J. Austral. Math. Soc. **73** (2002), 127–153

SOME CLASSES OF PSEUDO-BL ALGEBRAS

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(Received 7 August 2000; revised 22 August 2001)

Communicated by B. A. Davey

Abstract

Pseudo-BL algebras are noncommutative generalizations of BL-algebras and they include pseudo-MV algebras, a class of structures that are categorically equivalent to l-groups with strong unit. In this paper we characterize directly indecomposable pseudo-BL algebras and we define and study different classes of these structures: local, good, perfect, peculiar, and (strongly) bipartite pseudo-BL algebras.

2000 *Mathematics subject classification*: primary 06F99, 08A72.

Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [14]. The main example of a BL-algebra is the interval [0, 1] endowed with the structure induced by a t-norm. MV-algebras, Gödel algebras and product algebras are the most known classes of BL-algebras. Recent investigations are concerned with noncommutative generalizations for these structures.

In [4, 13], pseudo-BL algebras were defined as noncommutative generalizations of BL-algebras. The main source of examples of pseudo-BL algebras is l-group theory. In order to recapture some of the properties of pseudo-BL algebras a notion of pseudo-t-norm was introduced in [10]. For the interval [0, 1], this notion induces more general algebras named weak pseudo-BL algebras.

Pseudo-MV algebras were introduced as a noncommutative generalization of MV-algebras (see [11, 12]). Dvurecenskij proved in [9] that the category of pseudo-MV algebras is equivalent to the category of l-groups with strong unit. This theorem extends the fundamental result established by Mundici for the commutative case [16].

In [2], Belluce, Di Nola and Lettieri studied local MV-algebras, structures having a unique maximal ideal. An important class of local MV-algebras are perfect MV-algebras, which are MV-algebras generated by their radical. The category of perfect

MV-algebras is equivalent to the category of abelian l-groups [6]. All these results were extended in [15] to pseudo-MV algebras. Following [2], in [19] local BL-algebras were defined and classified.

Bipartite MV-algebras, defined in [7], are another important class of MV-algebras. Bipartite BL-algebras and strongly bipartite BL-algebras were defined in [17]. In [8] bipartite BL-algebras were classified and it was proved that the variety generated by perfect BL-algebras is exactly the variety of strongly bipartite BL-algebras. All these results are parallel to the ones already existing for MV-algebras (see [1, 7]).

In this paper we shall extend some of these results to pseudo-BL algebras. By [5], the congruences of a pseudo-BL algebra are in a bijective correspondence with the normal filters. Then, there are two possibilities to define a concept of *local* pseudo-BL algebra. The first one is to define a local pseudo-BL algebra as being a pseudo-BL algebra with a unique ultrafilter. This paper deals with this approach. Another way is to consider structures having a unique maximal normal filter. For the second case, we obtain the notion of *normal local* pseudo-BL algebra. The investigation of normal local pseudo-BL algebras seems to be a difficult problem, since we do not have a characterization of the normal filter generated by a set of elements.

The paper is divided into four sections. In the first section we recall some facts concerning pseudo-BL algebras and pseudo-MV algebras and we prove some properties used in the sequel. Following [3], we characterize directly indecomposable pseudo-BL algebras. In Section 2 we define and study local pseudo-BL algebras. Many of the results from local MV-algebras [2] and local BL-algebras [19] are extended to local pseudo-BL algebras. In the next section we study good pseudo-BL algebras, an important class of pseudo-BL algebras. We associate with any good pseudo-BL algebra a pseudo-MV algebra in a natural way. In Section 4 we investigate some classes of local pseudo-BL algebras, namely perfect, locally finite and peculiar pseudo-BL algebras. We give a classification of local pseudo-BL algebras and we give a simpler proof of the fact that locally finite pseudo-BL algebras are exactly locally finite MV-algebras. In the last section of the paper, following [17] we study (strongly) bipartite pseudo-BL algebras.

1. Definitions and first properties

A *pseudo*-BL *algebra* ([4, 13]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \rightsquigarrow, \rightarrow, 0, 1)$ with five binary operations $\wedge, \vee, \odot, \rightsquigarrow, \rightarrow$ and two constants 0, 1 such that:

(A1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

(A2) $(A, \odot, 1)$ is a monoid;

(A3) $a \odot b \leq c$ if and only if $a \leq b \rightsquigarrow c$ if and only if $b \leq a \rightarrow c$;

(A4) $a \wedge b = (a \rightsquigarrow b) \odot a = a \odot (a \rightarrow b)$;

(A5) $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = (a \rightarrow b) \vee (b \rightarrow a) = 1$.

https://doi.org/10.1017/S144678870000851X Published online by Cambridge University Press

In the sequel, we shall agree that the operations $\wedge, \vee, \odot$ have priority towards the operations $\rightsquigarrow, \rightarrow$. Sometimes, we shall put parenthesis even if this is not necessary.

It is proved in [4] that commutative pseudo-BL algebras are BL-algebras. For details on BL-algebras see [14, 18]. A pseudo-BL algebra **A** is nontrivial if and only if $0 \neq 1$. For any pseudo-BL algebra **A**, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *pseudo*-BL *chain* is a linear pseudo-BL algebra, that is a pseudo-BL algebra such that its lattice order is total.

For any $a \in A$, we define $a^{\sim} = a \rightsquigarrow 0$ and $a^{-} = a \rightarrow 0$. We shall write $a^{\approx}$ instead of $(a^{\sim})^{\sim}$ and $a^{=}$ instead of $(a^{-})^{-}$. We denote the set of natural numbers by ω. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $\text{ord}(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $\text{ord}(a) = \infty$.

The following properties hold in any pseudo-BL algebra **A** and will be used in the sequel. See [4] for details.

(1) $(a \odot b) \rightsquigarrow c = a \rightsquigarrow (b \rightsquigarrow c)$;
(2) $(b \odot a) \rightarrow c = a \rightarrow (b \rightarrow c)$;
(3) $a \leq b$ if and only if $a \rightsquigarrow b = 1$ if and only if $a \rightarrow b = 1$;
(4) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
(5) $a \odot b \leq a, b$;
(6) $a \odot b \leq a \wedge b$;
(7) $a \odot b = 0$ if and only if $a \leq b^{\sim}$ if and only if $b \leq a^{-}$;
(8) $a \odot 0 = 0 \odot a = 0$;
(9) $a^{\sim} \odot a = a \odot a^{-} = 0$;
(10) $1 \rightsquigarrow a = 1 \rightarrow a = a$;
(11) $a^{\sim} = 1$ if and only if $a^{-} = 1$ if and only if $a = 0$;
(12) $1^{\sim} = 1^{-} = 0$;
(13) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$ and $b^{-} \leq a^{-}$;
(14) $a \leq a^{\sim -}$ and $a \leq a^{-\sim}$;
(15) $a \rightsquigarrow b \leq b^{\sim} \rightarrow a^{\sim}$ and $a \rightarrow b \leq b^{-} \rightsquigarrow a^{-}$;
(16) $a^{\sim-\sim} = a^{\sim}$ and $a^{-\sim-} = a^{-}$;
(17) $(a \odot b)^{\sim} = a \rightsquigarrow b^{\sim}$ and $(a \odot b)^{-} = b \rightarrow a^{-}$;
(18) $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$ and $(a \vee b)^{-} = a^{-} \wedge b^{-}$;
(19) $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$ and $(a \wedge b)^{-} = a^{-} \vee b^{-}$;
(20) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$;
(21) $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$;
(22) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Let **A** be a pseudo-BL algebra. According to [4], a *filter* of **A** is a nonempty subset F of A such that for all $a, b \in A$,

(i) if $a, b \in F$, then $a \odot b \in F$;
(ii) if $a \in F$ and $a \leq b$, then $b \in F$.

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By (6), it is obvious that any filter of **A** is also a filter of the lattice $L(A)$. A filter F of **A** is *proper* if $F \neq A$. A proper filter P of **A** is *prime* if for all $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. We shall denote by $\mathrm{Spec}(A)$ the set of prime filters of the pseudo-BL algebra **A**.

A proper filter U of **A** is an *ultrafilter* (or a *maximal filter*) if it is not contained in any other proper filter. We shall denote by $\mathscr{M}(A)$ the intersection of all ultrafilters of **A**. Obviously, $\mathscr{M}(A)$ is a proper filter of **A**.

We recall some properties of filters that will be used in the sequel.

PROPOSITION 1.1 ([4, Theorem 3.25]). *Let F be a filter of the pseudo-*BL *algebra* **A** *and let S be a $\vee$-closed subset of A (that is, if $a, b \in S$, then $a \vee b \in S$) such that $F \cap S = \emptyset$. Then there exists a prime filter P of* **A** *such that $F \subseteq P$ and $P \cap S = \emptyset$.*

PROPOSITION 1.2. *Any proper filter of* **A** *can be extended to a prime filter.*

PROOF. Apply [4, Corollary 3.26]. □

PROPOSITION 1.3 ([4, Corollary 3.32]). *Any ultrafilter of* **A** *is a prime filter of* **A**.

PROPOSITION 1.4 ([4, Remark 3.33]). *Any proper filter of* **A** *can be extended to an ultrafilter.*

PROPOSITION 1.5. *Let* **A** *be a pseudo-*BL *algebra. The following are equivalent:*

(i) **A** *is a pseudo-*BL *chain;*
(ii) *any proper filter of* **A** *is prime.*

LEMMA 1.6. *If* **A** *is a pseudo-*BL *algebra, then the sets $A_0^{\sim} = \{a \in A \mid a^{\sim} = 0\}$ and $A_0^{-} = \{a \in A \mid a^{-} = 0\}$ are proper filters of* **A**.

PROOF. Let us prove that $A_0^{\sim}$ is a proper filter of **A**. By (12), $1 \in A_0^{\sim}$. Let $a, b \in A_0^{\sim}$, that is, $a^{\sim} = b^{\sim} = 0$. By (17), we get that $(a \odot b)^{\sim} = a \rightsquigarrow b^{\sim} = a \rightsquigarrow 0 = a^{\sim} = 0$, hence $a \odot b \in A_0^{\sim}$. Let $a \in A_0^{\sim}$ and $b \in A$ such that $a \leq b$. Then $a^{\sim} = 0$ and, by (13), $b^{\sim} \leq a^{\sim}$, so $b^{\sim} = 0$, that is, $b \in A_0^{\sim}$. Thus, $A_0^{\sim}$ is a filter of **A**. Since, by (11), $0^{\sim} = 1$, it follows that $0 \notin A_0^{\sim}$, hence $A_0^{\sim}$ is proper. Similarly we can show that A_0^{-} is a proper filter of **A**. □

Let $X \subseteq A$. The filter of **A** generated by X will be denoted by $\langle X \rangle$. We have that $\langle \emptyset \rangle = \{1\}$ and $\langle X \rangle = \{a \in A \mid x_1 \odot \cdots \odot x_n \leq a$ for some $n \in \omega - \{0\}$ and some $x_1, \ldots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $\langle a \rangle$ denotes the principal filter of **A** generated by $\{a\}$. Then, $\langle a \rangle = \{b \in A \mid a^n \leq b$ for some $n \in \omega - \{0\}\}$.

