta}_m) \in \mathbb{R}^{md}$, there must exist $\alpha \in \mathbb{R}^d$ and $\tilde{w}_i \in TW_i$ such that $D\phi_i \alpha + \tilde{w}_i = \tilde{\beta}_i$. So, setting $\beta_i = D\phi_i^{-1}(\tilde{\beta}_i)$ and $w_i = D\phi_i^{-1}(\tilde{w}_i)$, we must study the solutions of

$$\alpha + w_i = \beta_i. \tag{B.2}$$

Note that α is uniquely determined by $\alpha = \beta_1 - w_1$. Subtracting the second of the (B.2) from the first yields $w_1 - w_2 = \beta_1 - \beta_2$. If $d_1 + d_2 < d$, such an equation has no solution for all β_2, so the intersection must be empty. If $s_2 = d_1 + d_2 - d \geq 0$, then the dimension of $W_{1,2} := D\phi_i^{-1} W_1 \cap D\phi_i^{-1} W_2$ is s_2. We can then write $w_1 = \xi_1 + \hat{w}_1$ and $w_2 = \xi_1 + \hat{w}_2$ with $\hat{w}_i \in D\phi_i^{-1} W_i \cap W_{1,2}^\perp$. It follows that $\hat{w}_1 - \hat{w}_2 = \beta_1 - \beta_2$, which determines uniquely $\hat{w}_1, \hat{w}_2$. We can then write $w_3 - \xi_1 = \beta_3 - \beta_2 + \hat{w}_2$. Let $s_3 = d_3 + s_2 - d = d_1 + d_2 + d_3 - 2d$. If $s_3 < 0$, again, in general, there are no solutions. Otherwise, the dimension of $W_{1,2,3} = W_{1,2} \cap D\phi_i^{-1} W_3$ is s_3 and we can write $\xi_1 = \xi_2 + \hat{\xi}_1$, $w_3 = \xi_2 + \hat{w}_3$ with $\xi_2 \in W_{1,2,3}$ and $\hat{\xi} \in W_{1,2} \cap W_{1,2,3}^\perp$, $\hat{w}_3 = D\phi_i^{-1} W_3 \cap W_{1,2,3}^\perp$. Accordingly, we have $\hat{w}_3 - \hat{\xi}_1 = \beta_3 - \beta_2 + \hat{w}_2$, which determines uniquely $\hat{w}_3, \hat{\xi}_1$. Continuing in such a way, we have that $W_{1,\ldots,m} = D\phi_i^{-1} W_1 \cap \cdots \cap D\phi_i^{-1} W_m$ has dimension $\bar{d} - (m-1)d$. The case in which $F(y)$ belongs to a substrata of C is treated in exactly the same way and yields a lower dimension. □

C. *Appendix. Technical lemmata*

In this section, for the convenience of the reader, we collect some simple but boring technical results.

LEMMA C.1. *There exists $C_\# > 0$ such that, for given $\varepsilon \in (0, 1/2)$ and $f, h \in \mathcal{C}^3$, where h is a diffeomorphism such that $\|h - \mathrm{id}\|_{\mathcal{C}^2} < \varepsilon$, we have $h \circ f \circ h^{-1} \in \mathcal{C}^3$ and*

$$\|h^{-1} - \mathrm{id}\|_{\mathcal{C}^2} \le 6\varepsilon,$$
$$\|h \circ f \circ h^{-1} - f\|_{\mathcal{C}^2} < C_\# \|f\|_{\mathcal{C}^3}\varepsilon.$$

Proof. First, we claim that $\|h - \mathrm{id}\|_{\mathcal{C}^2} < \varepsilon$ implies that $\|h^{-1} - \mathrm{id}\|_{\mathcal{C}^1} < 2\varepsilon$. Indeed, there exists a transformation A, with $\|A\|_{\mathcal{C}^1} < 1$, such that we can write $Dh = \mathbb{1} + \varepsilon A$. Therefore,

$$(Dh)^{-1} = (\mathbb{1} + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k \varepsilon^k A^k.$$

That is,

$$\|(Dh)^{-1} - \mathbb{1}\|_{\mathcal{C}^0} \le \sum_{k=1}^{\infty} \varepsilon^k \|A\|_{\mathcal{C}^0}^k \le \frac{\varepsilon}{1-\varepsilon} \le 2\varepsilon,$$
$$\|D(Dh)^{-1}\|_{\mathcal{C}^0} \le \sum_{k=1}^{\infty} \varepsilon^k \|A\|_{\mathcal{C}^1}^k \le 2\varepsilon.$$

Moreover, the inverse function theorem implies that $h^{-1} \in \mathcal{C}^3$. Accordingly, since $Dh^{-1} = (Dh)^{-1} \circ h^{-1}$,

$$\begin{aligned}\|Dh^{-1} - \mathbb{1}\|_{\mathcal{C}^0} &= \|((Dh)^{-1} - h) \circ h^{-1}\|_{\mathcal{C}^0} = \|((Dh)^{-1} - \mathbb{1}) \circ h^{-1}\|_{\mathcal{C}^0} \\ &= \|(Dh)^{-1} - \mathbb{1}\|_{\mathcal{C}^0} < 2\varepsilon, \\ \|Dh^{-1} - \mathbb{1}\|_{\mathcal{C}^1} &\le 2\varepsilon + \|D(Dh)^{-1}\|_{\mathcal{C}^0}\|Dh^{-1}\|_{\mathcal{C}^0} \le 6\varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}\|h \circ f \circ h^{-1} - f\|_{\mathcal{C}^0} &\le \varepsilon + \|f\|_{\mathcal{C}^1}\varepsilon, \\ \|D(h \circ f \circ h^{-1} - f)\|_{\mathcal{C}^0} &= \|Dh \circ f \circ h^{-1} \cdot Df \circ h^{-1} \cdot Dh^{-1} - Df)\|_{\mathcal{C}^0} \\ &\le \|Df \circ h^{-1} - Df\|_{\mathcal{C}^0} + C_\#\varepsilon\|f\|_{\mathcal{C}^1} \le C_\#\|f\|_{\mathcal{C}^2}\varepsilon.\end{aligned}$$

Next,

$$\begin{aligned}\|\partial_x D(h \circ f \circ h^{-1} - f)\|_{\mathcal{C}^0} &= \|(\partial_y Dh) \circ f \circ h^{-1}(\partial_x f)_y \circ h^{-1}\partial_x(h_z^{-1})D(f \circ h^{-1}) \\ &\quad + Dh \circ f \circ h^{-1}(\partial_y Df) \circ h^{-1}\partial_x(h_y^{-1})Dh^{-1} \\ &\quad + Dh \circ f \circ h^{-1}Df \circ h^{-1}\partial_x(Dh^{-1}) - \partial_x Df\|_{\mathcal{C}^0} \\ &\le \|(\partial_x Df) \circ h^{-1} - \partial_x Df\|_{\mathcal{C}^0} + C_\#\|f\|_{\mathcal{C}^2}\varepsilon \\ &\le C_\#\|f\|_{\mathcal{C}^3}\varepsilon.\end{aligned}$$

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

Thus,

$$\|h\circ f\circ h^{-1}-f\|_{\mathcal{C}^2}=\|h\circ f\circ h^{-1}-f\|_{\mathcal{C}^0}+\|D(h\circ f\circ h^{-1}-f)\|_{\mathcal{C}^0}\\ +\|D^2(h\circ f\circ h^{-1}-f)\|_{\mathcal{C}^0}\le C_\#\|f\|_{\mathcal{C}^3}\varepsilon$$

for some constant $C_\# > 0$. Finally, $h\circ f\circ h^{-1}\in\mathcal{C}^3$ is the composition of $\mathcal{C}^3$ functions. □

LEMMA C.2. *For given $\varepsilon > 0$, let $g, h \in \mathcal{C}^3$ be such that g is invertible and $\|g-h\|_{\mathcal{C}^2}<\varepsilon$. Then $g^{-1}\circ h\in\mathcal{C}^3$ and*

$$\|g^{-1}\circ h-\mathrm{id}\|_{\mathcal{C}^2}\le C_\#\varepsilon\|g^{-1}\|^3_{\mathcal{C}^3}.$$

Additionally for $f\in\mathcal{C}^3$, $f\circ g^{-1}\circ h$, $(f\circ g^{-1}\circ h)^{-1}\in\mathcal{C}^3$ and

$$\|f\circ g^{-1}\circ h-\mathrm{id}\|_{\mathcal{C}^2}\le C_\#\|f\|_{\mathcal{C}^3}\varepsilon\|g^{-1}\|^3_{\mathcal{C}^3}.$$