LEMMA 1.7. *Let $a, b \in A$. Then*

(i) *$\langle a \rangle$ is proper if and only if* $\mathrm{ord}(a) = \infty$;

(ii) *if $a \leq b$ and* $\mathrm{ord}(b) < \infty$, *then* $\mathrm{ord}(a) < \infty$;

(iii) *if $a \leq b$ and* $\mathrm{ord}(a) = \infty$, *then* $\mathrm{ord}(b) = \infty$.

PROOF. (i) $\langle a \rangle$ is proper if and only if $0 \notin \langle a \rangle$ if and only if $a^n \neq 0$ for all $n \in \omega - \{0\}$ if and only if $\mathrm{ord}(a) = \infty$.

(ii), (iii) Applying (4), $a \leq b$ implies $a^n \leq b^n$ for all $n \in \omega$. □

A filter H of **A** is called *normal* ([5]) if for every $a, b \in A$ we have the equivalence:

$$(N) \qquad a \rightsquigarrow b \in H \quad \text{if and only if} \quad a \to b \in H.$$

It is easy to see that $\{1\}$ and A are normal filters of the pseudo-BL algebra **A**. We remark that if **A** is a BL-algebra, then $\rightsquigarrow = \to$, so the notions of filter and normal filter coincide.

For a filter F of **A** and $a \in A$, let us denote $a \odot F = \{a \odot x \mid x \in F\}$ and $F \odot a = \{x \odot a \mid x \in F\}$.

PROPOSITION 1.8 ([5]). *Let H be a a filter of* **A**. *The following are equivalent*:

(i) *H is a normal filter*;

(ii) *$a \odot H = H \odot a$ for any $a \in A$.*

With any normal filter H of **A** we can associate a congruence relation $\equiv_H$ on **A** by defining $a \equiv_H b$ if and only if $(a \rightsquigarrow b) \odot (b \rightsquigarrow a) \in H$ if and only if $(a \to b) \odot (b \to a) \in H$.

In [5] it is proved that the map $H \mapsto \equiv_H$ is an isomorphism between the lattice of normal filters of **A** and the lattice of congruences of **A**. If we denote by A/H the quotient set $A/_{\equiv_H}$, then A/H becomes a pseudo-BL algebra $\mathbf{A}/H$ with the natural operations induced from those of **A**.

PROPOSITION 1.9 ([5]). *Let H be a normal filter of* **A**. *Then* $\mathbf{A}/H$ *is a pseudo-*BL *chain if and only if H is a prime filter of* **A**.

The following lemma is implicitly contained in [5].

LEMMA 1.10. *Let H be a normal filter of* **A** *and $a, b \in A$. Then*

(i) *$a/H = 1/H$ if and only if $a \in H$*;

(ii) *$a/H = 0/H$ if and only if $a^{\sim} \in H$ if and only if $a^{-} \in H$*;

(iii) *$a/H \leq b/H$ if and only if $a \rightsquigarrow b \in H$ if and only if $a \to b \in H$.*

https://doi.org/10.1017/S144678870000851X Published online by Cambridge University Press

PROOF. (i) $a/H = 1/H$ if and only if $(a \rightsquigarrow 1) \odot (1 \rightsquigarrow a) \in H$ if and only if $1 \odot (1 \rightsquigarrow a) \in H$ if and only if $a \in H$, since $a \rightsquigarrow 1 = 1$ and $1 \rightsquigarrow a = a$, by (3) and (10).

(ii) $a/H = 0/H$ if and only if $(a \rightsquigarrow 0) \odot (0 \rightsquigarrow a) \in H$ if and only if $a^{\sim} \odot 1 \in H$ if and only if $a^{\sim} \in H$. Applying ((N)), $a^{\sim} \in H$ if and only if $a \rightsquigarrow 0 \in H$ if and only if $a \to 0 \in H$ if and only if $a^{-} \in H$.

(iii) By (3) and (i), $a/H \leq b/H$ if and only if $a/H \rightsquigarrow b/H = 1/H$ if and only if $(a \rightsquigarrow b)/H = 1/H$ if and only if $a \rightsquigarrow b \in H$. By (N), we have that $a \rightsquigarrow b \in H$ if and only if $a \to b \in H$. □

If $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism of pseudo-BL algebras, then the *kernel* of h is the set $\mathrm{Ker}(h) = \{a \in A \mid h(a) = 1\}$. For any normal filter H of $\mathbf{A}$, let us denote by $[\,]_H$ the natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/H$, defined by $[\,]_H(a) = a/H$ for any $a \in A$. Then $H = \mathrm{Ker}([\,]_H)$. The following propositions are easily obtained.

PROPOSITION 1.11. *Let $h : \mathbf{A} \to \mathbf{B}$ be a homomorphism of pseudo-*BL *algebras. Then the following properties hold*:

(i) *for any (normal) filter G of $\mathbf{B}$, the set $h^{-1}(G) =_{\mathrm{def}} \{a \in A \mid h(a) \in G\}$ is a (normal) filter of $\mathbf{A}$. Thus, in particular* $\mathrm{Ker}(h)$ *is a normal filter of* $\mathbf{A}$.

(ii) *h is injective if and only if* $\mathrm{Ker}(h) = \{1\}$.

PROPOSITION 1.12. *Let $\mathbf{A}$ be a pseudo-*BL *algebra and H be a normal filter of $\mathbf{A}$.*

(i) *The map $F \overset{\alpha}{\mapsto} [\,]_H(F)$ is an inclusion-preserving bijective correspondence between the filters of $\mathbf{A}$ containing H and the filters of $\mathbf{A}/H$. The inverse map is also inclusion-preserving.*

(ii) *F is a proper filter of $\mathbf{A}$ containing H if and only if $[\,]_H(F)$ is a proper filter of $\mathbf{A}/H$. Hence, there is a bijection between the proper filters of $\mathbf{A}$ containing H and the proper filters of $\mathbf{A}/H$.*

(iii) *There is a bijection between the ultrafilters of $\mathbf{A}$ containing H and the ultrafilters of $\mathbf{A}/H$.*

Let $\mathbf{A}$ be a pseudo-BL algebra and F be a filter of $\mathbf{A}$. We shall use the following notation:

$$F^*_{\sim} = \{a \in A \mid a \leq x^{\sim} \text{ for some } x \in F\} \quad \text{and}$$
$$F^*_{-} = \{a \in A \mid a \leq x^{-} \text{ for some } x \in F\}.$$

REMARK 1.13. Let $\mathbf{A}$ be a pseudo-BL algebra. Then

(i) $F^*_{\sim} = \{a \in A \mid a \odot x = 0 \text{ for some } x \in F\}$;

(i′) $F^*_{-} = \{a \in A \mid x \odot a = 0 \text{ for some } x \in F\}$;

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(ii) $F^*_\sim = \{a \in A \mid a^- \in F\}$;
(ii′) $F^*_- = \{a \in A \mid a^\sim \in F\}$.

PROOF. (i), (i′) Apply (7).

(ii) Let $a \in A$. If $a \le x^\sim$ for some $x \in F$ then, by (13) and (14), we get that $x \le x^{\sim -} \le a^-$. Since F is a filter, it follows that $a^- \in F$. Conversely, suppose that $a^- \in F$. Then, $a \le (a^-)^\sim$, hence $a \in F^*_\sim$.

(ii′) Similar to (ii). □

For any pseudo-BL algebra **A**, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$. Hence, $B(A) = B(L(A))$.

PROPOSITION 1.14 ([5]). *Let* **A** *be a pseudo-*BL *algebra and* $e \in A$. *The following are equivalent*:

(i) $e \in B(A)$;
(ii) $e \odot e = e$ *and* $e = e^{\sim -} = e^{-\sim}$;
(iii) $e \odot e = e$ *and* $e^\sim \rightsquigarrow e = e$;
(iii′) $e \odot e = e$ *and* $e^- \to e = e$;
(iv) $e \vee e^\sim = 1$;
(iv′) $e \vee e^- = 1$.

LEMMA 1.15 ([5]). *Let* **A** *be a pseudo-*BL *algebra and* $e \in B(A)$. *Then*

(i) $\langle e \rangle = \{a \in A \mid e \le a\}$;
(ii) $e \odot a = e \wedge a$ *for any* $a \in A$;
(iii) $e \vee (a \odot b) = (e \vee a) \odot (e \vee b)$ *for any* $a, b \in A$;
(iv) $e^\sim = e^-$ *is the complement of* e.

A pseudo-BL algebra **A** is called *directly indecomposable* if and only if **A** is nontrivial and whenever $\mathbf{A} \cong \mathbf{A}_1 \times \mathbf{A}_2$ then either $\mathbf{A}_1$ or $\mathbf{A}_2$ is trivial. In the sequel, in a similar manner as in [3, Chapter 6.4], we shall give a characterization of directly indecomposable pseudo-BL algebras. Let **A** be a pseudo-BL algebra. For each $x \in A$, let the functions $\rightsquigarrow_x : A \times A \to A$, $\to_x : A \times A \to A$ and $h_x : A \to A$ be defined by $a \rightsquigarrow_x b = x \vee (a \rightsquigarrow b)$, $a \to_x b = x \vee (a \to b)$, and $h_x(a) = x \vee a$.

PROPOSITION 1.16. *Let* **A** *be a pseudo-*BL *algebra and* $e \in B(A)$. *Then*

(i) $\langle \mathbf{e} \rangle = (\langle e \rangle, \wedge, \vee, \odot, \rightsquigarrow_e, \to_e, e, 1)$ *is a pseudo-*BL *algebra*;
(ii) $h_e(A) = \langle e \rangle$;
(iii) h_e *is a homomorphism of pseudo-*BL *algebras from* **A** *onto* $\langle \mathbf{e} \rangle$;
(iv) $\mathrm{Ker}(h_e) = \langle e^- \rangle$;
(v) $\langle \mathbf{e} \rangle$ *is nontrivial if and only if* $e \neq 1$;
(vi) $\langle e \rangle$ *is a subalgebra of* **A** *if and only if* $e = 0$ *if and only if* $\langle e \rangle = A$;

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(vii) $B(\langle e\rangle) = \langle e\rangle \cap B(A)$.

PROOF. (i) By Lemma 1.15 (i), we have that $\langle e\rangle = \{a \in A \mid e \leq a\}$. Let us verify the axioms from the definition of a pseudo-BL algebra.

(A1) It follows immediately that $(\langle e\rangle, \wedge, \vee, e, 1)$ is a bounded lattice.

(A2) Since $\langle e\rangle$ is a filter of **A**, $\langle e\rangle$ is $\odot$-closed and, obviously, $(\langle e\rangle, \odot, 1)$ is a monoid.

(A3) Let $a, b, c \geq e$. If $a \odot b \leq c$, then $a \leq b \rightsquigarrow c \leq e \vee (b \rightsquigarrow c) = b \rightsquigarrow_e c$ and $b \leq a \rightarrow c \leq e \vee (a \rightarrow c) = a \rightarrow_e c$.

Conversely, let us suppose that $a \leq b \rightsquigarrow_e c$, that is, $a \leq e \vee (b \rightsquigarrow c)$. Applying (4), (21), Lemma 1.15 (ii) and (A4), we get that $a \odot b \leq [e \vee (b \rightsquigarrow c)] \odot b = (e \odot b) \vee [(b \rightsquigarrow c) \odot b] = (e \wedge b) \vee (b \wedge c) = e \vee (b \wedge c) = b \wedge c \leq c$.

Now, let us suppose that $b \leq a \rightarrow_e c$, so $b \leq e \vee (a \rightarrow c)$. Then, by (4), (20), Lemma 1.15 (ii) and (A4), $a \odot b \leq a \odot [e \vee (a \rightarrow c)] = (a \odot e) \vee [a \odot (a \rightarrow c)] = (a \wedge e) \vee (a \wedge c) = e \vee (a \wedge c) = a \wedge c \leq c$.

(A4) Let $a, b \geq e$. We have that $(a \rightsquigarrow_e b) \odot a = [e \vee (a \rightsquigarrow b)] \odot a = (e \odot a) \vee [(a \rightsquigarrow b) \odot a] = (e \wedge a) \vee (a \wedge b) = e \vee (a \wedge b) = a \wedge b$ and, similarly, $a \odot (a \rightarrow_e b) = a \odot [e \vee (a \rightarrow b)] = (a \odot e) \vee [a \odot (a \rightarrow b)] = (a \wedge e) \vee (a \wedge b) = a \wedge b$.

(A5) Let $a, b \in A$. By (A5), we get that $(a \rightsquigarrow_e b) \vee (b \rightsquigarrow_e a) = e \vee (a \rightsquigarrow b) \vee e \vee (b \rightsquigarrow a) = e \vee 1 = 1$ and, similarly, $(a \rightarrow_e b) \vee (b \rightarrow_e a) = e \vee (a \rightarrow b) \vee e \vee (b \rightarrow a) = e \vee 1 = 1$. Hence, $(\langle e\rangle, \wedge, \vee, \odot, \rightsquigarrow_e, \rightarrow_e, e, 1)$ is a pseudo-BL algebra.

(ii) For any $a \in \langle e\rangle$, we have that $h_e(a) = e \vee a = a$. Hence, $\langle e\rangle \subseteq h_e(A)$. The other inclusion is obvious.

(iii) Let $a, b \in A$. It follows immediately that $h_e(a \rightsquigarrow b) = e \vee (a \rightsquigarrow b) = a \rightsquigarrow_e b$, $h_e(a \rightarrow b) = e \vee (a \rightarrow b) = a \rightarrow_e b$, $h_e(0) = 0 \vee e = e$, $h_e(1) = e \vee 1 = 1$, $h_e(a \vee b) = e \vee (a \vee b) = h_e(a) \vee h_e(b)$. By (22), $h_e(a \wedge b) = e \vee (a \wedge b) = (e \vee a) \wedge (e \vee b) = h_e(a) \wedge h_e(b)$. Applying Lemma 1.15 (iii), we also get that $h_e(a \odot b) = e \vee (a \odot b) = (e \vee a) \odot (e \vee b) = h_e(a) \odot h_e(b)$.

(iv) If $a \in \mathrm{Ker}(h_e)$, then $h_e(a) = a \vee e = 1$, so $e^- = e^- \wedge (a \vee e) = (e^- \wedge a) \vee 0 = e^- \wedge a$. It follows that $a \geq e^-$, hence $a \in \langle e^-\rangle$. Conversely, if $a \geq e^-$, we get that $h_e(a) = e \vee a \geq e \vee e^- = 1$, hence $h_e(a) = 1$, that is, $a \in \mathrm{Ker}(h_e)$.

(v), (vi) They are obvious.

(vii) Let $a \in \langle e\rangle$, that is, $e \leq a$. If $a \in B(\langle e\rangle)$, then there is $b \geq e$ such that $a \wedge b = e$ and $a \vee b = 1$. Taking $c = b \wedge e^{\sim}$, we get that $a \wedge c = 0$ and $a \vee c = a \vee (b \wedge e^{\sim}) = (a \vee b) \wedge (a \vee e^{\sim}) = 1 \wedge (a \vee e^{\sim}) = a \vee e^{\sim} \geq e \vee e^{\sim} = 1$, by (22) and Lemma 1.15 (iv). Conversely, suppose that $a \in B(A)$, hence there is $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. Let $c = e \vee b$. Then $c \geq e$ and $a \vee c = 1$, $a \wedge c = a \wedge (e \vee b) = (a \wedge e) \vee (a \wedge b) = e \vee 0 = e$. □

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PROPOSITION 1.17. *Let* $\{\mathbf{A}_i\}_{i\in I}$ *be a nonempty family of pseudo*-BL *algebras and let* $\mathbf{P} = \prod_{i\in I} \mathbf{A}_i$. *Then there exists a set* $\{\delta_i \mid i \in I\} \subseteq B(P)$ *satisfying the following conditions*:

(i) $\wedge_{i\in I}\delta_i = 0$;
(ii) $\delta_i \vee \delta_j = 1$, *whenever* $i \neq j$;
(iii) *each* $\mathbf{A}_i$ *is isomorphic to* $\langle \boldsymbol{\delta}_i \rangle$.

PROOF. Similar to the proof of [3, Lemma 6.4.4]. □

PROPOSITION 1.18. *Let* $\mathbf{A}$ *be a pseudo*-BL *algebra and* $e_1, \dots, e_n \in B(A)$, $n \geq 2$, *such that*

(i) $e_1 \wedge \cdots \wedge e_n = 0$; *and*
(ii) $e_i \vee e_j = 1$ *for* $i \neq j$, $i, j = 1, \dots, n$.

Then $\mathbf{A} \cong \langle \mathbf{e}_1 \rangle \times \cdots \times \langle \mathbf{e}_n \rangle$.

PROOF. Similar to the proof of [3, Lemma 6.4.5]. □

PROPOSITION 1.19. *A pseudo*-BL *algebra* $\mathbf{A}$ *is directly indecomposable if and only if* $B(A) = \{0, 1\}$.

PROOF. Similar to the proof of [3, Theorem 6.4.7]. □

It follows immediately that

PROPOSITION 1.20. *Any pseudo*-BL *chain is directly indecomposable.*

PROOF. Let $\mathbf{A}$ be a pseudo-BL chain and $e \in B(A)$. By Proposition 1.14, we get that $e \vee e^{\sim} = 1$. But $e \leq e^{\sim}$ or $e^{\sim} \leq e$, hence $e = 1$ or $e^{\sim} = 1$. By (11), it follows that $e \in \{0, 1\}$. □

In the sequel we shall recall some facts about pseudo-MV algebras, which are non-commutative generalizations of MV-algebras (see [11, 12]). A *pseudo*-MV *algebra* is an algebra $(A, \oplus, ^-, ^\sim, 0, 1)$ with one binary operation $\oplus$, two unary operations $^-, ^\sim$ and two constants 0, 1 such that:

(i) $(A, \oplus, 0)$ is a monoid;
(ii) $a \oplus 1 = 1 \oplus a = a$;
(iii) $1^\sim = 1^- = 0$;
(iv) $(a^- \oplus b^-)^\sim = (a^\sim \oplus b^\sim)^-$;
(v) $a \oplus (a^\sim \odot b) = b \oplus (b^\sim \odot a) = (a \odot b^-) \oplus b = (b \odot a^-) \oplus a$;
(vi) $a \odot (a^- \oplus b) = (a \oplus b^\sim) \odot b$;
(vii) $a^{-\sim} = a$,

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where $a \odot b \overset{\text{def}}{=} (b^- \oplus a^-)^\sim$. Let $\mathbf{A}$ be a pseudo-MV algebra. On A one can define an order relation '$\leq$' by

$$a \leq b \text{ if and only if } a^- \oplus b = 1 \text{ if and only if } b \oplus a^\sim = 1.$$

PROPOSITION 1.21 ([11, Proposition 1.13]). *Let* $\mathbf{A}$ *be a pseudo-*MV *algebra. Then* $(A, \leq)$ *is a lattice in which for all* $a, b \in A$,

$$a \vee b = a \oplus (a^\sim \odot b) = b \oplus (b^\sim \odot a) = (a \odot b^-) \oplus b = (b \odot a^-) \oplus a \quad \textit{and}$$
$$a \wedge b = (a \oplus b^\sim) \odot b = (b \oplus a^\sim) \odot a = a \odot (a^- \oplus b) = b \odot (b^- \oplus a).$$

For any $a \in A$, we define $0a = 0$ and $na = (n-1)a \oplus a$ for $n \in \omega - \{0\}$. The MV-*order* of $a \in A$, in symbols MV-ord(a), is the smallest $n \in \omega$ such that $na = 1$. If no such n exists, then MV-ord$(a) = \infty$.

LEMMA 1.22 ([15, Lemma 14]). *Let* $\mathbf{A}$ *be a pseudo-*MV *algebra. For any* $a \in A$, MV-ord(a^-) = MV-ord$(a^\sim)$.

We shall denote by $D(A)$ the set $\{a \in A \mid \text{MV-ord}(a) = \infty\}$. A pseudo-MV algebra $\mathbf{A}$ is *locally finite* if for all $a \in A$, $a \neq 0$ implies MV-ord$(a) < \infty$. According to [15], a pseudo-MV algebra $\mathbf{A}$ is *strong* if for all $a \in A$, $a^- = a^\sim$. According to [11], an *ideal* of $\mathbf{A}$ is is a nonempty subset I of A such that for all $a, b \in A$,

(i) if $a, b \in I$, then $a \oplus b \in I$;
(ii) if $b \in I$ and $a \leq b$, then $a \in I$.

An ideal I is *proper* if $I \neq A$. A proper ideal of $\mathbf{A}$ is called a *maximal ideal* if it is not contained in any other proper ideal. An ideal H of a pseudo-MV algebra $\mathbf{A}$ is called *normal* (see [12]) if for all $a, b \in A$, $a^\sim \odot b \in H$ if and only if $b \odot a^- \in H$.

LEMMA 1.23 ([12, Lemma 3.2]). *Let* H *be a normal ideal of* $\mathbf{A}$ *and* $a \in A$. *Then* $a \in H$ *if and only if* $a^{=} \in H$ *if and only if* $a^{\approx} \in H$.

PROPOSITION 1.24 ([4, Corollary 2.34]). *A pseudo-*BL *algebra* $\mathbf{A}$ *is a pseudo-*MV *algebra if and only if* $a^{\sim -} = a^{-\sim} = a$ *for all* $a \in A$.

Following [2], in [15] local pseudo-MV algebras were defined and some classes of local pseudo-MV algebras were studied. Thus, a pseudo-MV algebra is *local* if and only if it has a unique maximal ideal and a local pseudo-MV algebra is:

- *perfect* if for any $a \in A$, MV-ord$(a) < \infty$ if and only if MV-ord$(a^-) = \infty$;
- *singular* if there exist $a, b \in A$ such that MV-ord$(a) < \infty$, MV-ord$(b) < \infty$ and MV-ord$(a \odot b) = \infty$.