Proof. Since $g\in\mathcal{C}^3$ is invertible, by the inverse function theorem, $g^{-1}\in\mathcal{C}^3$ and therefore there is a composition of $\mathcal{C}^3$ functions $g^{-1}\circ h\in\mathcal{C}^3$. Next, let $\Psi=g^{-1}\circ h$. Then

$$\begin{aligned}\|\Psi-\mathrm{id}\|_{\mathcal{C}^0}&=\|g^{-1}\circ h-g^{-1}\circ g\|_{\mathcal{C}^0}\le\|g^{-1}\|_{\mathcal{C}^1}\varepsilon,\\ \|D(\Psi-\mathrm{id})\|_{\mathcal{C}^0}&=\|(Dg)^{-1}\circ\Psi\cdot Dh-\mathbb{1}\|_{\mathcal{C}^0}\le\|g^{-1}\|_{\mathcal{C}^1}\|g^{-1}\|_{\mathcal{C}^2}\varepsilon,\\ \|\partial_x D(\Psi-\mathrm{id})\|_{\mathcal{C}^0}&=\|\partial_y[(Dg)^{-1}]\circ\Psi\partial_x\Psi_y\cdot Dh+(Dg)^{-1}\circ\Psi\cdot\partial_x Dh\|_{\mathcal{C}^0}\\ &\le\|\partial_y[(Dg)^{-1}]\circ\Psi\partial_x\Psi_y\cdot Dg+(Dg)^{-1}\circ\Psi\cdot\partial_x Dg\|_{\mathcal{C}^0}+C_\#\varepsilon\|g^{-1}\|^2_{\mathcal{C}^2}\\ &\le\|\partial_x[(Dg)^{-1}\circ\Psi\cdot Dg]\|_{\mathcal{C}^0}+C_\#\varepsilon\|g^{-1}\|^2_{\mathcal{C}^2}\\ &\le\|\partial_x[(Dg)^{-1}\cdot Dg]\|_{\mathcal{C}^0}+C_\#\varepsilon\|(Dg)^{-1}\|^3_{\mathcal{C}^3}=C_\#\varepsilon\|g^{-1}\|^3_{\mathcal{C}^3}.\end{aligned}$$

Accordingly, $\|\Psi-\mathrm{id}\|_{\mathcal{C}^2}\le C_\#\varepsilon\|(Dg)^{-1}\|_{\mathcal{C}^3}$. Moreover, the first part of Lemma C.1 implies that Ψ is invertible and $\|\Psi^{-1}-\mathrm{id}\|_{\mathcal{C}^2}\le C_\#\varepsilon\|(Dg)^{-1}\|_{\mathcal{C}^3}$. This implies that $f\circ\Psi$, $\Psi^{-1}\circ f^{-1}\in\mathcal{C}^3$ and

$$\begin{aligned}\|f\circ\Psi-f\|_{\mathcal{C}^0}&\le C_\#\varepsilon\|f\|_{\mathcal{C}^1}\|g^{-1}\|_{\mathcal{C}^1},\\ \|D(f\circ\Psi-f)\|_{\mathcal{C}^0}&=\|(Df)\circ\Psi\cdot D\Psi-Df\|_{\mathcal{C}^0}\le C_\#\varepsilon\|f\|_{\mathcal{C}^2}\|g^{-1}\|^2_{\mathcal{C}^2},\\ \|\partial_x D(f\circ\Psi-f)\|_{\mathcal{C}^0}&=\|[\partial_y(Df)]\circ\Psi\cdot\partial_x\Psi_y\cdot D\Psi+(Df)\circ\Psi\cdot\partial_x D\Psi-\partial_x Df\|_{\mathcal{C}^0}\\ &\le C_\#\|f\|_{\mathcal{C}^3}\|g^{-1}\|^3_{\mathcal{C}^3},\end{aligned}$$

from which the Lemma follows. □

Let f be a piecewise smooth contraction with an associated IFS $\Phi=\{\phi_1,\ldots,\phi_m\}$, as in equation (4.11), let $\Sigma^m_{n,i}(\Phi)$ be the set of i-admissible sequences and, as in equation (4.15), let $D^N_\delta(\Phi)$ be a $\delta-$neighbourhood of the boundary of partition $\boldsymbol{P}(f^N)$ for $\delta>0$. We have the following result.