By Lemma 1.22, it follows that a local pseudo-MV algebra **A** is perfect if and only if for any $a \in A$, MV-ord$(a) < \infty$ if and only if MV-ord$(a^{\sim}) = \infty$

PROPOSITION 1.25 ([15]). *Every local pseudo-*MV *algebra is either perfect or singular. There is no local pseudo-*MV *algebra which is both perfect and singular.*

PROPOSITION 1.26 ([15]). *Every locally finite pseudo-*MV *algebra different from* $\{0, 1\}$ *is singular.*

2. Local pseudo-BL algebras

Local rings play an important role in ring theory. On the other hand, the study of local objects became a standard problem for other classes of structures (MV-algebras [2], BL-algebras [19], pseudo-MV algebras [15]). In this section we shall study local pseudo-BL algebras.

A pseudo-BL algebra is called *local* if and only if it has a unique ultrafilter.

LEMMA 2.1. *Let* **A** *be a local pseudo-*BL *algebra. Then*

(i) *any proper filter of* **A** *is included in the unique ultrafilter of* **A**;
(ii) $A_0^{\sim}$, A_0^{-} *are included in the unique ultrafilter of* **A**.

PROOF. (i) Apply Proposition 1.4 and the fact that **A** has a unique ultrafilter.
(ii) Apply Lemma 1.6 and (i). □

In the sequel, we shall use the following notation:

$$D(A) = \{a \in A \mid \text{ord}(a) = \infty\} \quad \text{and} \quad D(A)^* = \{a \in A \mid \text{ord}(a) < \infty\}.$$

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.

PROPOSITION 2.2. *Let* **A** *be a pseudo-*BL *algebra. The following are equivalent*:

(i) $D(A)$ *is a filter of* **A**;
(ii) $D(A)$ *is a proper filter of* **A**;
(iii) **A** *is local*;
(iv) $D(A)$ *is the unique ultrafilter of* **A**;
(v) *for all* $a, b \in A$, $\text{ord}(a \odot b) < \infty$ *implies* $\text{ord}(a) < \infty$ *or* $\text{ord}(b) < \infty$.

PROOF. (i) if and only if (ii). We have that $\text{ord}(0) = 1$, hence $0 \notin D(A)$.

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(i) implies (v). Let $a, b \in A$ such that $\operatorname{ord}(a \odot b) < \infty$, so $a \odot b \notin D(A)$. Since $D(A)$ is a filter of $\mathbf{A}$, we get that $a \notin D(A)$ or $b \notin D(A)$. Hence, $\operatorname{ord}(a) < \infty$ or $\operatorname{ord}(b) < \infty$.

(v) implies (i). Since $1 \in D(A)$, we have that $D(A)$ is nonempty. Let $a, b \in D(A)$, that is $\operatorname{ord}(a) = \operatorname{ord}(b) = \infty$. It follows that $\operatorname{ord}(a \odot b) = \infty$, that is $a \odot b \in D(A)$. If $a \leq b$ and $a \in D(A)$, then $a^n > 0$ for all $n \in \omega$. Since $a^n \leq b^n$, we have that $b^n > 0$ for all $n \in \omega$. That is, $\operatorname{ord}(b) = \infty$, hence $b \in D(A)$. Thus, we have proved that $D(A)$ is a filter of $\mathbf{A}$.

(iv) implies (iii). It is immediate.

(iii) implies (iv). Let U be the unique ultrafilter of $\mathbf{A}$. Applying Lemma 2.1 (i) and Lemma 1.7 (i), we get that $a \in U$ if and only if $\langle a \rangle \subseteq U$ if and only if $\langle a \rangle$ is proper if and only if $\operatorname{ord}(a) = \infty$ if and only if $a \in D(A)$. Hence, $U = D(A)$.

(iv) implies (i). It is obvious.

(i) implies (iv). Since $0 \notin D(A)$, we have that $D(A)$ is proper. Let F be a proper filter of $\mathbf{A}$. If $a \in F$, then $\langle a \rangle \subseteq F$, so $\langle a \rangle$ is a proper filter of $\mathbf{A}$. Applying Lemma 1.7 (i), it follows that $\operatorname{ord}(a) = \infty$, hence $a \in D(A)$. Thus, we have got that any proper filter F of $\mathbf{A}$ is included in $D(A)$. From this fact it follows that $D(A)$ is the unique ultrafilter of $\mathbf{A}$. □

COROLLARY 2.3. *Let* $\mathbf{A}$ *be a local pseudo*-BL *algebra. Then*

(i) *for any* $a \in A$, $\operatorname{ord}(a) < \infty$ *or* $(\operatorname{ord}(a^{\sim}) < \infty$ *and* $\operatorname{ord}(a^{-}) < \infty)$;

(ii) $D(A)^{*}_{\sim} \subseteq D(A)^{*}$ *and* $D(A)^{*}_{-} \subseteq D(A)^{*}$;

(iii) $D(A) \cap D(A)^{*}_{\sim} = D(A) \cap D(A)^{*}_{-} = \emptyset$.

PROOF. (i) Let $a \in A$. By (9), we have that $a^{\sim} \odot a = a \odot a^{-} = 0$, so $\operatorname{ord}(a^{\sim} \odot a) = \operatorname{ord}(a \odot a^{-}) = \operatorname{ord}(0) = 1 < \infty$. Apply now Proposition 2.2 (v) to get (i).

(ii) Let $a \in D(A)^{*}_{\sim}$, so there is $x \in D(A)$ such that $a \leq x^{\sim}$. Since $\operatorname{ord}(x) = \infty$, applying (i), we get that $\operatorname{ord}(x^{\sim}) < \infty$. Applying now Lemma 1.7(ii), we get that $\operatorname{ord}(a) < \infty$. Hence, $a \in D(A)^{*}$. We obtain similarly that $D(A)^{*}_{-} \subseteq D(A)^{*}$.

(iii) Apply (ii) and the fact that $D(A) \cap D(A)^{*} = \emptyset$. □

PROPOSITION 2.4. *Any pseudo*-BL *chain is a local pseudo*-BL *algebra.*

PROOF. Let $\mathbf{A}$ be a pseudo-BL chain. We apply Proposition 2.2 (v) to obtain that $\mathbf{A}$ is local. Let $a, b \in A$ such that $\operatorname{ord}(a \odot b) < \infty$. Since $\mathbf{A}$ is a chain, we have that $a \leq b$ or $b \leq a$. Suppose that $a \leq b$. Then $a \odot a \leq a \odot b$, so, by Lemma 1.7 (ii), we get that $\operatorname{ord}(a \odot a) < \infty$, hence $\operatorname{ord}(a) < \infty$. Similarly, from $b \leq a$ it follows that that $\operatorname{ord}(b) < \infty$. □

A proper normal filter P of a pseudo-BL algebra $\mathbf{A}$ is called *primary* if for all

$a, b \in A$,

$$((a \odot b)^n)^{\sim} \in P \text{ for some } n \in \omega \text{ implies } (a^m)^{\sim} \in P \text{ or } (b^m)^{\sim} \in P \text{ for some } m \in \omega.$$

Applying the definition of a normal filter, we get that a proper normal filter P of $\mathbf{A}$ is primary if and only if for all $a, b \in A$, $((a \odot b)^n)^- \in P$ for some $n \in \omega$ implies $(a^m)^- \in P$ or $(b^m)^- \in P$ for some $m \in \omega$.

REMARK 2.5. Suppose that $\mathbf{A}$ is a BL-algebra and let P be a proper filter of $\mathbf{A}$. The following are equivalent:

(i) P is primary;
(ii) for all $a, b \in A$, $(a \odot b)^- \in P$ implies $(a^m)^- \in P$ or $(b^m)^- \in P$ for some $m \in \omega$.

PROOF. (i) implies (ii). It follows immediately from the definition of a primary filter.

(ii) implies (i). Let $a, b \in A$ such that $((a \odot b)^n)^- \in P$ for some $n \in \omega$. Since $\odot$ is commutative, we get that $((a \odot b)^n)^- = (a^n \odot b^n)^- \in P$. Applying now (ii), it follows that there is $p \in \omega$ such that $(a^{np})^- \in P$ or $(b^{np})^- \in P$. Hence, letting $m = np$, we have that $(a^m)^- \in P$ or $(b^m)^- \in P$. □

Hence, in the case that $\mathbf{A}$ is a BL-algebra, the notion of primary filter defined here coincides with the notion of primary filter defined in [19].

PROPOSITION 2.6. *Let* $\mathbf{A}$ *be a pseudo*-BL *algebra and* P *be a proper normal filter of* $\mathbf{A}$. *The following are equivalent*:

(i) $\mathbf{A}/P$ *is a local pseudo*-BL *algebra*;
(ii) P *is a primary filter of* $\mathbf{A}$.

PROOF. Applying Proposition 2.2 (v) and Lemma 1.10 (ii), we have that $\mathbf{A}/P$ is local if and only if for all $a, b \in A$, $\text{ord}(a/P \odot b/P) < \infty$ implies $\text{ord}(a/P) < \infty$ or $\text{ord}(b/P) < \infty$ if and only if for all $a, b \in A$, $(a/P \odot b/P)^n = 0/P$ for some $n \in \omega$ implies $(a/P)^m = 0/P$ or $(b/P)^m = 0/P$ for some $m \in \omega$ if and only if for all $a, b \in A$, $((a \odot b)^n)/P = 0/P$ for some $n \in \omega$ implies $a^m/P = 0/P$ or $b^m/P = 0/P$ for some $m \in \omega$ if and only if for all $a, b \in A$, $((a \odot b)^n)^{\sim} \in P$ for some $n \in \omega$ implies $(a^m)^{\sim} \in P$ or $(b^m)^{\sim} \in P$ for some $m \in \omega$ if and only if P is primary. □

PROPOSITION 2.7. *Any prime normal filter of a pseudo*-BL *algebra* $\mathbf{A}$ *is primary*.

PROOF. Let P be a prime normal filter of $\mathbf{A}$. Applying Proposition 1.9, we get that $\mathbf{A}/P$ is a pseudo-BL chain, hence $\mathbf{A}/P$ is local, by Proposition 2.4. Apply now Proposition 2.6 to get that P is primary. □

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PROPOSITION 2.8. *Let* **A** *be a pseudo-*BL *algebra. A proper normal filter of* **A** *is primary if and only if it is contained in a unique ultrafilter of* **A**.

PROOF. Let H be a proper normal filter of **A**. By Proposition 2.6, H is primary if and only if $\mathbf{A}/H$ is a local algebra if and only if $\mathbf{A}/H$ has a unique ultrafilter. Applying Proposition 1.12 (iii), there is a bijection between the set of ultrafilters of $\mathbf{A}/H$ and the set of ultrafilters of **A** that contain H. Hence, H is primary if and only if there is a unique ultrafilter of **A** that contains H. □

PROPOSITION 2.9. *Let* **A** *be a pseudo-*BL *algebra. The following are equivalent*:

(i) **A** *is local*;
(ii) *any proper normal filter of* **A** *is primary*;
(iii) $\{1\}$ *is a primary filter of* **A**.

PROOF. (i) implies (ii). Let H be a proper normal filter of **A**. Since **A** is local, by Lemma 2.1 (i) and Proposition 2.2 (iv) it follows that $D(A)$ is the unique ultrafilter of **A** containing H. Applying Proposition 2.8, we get that H is primary.

(ii) implies (iii). Apply the fact that $\{1\}$ is a proper normal filter of **A**.

(iii) implies (i). Since $\{1\}$ is a primary filter of **A**, by Proposition 2.6, we get that $\mathbf{A}/\{1\}$ is local. But $\mathbf{A} \cong \mathbf{A}/\{1\}$, hence **A** is local. □

PROPOSITION 2.10. *Any local pseudo-*BL *algebra is directly indecomposable.*

PROOF. Let **A** be a local pseudo-BL algebra. We shall prove that $B(A) = \{0, 1\}$ and then apply Proposition 1.19. Let $e \in B(A)$. Applying Corollary 2.3 (i), we get that $\mathrm{ord}(e) < \infty$ or $\mathrm{ord}(e^{\sim}) < \infty$, that is, there is $n \in \omega - \{0\}$ such that $e^n = 0$ or $(e^{\sim})^n = 0$. But $e^n = e$ and $(e^{\sim})^n = e^{\sim}$, by Proposition 1.14 (ii) and the fact that $e^{\sim}$ is the complement of e, so $e, e^{\sim} \in B(A)$. It follows that $e = 0$ or $e^{\sim} = 0$. By Proposition 1.14 (ii) and (11), from $e^{\sim} = 0$ we get that $e = (e^{\sim})^{-} = 0^{-} = 1$. That is, $e \in \{0, 1\}$. Hence, $B(A) = \{0, 1\}$. □

3. Good pseudo-BL algebras

A *good* pseudo-BL algebra is a pseudo-BL algebra **A** satisfying the following identity

$$(*) \qquad a^{\sim -} = a^{-\sim}.$$

Pseudo-MV algebras are particular cases of good pseudo-BL algebras. In [5] it is proved that any pseudo-product algebra is also a good pseudo-BL algebra. A *strong*

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pseudo-BL algebra is a pseudo-BL algebra **A** such that $a^{\sim} = a^{-}$ for all $a \in A$. Obviously, every strong pseudo-BL algebra is a good pseudo-BL algebra.

In the sequel, if not otherwise specified, **A** is a good pseudo-BL algebra. Let us consider the subset $M(A) = \{a \in A \mid a^{\sim -} = a^{-\sim} = a\}$.

LEMMA 3.1. *Let* **A** *be a good pseudo-*BL *algebra. Then*

(i) $0, 1 \in M(A)$;

(ii) $a^{\sim}, a^{-} \in M(A)$ *for all* $a \in A$;

(iii) *if* $a, b \in M(A)$, *then* $a \rightsquigarrow b = b^{\sim} \to a^{\sim}$ *and* $a \to b = b^{-} \rightsquigarrow a^{-}$;

(iv) *if* $a, b \in M(A)$, *then* $(a^{\sim} \odot b^{\sim})^{-} = (a^{-} \odot b^{-})^{\sim} = a^{-} \rightsquigarrow b = b^{\sim} \to a$.

PROOF. (i) Apply (11) and (12).

(ii) Let $a \in A$. Applying $(*)$ for $a^{\sim}$ and a^{-} and (16), we have that $(a^{\sim})^{\sim -} = (a^{\sim})^{-\sim} = a^{\sim}$ and $(a^{-})^{-\sim} = (a^{-})^{\sim -} = a^{-}$. It follows that $a^{\sim}, a^{-} \in M(A)$.

(iii), (iv) See [4, Lemma 2.31]. □

For any $a, b \in A$, let us define $a \oplus b \overset{\text{def}}{=} (b^{\sim} \odot a^{\sim})^{-}$.

LEMMA 3.2. *Let* **A** *be a good pseudo-*BL *algebra. Then*

(i) $a \oplus b \in M(A)$ *for any* $a, b \in A$;

(ii) *if* $a, b \in M(A)$, *then* $a \oplus b = (b^{\sim} \odot a^{\sim})^{-} = (b^{-} \odot a^{-})^{\sim} = b^{-} \rightsquigarrow a = a^{\sim} \to b$;

(iii) *if* $a, b \in M(A)$, *then* $a \oplus b^{-} = a^{\sim} \to b^{-}$, $a \oplus b^{\sim} = b \rightsquigarrow a$, $a^{-} \oplus b = a \to b$ *and* $a^{\sim} \oplus b = b^{-} \rightsquigarrow a^{\sim}$;

(iv) *if* $a, b \in M(A)$, *then* $a^{\sim} \oplus b^{\sim} = (b \odot a)^{\sim}$ *and* $a^{-} \oplus b^{-} = (b \odot a)^{-}$.

PROOF. (i) Apply Lemma 3.1 (ii).

(ii) Apply Lemma 3.1 (iv).

(iii) Apply (ii).

(iv) By (iii), (1) and (2), we have that $a^{\sim} \oplus b^{\sim} = b \rightsquigarrow a^{\sim} = b \rightsquigarrow (a \rightsquigarrow 0) = (b \odot a) \rightsquigarrow 0 = (b \odot a)^{\sim}$ and $a^{-} \oplus b^{-} = a \to b^{-} = a \to (b \to 0) = (b \odot a) \to 0 = (b \odot a)^{-}$. □

The following proposition extends a result from [19].

PROPOSITION 3.3. *Let* **A** *be a good pseudo-*BL *algebra. The structure* $\mathbf{M(A)} = (M(A), \oplus, ^{-}, ^{\sim}, 0, 1)$ *is a pseudo-*MV *algebra. The order on* **A** *agrees with the one of* $\mathbf{M(A)}$, *defined by* $a \le_{M(A)} b$ *if and only if* $a^{-} \oplus b = 1$.

PROOF. By Lemma 3.1 and Lemma 3.2, it follows that the operations $\oplus, ^{-}, ^{\sim}$ are well defined on $M(A)$ and that $0, 1 \in M(A)$. Let us denote by $\odot_{M(A)}$ the product on $M(A)$. Hence, for all $a, b \in M(A)$, we have that $a \odot_{M(A)} b = (b^{-} \oplus a^{-})^{\sim} =$

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$(b^{\sim} \oplus a^{\sim})^{-} \in M(A)$. We shall verify the axioms from the definition of a pseudo-MV algebra. In the proof we use Lemma 3.1 and Lemma 3.2. Let $a, b, c \in M(A)$.

(i) We have that $(a \oplus b) \oplus c = (b^{\sim} \odot a^{\sim})^{-} \oplus (c^{\sim})^{-} = (c^{\sim} \odot (b^{\sim} \odot a^{\sim}))^{-} = ((c^{\sim} \odot b^{\sim}) \odot a^{\sim})^{-} = (a^{\sim})^{-} \oplus (c^{\sim} \odot b^{\sim})^{-} = a \oplus (b \oplus c)$. We also get that $a \oplus 0 = (0^{\sim} \odot a^{\sim})^{-} = (1 \odot a^{\sim})^{-} = a^{\sim -} = a$. Similarly, $0 \oplus a = a$. Hence $(M(A), \oplus, 0)$ is a monoid.

(ii) By (8), (11) and (12), $a \oplus 1 = (1^{\sim} \odot a^{\sim})^{-} = (0 \odot a^{\sim})^{-} = 0^{-} = 1$. We obtain $1 \oplus a = 1$ similarly.

(iii) Apply (12).

(iv) By (*), $(a^{-} \oplus b^{-})^{\sim} = (b \odot a)^{-\sim} = (b \odot a)^{\sim -} = (a^{\sim} \oplus b^{\sim})^{-}$.

(v) We have to prove that $a \oplus (a^{\sim} \odot_{M(A)} b) = b \oplus (b^{\sim} \odot_{M(A)} a) = (a \odot_{M(A)} b^{-}) \oplus b = (b \odot_{M(A)} a^{-}) \oplus a$. Applying (18) and (A4), we get that

$$\begin{aligned} a \oplus (a^{\sim} \odot_{M(A)} b) &= a \oplus (b^{-} \oplus a)^{\sim} = (a^{-})^{\sim} \oplus (b^{-} \oplus a)^{\sim} \\ &= ((b^{-} \oplus a) \odot a^{-})^{\sim} = ((b \to a) \odot a^{-})^{\sim} \\ &= ((a^{-} \rightsquigarrow b^{-}) \odot a^{-})^{\sim} = (a^{-} \wedge b^{-})^{\sim} \\ &= a^{-\sim} \vee b^{-\sim} = a \vee b \end{aligned}$$

and

$$\begin{aligned} (b \odot_{M(A)} a^{-}) \oplus a &= (a \oplus b^{\sim})^{-} \oplus a = (a \oplus b^{\sim})^{-} \oplus (a^{\sim})^{-} \\ &= (a^{\sim} \odot (a \oplus b^{\sim}))^{-} = (a^{\sim} \odot (b \rightsquigarrow a))^{-} \\ &= (a^{\sim} \odot (a^{\sim} \to b^{\sim}))^{-} = (a^{\sim} \wedge b^{\sim})^{-} \\ &= a^{\sim -} \vee b^{\sim -} = a \vee b. \end{aligned}$$

Similarly we get $b \oplus (b^{\sim} \odot_{M(A)} a) = (a \odot_{M(A)} b^{-}) \oplus b = b \vee a = a \vee b$.

(vi)
$$\begin{aligned} a \odot_{M(A)} (a^{-} \oplus b) &= a \odot_{M(A)} (a \to b) = ((a \to b)^{-} \oplus a^{-})^{\sim} \\ &= (a \odot (a \to b))^{-\sim} = (a \wedge b)^{-\sim} = (b \wedge a)^{-\sim} \\ &= ((b \rightsquigarrow a) \odot b)^{-\sim} = (b^{-} \oplus (b \rightsquigarrow a)^{-})^{\sim} \\ &= (b \rightsquigarrow a) \odot_{M(A)} b = (a \oplus b^{\sim}) \odot_{M(A)} b. \end{aligned}$$

(vii) It follows from the definition of $M(A)$.

Hence, **M(A)** is a pseudo-MV algebra. By Lemma 3.2 (iii), we have that for all $a, b \in M(A)$, $a \leq_{M(A)} b$ if and only if $b \oplus a^{\sim} = 1$ if and only if $a \rightsquigarrow b = 1$ if and only if $a \leq b$. □

As a consequence of this proposition we obtain [4, Corollary 2.34]:

COROLLARY 3.4. *A pseudo-BL algebra* **A** *is a pseudo-MV algebra if and only if* $a^{\sim -} = a^{-\sim} = a$ *for all* $a \in A$.

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REMARK 3.5. Let **A** be a good pseudo-BL algebra. For any $a, b \in M(A)$,

$$a \odot_{M(A)} b = (b^{\sim} \oplus a^{\sim})^{-} = (b^{-} \oplus a^{-})^{\sim} = (a \odot b)^{\sim -} = (a \odot b)^{-\sim}.$$

PROOF. Apply the definitions of $\oplus$ and $\odot_{M(A)}$. □

Since, $a, b \in M(A)$ does not imply $a \odot b \in M(A)$, it follows that, generally, $(a \odot b)^{-\sim} \neq a \odot b$. Hence, the product on the pseudo-MV algebra **M(A)** does not coincide with the product on the pseudo-BL algebra **A**. In the case of BL-algebras, the product is the same (see [19]).