LEMMA C.3. *For a piecewise smooth contraction f with IFS $\Phi=\{\phi_1,\ldots,\phi_m\}$ and $N\in\mathbb{N}$, there exists $\varepsilon>0$ such that, for $\tilde f$ with associated IFS $\tilde\Phi=\{\tilde\phi_1,\ldots,\tilde\phi_m\}$ satisfying $d_2(f,\tilde f)<\varepsilon$, we have $\Sigma^m_{n,i}(\Phi)=\Sigma^m_{n,i}(\tilde\Phi)$. Moreover, there exists $\delta>0$ such that $D^N_{\delta/4}(\tilde\Phi)\subset D^N_{\delta/2}(\Phi)$.*

Proof. By hypothesis, there exists $M > 0$ such that $\|\phi_i|_{U_i}^{-1}\|_{\mathcal{C}^1} \leq M$ and we can restrict to such a set by the definition of D^N which entails only admissible sequences. Thus,

$$\|\tilde{\phi}_i^{-1} - \phi_i^{-1}\|_{\mathcal{C}^0} = \|\mathrm{id} - \phi_i^{-1} \circ \tilde{\phi}_i\|_{\mathcal{C}^0} = \|\phi_i^{-1} \circ \phi_i - \phi_i^{-1} \circ \tilde{\phi}_i\|_{\mathcal{C}^0}$$
$$\leq \|\phi_i^{-1}\|_{\mathcal{C}^1}\|\phi_i - \tilde{\phi}_i\|_{\mathcal{C}^0} \leq M\|\phi_i - \tilde{\phi}_i\|_{\mathcal{C}^0}.$$

For $n \leq N$ and admissible sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma_n^m$,

$$\begin{aligned}
&\|\tilde{\phi}_{\sigma_n}^{-1} \circ \tilde{\phi}_{\sigma_{n-1}}^{-1} \circ \cdots \circ \tilde{\phi}_{\sigma_1}^{-1} - \phi_{\sigma_n}^{-1} \circ \phi_{\sigma_{n-1}}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1}\|_{\mathcal{C}^0} \\
&\quad = \|\mathrm{id} - \phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1} \circ \tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_n}\|_{\mathcal{C}^0} \\
&\quad = \|\phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1} \circ \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_n} - \phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1} \circ \tilde{\phi}_{\sigma_1} \circ \cdots \circ \tilde{\phi}_{\sigma_n}\|_{\mathcal{C}^0} \\
&\quad \leq M^n \sum_{i=1}^{n} \|\phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_i} \circ \tilde{\phi}_{\sigma_{i+1}} \circ \cdots \circ \tilde{\phi}_{\sigma_n} - \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_{i-1}} \circ \tilde{\phi}_{\sigma_i} \circ \cdots \circ \tilde{\phi}_{\sigma_n}\|_{\mathcal{C}^0} \\
&\quad \leq M^n (1 - \lambda)^{-1} \sup_i \|\phi_i - \tilde{\phi}_i\|_{\mathcal{C}^0}.
\end{aligned}$$

Note that, by definition (4.11), there exists $\varepsilon_0 > 0$ such that, for each ε-perturbation $\tilde{\Phi}$ of Φ, with $\varepsilon \in (0, \varepsilon_0)$, $\Sigma_{n,i}^m(\Phi) = \Sigma_{n,i}^m(\tilde{\Phi})$. Moreover, for each $\sigma \in \Sigma_{n,i}^m(\Phi)$ and $\xi \in \psi_i^{-1}(M_{\sigma,i}(\Phi))$,

$$\|\phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1} \circ \psi_i(\xi) - \tilde{\phi}_{\sigma_n}^{-1} \circ \cdots \circ \tilde{\phi}_{\sigma_1}^{-1} \circ \psi_i(\xi)\| \leq M^n (1 - \lambda)^{-1} \varepsilon.$$

Thus, for ε small enough, $D^N(\tilde{\Phi}) \subset D_{\delta/4}^N(\Phi)$, and hence $D_{\delta/4}^N(\tilde{\Phi}) \subset D_{\delta/2}^N(\Phi)$. □

REFERENCES

[1] R. Abraham. Transversality in manifolds of mappings. *Bull. Amer. Math. Soc. (N.S.)* **69** (1963), 470–474.