PROPOSITION 3.6. *Let* **A** *be a good pseudo*-BL *algebra. Then* **A** *is a strong pseudo*-BL *algebra if and only if* **M(A)** *is a strong pseudo*-MV *algebra.*

PROOF. If $a^{\sim} = a^{-}$ for all $a \in A$, then $a^{\sim} = a^{-}$ for all $a \in M(A)$. Conversely, suppose that $a^{\sim} = a^{-}$ for all $a \in M(A)$. Let $a \in A$. By (16) and (*), $a^{\sim} = a^{\sim-\sim} = (a^{\sim-})^{\sim} = (a^{-\sim})^{\sim}$. But, by Lemma 3.1 (ii), we have that $a^{-\sim} \in M(A)$, so $(a^{-\sim})^{\sim} = (a^{-\sim})^{-} = a^{-}$, by (16). Thus, for all $a \in A$, we have that $a^{\sim} = a^{-}$. □

Let **A** be a good pseudo-BL algebra. Since, by Lemma 3.1 (ii), $a^{-}, a^{\sim} \in M(A)$ for any $a \in A$, we can define the maps $\varphi_1 : A \to M(A)$ by $\varphi_1(a) = a^{\sim}$ for any $a \in A$, and $\varphi_2 : A \to M(A)$ by $\varphi_2(a) = a^{-}$ for any $a \in A$.

LEMMA 3.7. *Let* **A** *be a good pseudo*-BL *algebra. The following properties hold for all* $a, b \in A$:

(i) φ_1, φ_2 *are onto*;
(ii) $\varphi_1(a \vee b) = \varphi_1(a) \wedge \varphi_1(b)$ *and* $\varphi_2(a \vee b) = \varphi_2(a) \wedge \varphi_2(b)$;
(iii) $\varphi_1(a \wedge b) = \varphi_1(a) \vee \varphi_1(b)$ *and* $\varphi_2(a \wedge b) = \varphi_2(a) \vee \varphi_2(b)$;
(iv) $\varphi_1(a) \leq \varphi_1(b)$ *if and only if* $\varphi_2(a) \leq \varphi_2(b)$;
(v) $a \leq b$ *implies* $\varphi_1(a) \geq \varphi_1(b)$ *and* $\varphi_2(a) \geq \varphi_2(b)$;
(vi) $\varphi_1(a) = 1$ *if and only if* $\varphi_2(a) = 1$ *if and only if* $a = 0$;
(vii) $\varphi_1(1) = \varphi_2(1) = 0$;
(viii) $\varphi_1(a) = 0$ *if and only if* $\varphi_2(a) = 0$;
(ix) $\varphi_1(a \odot b) = \varphi_1(b) \oplus \varphi_1(a)$ *and* $\varphi_2(a \odot b) = \varphi_2(b) \oplus \varphi_2(a)$;
(x) *for any* $n \in \omega$, $\varphi_1(a^n) = n\varphi_1(a)$ *and* $\varphi_2(a^n) = n\varphi_2(a)$.

PROOF. (i) Let $a \in M(A)$. Then $a = a^{-\sim} = \varphi_1(a^{-})$ and $a = a^{\sim-} = \varphi_2(a^{\sim})$.
(ii) Apply (18).
(iii) Apply (19).
(iv) Suppose that $\varphi_1(a) \leq \varphi_1(b)$, that is, $a^{\sim} \leq b^{\sim}$. Applying (13) and (16), it follows that $b^{-\sim} = b^{\sim-} \leq a^{\sim-} = a^{-\sim}$, so $a^{-} = a^{-\sim-} \leq b^{-\sim-} = b^{-}$. Hence, $\varphi_2(a) \leq \varphi_2(b)$. We prove similarly that $\varphi_2(a) \leq \varphi_2(b)$ implies $\varphi_1(a) \leq \varphi_1(b)$.

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(v) Apply (13).

(vi) Apply (11).

(vii) Apply (12).

(viii) Suppose that $a^{\sim} = 0$, so $a^{-\sim} = a^{\sim-} = 1$, hence, $a^{-} = a^{-\sim-} = 0$. We get similarly that $a^{-} = 0$ implies $a^{\sim} = 0$.

(ix) Apply Lemma 3.2 (iv).

(x) By induction on n. For $n = 0$, we have that $a^0 = 1$, so $\varphi_1(1) = 0$ and $0\varphi_1(a) = 0$. Suppose that $\varphi_1(a^n) = n\varphi_1(a)$. By (ix), it follows that $\varphi_1(a^{n+1}) = \varphi_1(a^n \odot a) = \varphi_1(a) \oplus \varphi_1(a^n) = \varphi_1(a) \oplus n\varphi_1(a) = (n+1)\varphi_1(a)$. Similarly for φ_2. □

LEMMA 3.8. *Let* **A** *be a nontrivial good pseudo*-BL *algebra. Then*

(i) $A_0^{-} = A_0^{\sim} \overset{\text{not}}{=} A_0$;

(ii) *if* $a \in A_0$, *then* $\mathrm{ord}(a) = \infty$;

(iii) *for any* $a \in A$, $\mathrm{ord}(a) = \text{MV-ord}(\varphi_1(a)) = \text{MV-ord}(\varphi_2(a))$;

(iv) *for any* $a \in A$, $\mathrm{ord}(a^{\sim}) = \mathrm{ord}(a^{-})$;

(v) $\varphi_1(D(A)) = \varphi_2(D(A)) = D(M(A))$ *and* $\varphi_1^{-1}(D(M(A))) = \varphi_2^{-1}(D(M(A))) = D(A)$.

PROOF. (i) Apply Lemma 3.7 (viii).

(ii) Suppose that there is $n \in \omega$ such that $a^n = 0$. Then, applying Lemma 3.7 (vi) and (x), we get that $\varphi_1(a^n) = \varphi_1(0) = 1$ and $\varphi_1(a^n) = n\varphi_1(a) = na^{\sim} = n0 = 0$. We get that $0 = 1$, a contradiction. Hence, $a^n \neq 0$ for all $n \in \omega$, so $\mathrm{ord}(a) = \infty$.

(iii) Let $a \in A$ and $n \in \omega$. By Lemma 3.7 (vi), we have that $a^n = 0$ if and only if $\varphi_1(a^n) = 1$ if and only if $n\varphi_1(a) = 1$. Hence, $\mathrm{ord}(a) = \text{MV-ord}(\varphi_1(a))$. Similarly for φ_2.

(iv) Let $a \in A$. Applying (i) and (∗), we get that $\mathrm{ord}(a^{\sim}) = \text{MV-ord}(\varphi_2(a^{\sim})) = \text{MV-ord}(a^{\sim-}) = \text{MV-ord}(a^{-\sim}) = \text{MV-ord}(\varphi_1(a^{-})) = \mathrm{ord}(a^{-})$.

(v) Apply (iii) and the fact that φ_1, φ_2 are onto. □

LEMMA 3.9. *Let* **A** *be a good pseudo*-BL *algebra. Suppose that I is an ideal of* **M(A)** *and F is a filter of* **A**. *Then*

(i) $\varphi_1^{-1}(I), \varphi_2^{-1}(I)$ *are filters of* **A**;

(ii) $\varphi_1(F), \varphi_2(F)$ *are ideals of* **M(A)**;

(iii) $F \subseteq \varphi_1^{-1}(\varphi_1(F))$ *and* $F \subseteq \varphi_2^{-1}(\varphi_2(F))$;

(iv) $I = \varphi_1(\varphi_1^{-1}(I))$ *and* $I = \varphi_2(\varphi_2^{-1}(I))$;

(v) *I is proper if and only if $\varphi_1^{-1}(I)$ is proper if and only if $\varphi_2^{-1}(I)$ is proper;*

(vi) *F is proper if and only if $\varphi_1(F)$ is proper if and only if $\varphi_2(F)$ is proper;*

(vii) *if F is an ultrafilter of* **A**, *then* $F = \varphi_1^{-1}(\varphi_1(F))$ *and* $F = \varphi_2^{-1}(\varphi_2(F))$;

(viii) *if I is a maximal ideal of* **M(A)**, *then* $\varphi_1^{-1}(I), \varphi_2^{-1}(I)$ *are ultrafilters of* **A**;

(ix) *if F is an ultrafilter of* **A**, *then* $\varphi_1(F), \varphi_2(F)$ *are maximal ideals of* **M(A)**.

PROOF. (i) Let us prove that $\varphi_1^{-1}(I)$ is a filter of $\mathbf{A}$. Since $\varphi_1(1) = 0 \in I$, we have $1 \in \varphi_1^{-1}(I)$. Let $a_1, a_2 \in \varphi_1^{-1}(I)$. It follows that $\varphi_1(a_1), \varphi_1(a_2) \in I$, so $\varphi_1(a_1 \odot a_2) = \varphi_1(a_2) \oplus \varphi_1(a_1) \in I$. Hence, $a_1 \odot a_2 \in \varphi_1^{-1}(I)$. Let $a_1 \in \varphi_1^{-1}(I)$, $a_2 \in A$ be such that $a_1 \le a_2$. By Lemma 3.7 (v), we get $\varphi_1(a_2) \le \varphi_1(a_1) \in I$, so $\varphi_1(a_2) \in I$, that is, $a_2 \in \varphi_1^{-1}(I)$. Thus, we have proved that $\varphi_1^{-1}(I)$ is a filter of $\mathbf{A}$. We get similarly that $\varphi_2^{-1}(I)$ is a filter of $\mathbf{A}$.

(ii) Let us prove that $\varphi_1(F)$ is an ideal of $\mathbf{M}(\mathbf{A})$. We have that $0 = \varphi_1(1) \in \varphi_1(F)$. Let $b_1, b_2 \in \varphi_1(F)$. That is, there are $a_1, a_2 \in F$ such that $b_1 = \varphi_1(a_1)$ and $b_2 = \varphi_1(a_2)$. We have $a_2 \odot a_1 \in F$ and $b_1 \oplus b_2 = \varphi_1(a_2 \odot a_1) \in \varphi_1(F)$. Let $b_1, b_2 \in M(A)$ be such that $b_1 \le b_2$ and $b_2 \in \varphi_1(F)$. It follows that $b_2 = \varphi_1(a_2)$ with $a_2 \in F$ and, since φ_1 is onto, there is $a \in A$ such that $\varphi_1(a) = b_1$. Let $a_1 = a \vee a_2$. Then $a_2 \le a_1$ and $a_2 \in F$, so $a_1 \in F$ and $\varphi_1(a_1) = \varphi_1(a) \wedge \varphi_1(a_2) = b_1 \wedge b_2 = b_1$, by Lemma 3.7 (iii). Hence, $b_1 \in \varphi_1(F)$. We obtain in the same manner that $\varphi_2(F)$ is an ideal of $\mathbf{M}(\mathbf{A})$.

(iii) It is obvious.

(iv) It follows from the fact that φ_1 and φ_2 are onto.

(v) I is not proper if and only if $1 \in I$ if and only if $\varphi_1(0) \in I$ if and only if $0 \in \varphi_1^{-1}(I)$ if and only if $\varphi_1^{-1}(I)$ is not proper.

(vi) If $0 \in F$, then $1 = \varphi_1(0) \in \varphi_1(F)$. Suppose that $1 \in \varphi_1(F)$. Then, there is $a \in F$ such that $\varphi_1(a) = 1$. Applying Lemma 3.7 (vi), we get that $a = 0$, hence $0 \in F$.

(vii) Suppose that F is an ultrafilter of $\mathbf{A}$. Then, by (v) and (vi), $\varphi_1^{-1}(\varphi_1(F))$ is a proper filter of $\mathbf{A}$ and, by (iii), $F \subseteq \varphi_1^{-1}(\varphi_1(F))$. Since F is ultrafilter, we get that $F = \varphi_1^{-1}(\varphi_1(F))$.

(viii) Suppose that $\varphi_1^{-1}(I) \subseteq F$, where F is a proper filter of $\mathbf{A}$. It follows that $I = \varphi_1(\varphi_1^{-1}(I)) \subseteq \varphi_1(F)$. Since $\varphi_1(F)$ is proper, we get that $I = \varphi_1(F)$, so $\varphi_1^{-1}(I) = \varphi_1^{-1}(\varphi_1(F)) \supseteq F$. Hence, $\varphi_1^{-1}(I) = F$.

(ix) Suppose that $\varphi_1(F) \subseteq I$, where I is a proper ideal of $\mathbf{M}(\mathbf{A})$. It follows that $F = \varphi_1^{-1}(\varphi(F)) \subseteq \varphi_1^{-1}(I)$. Since $\varphi_1^{-1}(I)$ is proper, we get that $F = \varphi_1^{-1}(I)$, so $\varphi_1(F) = \varphi_1(\varphi_1^{-1}(I)) = I$. □

The next result is a consequence of the above proposition.

PROPOSITION 3.10. *The maps φ_1, φ_2 are bijections between the set of ultrafilters of* $\mathbf{A}$ *and the set of maximal ideals of* $\mathbf{M}(\mathbf{A})$.

COROLLARY 3.11. *Let* $\mathbf{A}$ *be a good pseudo-*BL *algebra. Then* $\mathbf{A}$ *is local if and only if* $\mathbf{M}(\mathbf{A})$ *is local.*

We remark that if $\mathbf{A}$ is a BL-algebra, then $\varphi_1 = \varphi_2$ and the results obtained above extend some results from [19, 8].

PROPOSITION 3.12. *Let* **A** *be a good pseudo*-BL *algebra. Suppose that* I *is an ideal of* $\mathbf{M}(\mathbf{A})$ *and* F *is a filter of* **A**. *Then*

(i) *if* F *is normal in* **A**, *then* $\varphi_1(F) = \varphi_2(F) \overset{\text{not}}{=} \varphi(F)$;

(ii) *if* F *is normal in* **A**, *then* $\varphi(F)$ *is normal in* $\mathbf{M}(\mathbf{A})$;

(iii) *if* I *is normal in* $\mathbf{M}(\mathbf{A})$, *then* $\varphi_1^{-1}(I) = \varphi_2^{-1}(I)$.

PROOF. (i) Let $b \in \varphi_1(F)$, that is, $b = \varphi_1(a)$ with $a \in F$. By (14), we have that $a \leq a^{\sim -}$, so $a^{\sim -} \in F$, hence $a^{\approx} \in F$, since F is a normal filter of **A**. Since $b \in M(A)$, we also get that $b = b^{\sim -} = a^{\approx -} = \varphi_2(a^{\approx})$, hence $b \in \varphi_2(F)$. Thus, $\varphi_1(F) \subseteq \varphi_2(F)$. The other inclusion is proved similarly.

(ii) Let $b, c \in M(A)$. By Lemma 3.2 (iii), we have $b^{\sim} \odot_{M(A)} c = (c^{-} \oplus b)^{\sim} = (c \rightarrow b)^{\sim}$ and $c \odot_{M(A)} b^{-} = (b \oplus c^{\sim})^{-} = (c \rightsquigarrow b)^{-}$. Suppose that $b^{\sim} \odot_{M(A)} c \in \varphi(F)$, so there is $a \in F$ such that $(c \rightarrow b)^{\sim} = a^{\sim}$. But $c \rightarrow b = c^{-} \oplus b \in M(A)$, hence $c \rightarrow b = (c \rightarrow b)^{\sim -} = a^{\sim -} \geq a$, by (14). Since $a \in F$ and F is a filter, we get that $c \rightarrow b \in F$. But F is normal, hence $c \rightsquigarrow b \in F$. We obtain that $c \odot_{M(A)} b^{-} = (c \rightsquigarrow b)^{-} \in \varphi(F)$. We get similarly that $c \odot_{M(A)} b^{-} \in \varphi(F)$ implies $b^{\sim} \odot_{M(A)} c \in \varphi(F)$.

(iii) Let $a \in \varphi_1^{-1}(I)$, so $a^{\sim} \in I$. Since I is normal, from $a^{\sim} \in I$ and Lemma 1.23 we get that $a^{\sim =} \in I$. But, by (∗) and (16), $a^{\sim =} = a^{- \sim -} = a^{-} = \varphi_2(a)$. We have got that $\varphi_2(a) \in I$, that is $a \in \varphi_2^{-1}(I)$. We prove similarly that $a \in \varphi_2^{-1}(I)$ implies $a \in \varphi_1^{-1}(I)$. □

4. Some classes of local pseudo-BL algebras

Perfect pseudo-BL algebras A pseudo-BL algebra **A** is called *perfect* if

(i) **A** is a local good pseudo-BL algebra, and

(ii) for any $a \in A$, $\text{ord}(a) < \infty$ if and only if $\text{ord}(a^{\sim}) = \infty$.

PROPOSITION 4.1. *Let* **A** *be a good pseudo*-BL *algebra. Then* **A** *is perfect if and only if* $\mathbf{M}(\mathbf{A})$ *is perfect.*

PROOF. We have that **A** is local if and only if $\mathbf{M}(\mathbf{A})$ is local, by Corollary 3.11. In the sequel, we shall apply repeatedly Proposition 3.8 (iii). Suppose that **A** is perfect and let $a \in M(A)$, so $a = a^{\sim -}$. We get that $\text{MV-ord}(a) < \infty$ if and only if $\text{MV-ord}(a^{\sim -}) < \infty$ if and only if $\text{ord}(a^{\sim}) < \infty$ if and only if $\text{ord}(a) = \infty$ if and only if $\text{MV-ord}(a^{\sim}) = \infty$. Hence, $\mathbf{M}(\mathbf{A})$ is perfect. Conversely, suppose that $\mathbf{M}(\mathbf{A})$ is perfect and let $a \in A$. Then, by Lemma 3.1 (i), $a^{\sim} \in M(A)$. It follows that $\text{ord}(a) < \infty$ if and only if $\text{MV-ord}(a^{\sim}) < \infty$ if and only if $\text{MV-ord}(a^{\approx}) = \infty$ if and only if $\text{ord}(a^{\sim}) = \infty$. Hence, **A** is perfect. □

PROPOSITION 4.2. *Let* **A** *be a local good pseudo*-BL *algebra. The following are equivalent*:

(i) **A** *is perfect*;
(ii) *for any* $a \in A$, $\mathrm{ord}(a) < \infty$ *implies* $\mathrm{ord}(a^{\sim}) = \infty$;
(ii′) *for any* $a \in A$, $\mathrm{ord}(a) < \infty$ *implies* $\mathrm{ord}(a^{-}) = \infty$;
(iii) $D(A)^*_{\sim} = D(A)^*$;
(iii′) $D(A)^*_{-} = D(A)^*$.

PROOF. (i) if and only if (ii). Let $a \in A$. Since **A** is local, by Corollary 2.3, we have that $\mathrm{ord}(a) = \infty$ implies $\mathrm{ord}(a^{\sim}) < \infty$, hence $\mathrm{ord}(a^{\sim}) = \infty$ implies $\mathrm{ord}(a) < \infty$. It follows that **A** is perfect if and only if ($\mathrm{ord}(a) < \infty$ implies $\mathrm{ord}(a^{\sim}) = \infty$).

(ii) if and only if (ii′). Apply Lemma 3.8 (iv).

(ii′) implies (iii). Since **A** is local, $D(A)^*_{\sim} \subseteq D(A)^*$, by Corollary 2.3 (ii). Let us prove the converse inclusion. Let $a \in A$ be such that $\mathrm{ord}(a) < \infty$. From (ii′) we get that $\mathrm{ord}(a^{-}) = \infty$, so $a^{-} \in D(A)$ and, by (14), $a \leq a^{-\sim}$. Hence, $a \in D(A)^*_{\sim}$.

(iii) implies (ii′). Suppose that $D(A)^*_{\sim} = D(A)^*$ and let $a \in A$ with $\mathrm{ord}(a) < \infty$, that is, $a \in D(A)^*$. It follows that there is $x \in D(A)$ such that $a \leq x^{\sim}$, so $x^{\sim-} \leq a^{-}$, by (13). Since $x \leq x^{\sim-}$ and $\mathrm{ord}(x) = \infty$, applying Lemma 1.7 (iii), we get that $\mathrm{ord}(x^{\sim-}) = \infty$. Applying again Lemma 1.7 (iii), from $x^{\sim-} \leq a^{-}$ it follows that $\mathrm{ord}(a^{-}) = \infty$.

(ii) if and only if (iii′). It is similar to '(ii′) if and only if (iii)'. □

A primary filter P of a pseudo-BL algebra **A** is called *perfect* if for all $a \in A$, $(a^n)^{\sim} \in P$ for some $n \in \omega$ implies $((a^{\sim})^m)^{\sim} \notin P$ for all $m \in \omega$.

LEMMA 4.3. *Let* **A** *be a pseudo*-BL *algebra and* P *be a perfect filter of* **A**. *Then for all* $a \in A$, $(a^n)^{\sim} \in P$ *for some* $n \in \omega$ *if and only if* $((a^{\sim})^m)^{\sim} \notin P$ *for all* $m \in \omega$.

PROOF. Let $a \in A$ such that $((a^{\sim})^m)^{\sim} \notin P$ for all $m \in \omega$. We have to prove that $(a^n)^{\sim} \in P$ for some $n \in \omega$. By (9), $a^{\sim} \odot a = 0$, hence $((a^{\sim} \odot a)^n)^{\sim} = 0^{\sim} = 1 \in P$ for all $n \in \omega$. Apply now the fact that P is primary and the hypothesis to get that $(a^n)^{\sim} \in P$ for some $n \in \omega$. □

PROPOSITION 4.4. *Let* **A** *be a good pseudo*-BL *algebra and* P *be a proper normal filter of* **A**. *The following are equivalent*:

(i) $\mathbf{A}/P$ *is a perfect pseudo*-BL *algebra*;
(ii) P *is a perfect filter of* **A**;
(iii) P *is primary and for all* $a \in A$, $(a^n)^{-} \in P$ *for some* $n \in \omega$ *implies* $((a^{-})^m)^{-} \notin P$ *for all* $m \in \omega$.