[2] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York, 1988.

[3] H. Bruin and J. H. B. Deane. Piecewise contractions are asymptotically periodic. *Proc. Amer. Math. Soc.* **137**(4) (2009), 1389–1395.

[4] A. Calderon, E. Catsigeras and P. Guiraud. A spectral decomposition of the attractor of piecewise-contracting maps of the interval. *Ergod. Th. & Dynam. Sys.* **41**(7) (2021), 1940–1960.

[5] R. Castorrini and C. Liverani. Quantitative statistical properties of two-dimensional partially hyperbolic systems. *Adv. Math.* **409**(Part A) (2022), 108625.

[6] E. Catsigeras and R. Budelli. Topological dynamics of generic piecewise continuous contractive maps in n dimensions. *Int. J. Pure Appl. Math.* **68** (2011), 61–83.

[7] E. Catsigeras, P. Guiraud and A. Meyroneinc. Complexity of injective piecewise contracting interval maps. *Ergod. Th. & Dynam. Sys.* **40** (2020), 64–88.

[8] E. Catsigeras, P. Guiraud, A. Meyroneinc and E. Ugalde. On the asymptotic properties of piecewise contracting maps. *Dyn. Syst.* **67** (2016), 609–655.

[9] R. Coutinho. Dinàmica simbòlica linear. *Thesis*, Universidade Técnica de Lisboa, 1999.

[10] G. Del Magno, J. P. Gaivao and E. Gutkin. Dissipative outer billiards: a case study. *Dyn. Syst.* **30**(1) (2010), 45–69.

[11] J. P. Gaivão. Assymptotic periodicity in outer billiards. *Ergod. Th. & Dynam. Sys.* **40**(2) (2020), 402–417.

[12] J. P. Gaivão and A. Nogueira. Dynamics of piecewise increasing contractions. *Bull. Lond. Math. Soc.* **54**(2) (2022), 482–500.

[13] J. P. Gaivão and B. Pires. Multi-dimensional piecewise contractions are asymptotically periodic. *Preprint*, 2024, arXiv:2405.08444v1.

[14] J. Gambaudo and C. Tresser. On the dynamics of quasi-contractions. *Bull. Braz. Math. Soc. (N.S.)* **19** (1988), 61–141.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press

[15] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities (Graduate Texts in Mathematics)*. Springer-Verlag, New York, 1973.

[16] S. Janson and C. Oberg. A piecewise contractive dynamical system and election methods. *Bull. Soc. Math. France* **147**(3) (2019), 395–441.

[17] I.-J. Jeong. Outer billiards with contractions. *Senior Thesis*, Brown University, 2012.

[18] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.

[19] B. Kruglikov and M. Rypdal. A piece-wise affine contracting map with positive entropy. *Discrete Contin. Dyn. Syst.* **16**(2) (2006), 393–394.

[20] M. Laurent and A. Nogueira. Rotation number of contracted rotations. *J. Mod. Dyn.* **12** (2018), 175–191.

[21] A. Nogueira and B. Pires. Dynamics of piecewise contractions of the interval. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2198–2215.

[22] A. Nogueira, B. Pires and R. Rosales. Asymptotically periodic piecewise contractions of the interval. *Nonlinearity* **27**(7) (2014), 1603–1610.

[23] A. Nogueira, B. Pires and R. Rosales. Piecewise contractions defined by iterated function systems. *Preprint*, 2014, arXiv:1408.1663.

[24] A. Nogueira, B. Pires and R. Rosales. Topological dynamics of piecewise λ-affine maps. *Ergod. Th. & Dynam. Sys.* **38**(5) (2018), 1876–1893.

[25] B. Pires. Symbolic dynamics of piecewise contractions. *Nonlinearity* **32** (2019), 4871–4889.

[26] F. A. Valentine. A Lipschitz condition preserving extension for a vector function. *Amer. J. Math.* **67**(1) (1945), 83–93.

https://doi.org/10.1017/etds.2024.78 Published online by Cambridge University Press