PROOF. Since good pseudo-BL algebras form a variety, it follows that $\mathbf{A}/P$ is a good pseudo-BL algebra. By Proposition 2.6, we have that $\mathbf{A}/P$ is local if and only if P is primary. Let $a \in A$. Applying Lemma 1.10, we get that $\mathrm{ord}(a/P) < \infty$ if and only if $(a/P)^n = 0/P$ for some $n \in \omega$ if and only if $(a^n)^{\sim} \in P$ for some $n \in \omega$ if and only if $(a^n)^{-} \in P$ for some $n \in \omega$, that $\mathrm{ord}((a/P)^{\sim}) = \infty$ if and only if $((a/P)^{\sim})^m \neq 0/P$ for all $m \in \omega$ if and only if $((a^{\sim})^m)^{\sim} \notin P$ for all $m \in \omega$ and that $\mathrm{ord}((a/P)^{-}) = \infty$ if and only if $((a/P)^{-})^m \neq 0/P$ for all $m \in \omega$ if and only if $((a^{-})^m)^{-} \notin P$ for all $m \in \omega$. Apply now Proposition 4.2 (ii) and (iii) to get that (i) if and only if (ii) and (i) if and only if (iii). □

PROPOSITION 4.5. *Let* $\mathbf{A}$ *be a* BL-*algebra and* P *be a proper filter of* $\mathbf{A}$. *The following are equivalent*:

(i) *P is a perfect filter of* $\mathbf{A}$;

(ii) *for all $a \in A$, $(a^n)^{-} \in P$ for some $n \in \omega$ if and only if $((a^{-})^m)^{-} \notin P$ for all $m \in \omega$.*

PROOF. (i) implies (ii). Apply Lemma 4.3.

(ii) implies (i). We shall prove that $\mathbf{A}/P$ is local and apply Proposition 2.6 to get that P is a primary filter. Let $a \in A$ and suppose that $\mathrm{ord}(a^{-}/P) = \infty$. As in the proof of Proposition 4.4, we get $((a^{-})^m)^{-} \notin P$ for all $m \in \omega$. Applying (i), it follows that $(a^n)^{-} \in P$ for some $n \in \omega$, that is, $\mathrm{ord}(a/P) < \infty$. Thus, we have proved that for all $a \in A$, $\mathrm{ord}(a/P) < \infty$ or $\mathrm{ord}(a^{-}/P) < \infty$. Apply now [19, Proposition 1] to obtain that $\mathbf{A}/P$ is local. □

Hence, in the case that $\mathbf{A}$ is a BL-algebra, the notion of perfect filter defined above coincides with the notion of perfect filter defined in [19].

PROPOSITION 4.6. *Let* $\mathbf{A}$ *be a good pseudo*-BL *algebra. The following are equivalent*:

(i) $\mathbf{A}$ *is perfect*;

(ii) *any proper normal filter of* $\mathbf{A}$ *is perfect*;

(iii) $\{1\}$ *is a perfect filter of* $\mathbf{A}$.

PROOF. (i) implies (ii). Let F be a proper normal filter of $\mathbf{A}$. Since $\mathbf{A}$ is local, by Proposition 2.9 it follows that F is primary. Let $a \in A$ such that $(a^n)^{\sim} \in F$ for some $n \in \omega$. Suppose that $((a^{\sim})^k)^{\sim} \in F$ for some $k \in \omega$. We get that $\langle (a^n)^{\sim} \rangle, \langle ((a^{\sim})^k)^{\sim} \rangle \subseteq F$ and, since F is proper, it follows that $\langle (a^n)^{\sim} \rangle$ and $\langle ((a^{\sim})^k)^{\sim} \rangle$ are also proper filters of $\mathbf{A}$. Applying Lemma 1.7 (i), we get that $\mathrm{ord}((a^n)^{\sim}) = \mathrm{ord}(((a^{\sim})^k)^{\sim}) = \infty$. Since $\mathbf{A}$ is perfect, we obtain that $\mathrm{ord}(a^n) < \infty$ and $\mathrm{ord}((a^{\sim})^k) < \infty$, hence, $\mathrm{ord}(a) < \infty$ and $\mathrm{ord}(a^{\sim}) < \infty$, a contradiction with the fact that $\mathbf{A}$ is perfect. Thus, $(a^n)^{\sim} \in F$ for some $n \in \omega$ implies $((a^{\sim})^m)^{\sim} \notin F$ for all $m \in \omega$.

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(ii) implies (iii). It is obvious, since $\{1\}$ is a proper normal filter of $\mathbf{A}$.

(iii) implies (i). Since $\{1\}$ is a perfect filter of $\mathbf{A}$, applying Proposition 4.4, we get that $\mathbf{A}/\{1\}$ is perfect. But $\mathbf{A} \cong \mathbf{A}/\{1\}$, hence $\mathbf{A}$ is perfect. □

Locally finite pseudo-BL algebras According to [5], a pseudo-BL algebra $\mathbf{A}$ is *locally finite* if for any $a \in A$, $a \neq 1$ implies $\mathrm{ord}(a) < \infty$.

PROPOSITION 4.7. *Let* $\mathbf{A}$ *be a pseudo*-BL *algebra. The following are equivalent*:

(i) $\mathbf{A}$ *is locally finite*;

(ii) $\{1\}$ *is the unique proper filter of* $\mathbf{A}$.

PROOF. Applying Lemma 1.7 (i), it follows that A is locally finite if and only if for every $a \in A$, if $a \neq 1$ then $\langle a \rangle = A$ if and only if $\{1\}$ is the unique proper filter of A. □

PROPOSITION 4.8. *Every locally finite pseudo*-BL *algebra* $\mathbf{A}$ *is a local pseudo*-BL *algebra.*

PROOF. We have that $D(A) = \{1\}$, hence $D(A)$ is a filter of $\mathbf{A}$. Apply Proposition 2.2 to get that $\mathbf{A}$ is local. □

In [5] it is proved that locally finite pseudo-BL algebras are locally finite MV-algebras. We shall give a simpler proof of this fact.

PROPOSITION 4.9. *Let* $\mathbf{A}$ *be a locally finite pseudo*-BL *algebra. Then for all* $a \in A$, $a^{\sim -} = a^{-\sim} = a$. *Hence,* $A = M(A)$.

PROOF. If $a = 0$, then it follows immediately that $0^{\sim -} = 0^{-\sim} = 0$. Suppose that $a \neq 0$. Let us prove that $a^{-\sim} = a$. By (14), we have that $a \leq a^{-\sim}$. Suppose that $a^{-\sim} \not\leq a$, hence $a^{-\sim} \to a \neq 1$. Since $\mathbf{A}$ is locally finite, it follows that $\mathrm{ord}(a^{-\sim} \to a) < \infty$, hence $(a^{-\sim} \to a)^n = 0$ for some $n \in \omega - \{0\}$. By (16), (2), (A4) and (14), we get

$$\begin{aligned}(a^{-\sim} \to a) \to a^- &= (a^{-\sim} \to a) \to a^{-\sim -} = (a^{-\sim} \to a) \to (a^{-\sim} \to 0)\\ &= a^{-\sim} \odot (a^{-\sim} \to a) \to 0 = (a \wedge a^{-\sim}) \to 0 = a \to 0 = a^-.\end{aligned}$$

Applying repeatedly this procedure, it follows that $(a^{-\sim} \to a)^n \to a^- = a^-$, hence $a^- = 0 \to a^- = 1$, so, by (11), $a = 0$. We have got a contradiction, since $a \neq 0$. Hence, $a^{-\sim} = a$. We prove similarly that $a^{\sim -} = a$. □

COROLLARY 4.10 ([5]). *Every locally finite pseudo-*BL *algebra* **A** *is a locally finite* MV*-algebra.*

PROOF. Applying Proposition 4.9 and Proposition 1.24, we get that **A** is a pseudo-MV algebra. Let $a \in A$, $a \neq 0$, so $a^{\sim} \neq 1$, by (11). By Proposition 3.8 (i), we obtain that MV-ord(a) = MV-ord$(a^{\sim -})$ = ord$(a^{\sim}) < \infty$. Thus, we have proved that **A** is a locally finite pseudo-MV algebra. Apply now [15, Proposition 39] to get that **A** is a locally-finite MV-algebra. □

Peculiar pseudo-BL algebras A pseudo-BL algebra **A** is called *peculiar* if

(i) **A** is a local good pseudo-BL algebra;
(ii) there is $a \in A - \{1\}$ such that ord$(a) = \infty$;
(iii) there is $a \in A$ such that ord$(a) < \infty$ and ord$(a^{\sim}) < \infty$.

Let us denote by $\mathscr{PF}$ the class of perfect pseudo-BL algebras, by $\mathscr{LF}$ the class of locally finite pseudo-BL algebras and by $\mathscr{PC}$ the class of peculiar pseudo-BL algebras. The following proposition is similar to [2, Theorem 5.1].

PROPOSITION 4.11. *Let* **A** *be a local good pseudo-*BL *algebra different from* $\mathbf{L}_2 = \{0, 1\}$. *Then exactly one of the following holds*:

(i) $\mathbf{A} \in \mathscr{PF}$;
(ii) $\mathbf{A} \in \mathscr{LF}$;
(iii) $\mathbf{A} \in \mathscr{PC}$.

PROOF. By the definitions, if $\mathbf{A} \notin \mathscr{PF} \cup \mathscr{LF}$, then $\mathbf{A} \in \mathscr{PC}$. Hence, one of (i), (ii) or (iii) holds. It is easy to see that $\mathscr{PC} \cap \mathscr{LF} = \mathscr{PC} \cap \mathscr{PF} = \emptyset$. Let us prove that $\mathscr{PF} \cap \mathscr{LF} = \{\mathbf{L}_2\}$. Obviously, $\mathbf{L}_2$ is perfect and locally finite. Now, let $\mathbf{A} \neq \mathbf{L}_2$ be a locally finite pseudo-BL algebra. Since $A \neq \{0, 1\}$, there is $a \in A$ such that $a \neq 0$ and $a \neq 1$. From $a \neq 0$ and (11) we get that $a^{\sim} \neq 1$. Applying now the fact that **A** is locally finite, it follows that ord$(a) < \infty$ and ord$(a^{\sim}) < \infty$. Hence, **A** is not perfect. That is, exactly one of (i), (ii), (iii) holds. □

PROPOSITION 4.12. *Let* **A** *be a locally good pseudo-*BL *algebra such that* $A \neq M(A)$. *Then* **A** *is a peculiar pseudo-*BL *algebra if and only if* $\mathbf{M(A)} \neq \mathbf{L}_2$ *is a singular pseudo-*MV *algebra.*

PROOF. Suppose that **A** is peculiar. Then **A** is not perfect, hence, by Proposition 4.1, **M(A)** is not a perfect pseudo-MV algebra. Since $\mathbf{L}_2$ is a perfect pseudo-MV algebra, it follows that $\mathbf{M(A)} \neq \mathbf{L}_2$. Applying Proposition 1.25, we also get that **M(A)** is singular. Conversely, suppose that $\mathbf{M(A)} \neq \mathbf{L}_2$ and that **M(A)** is a singular pseudo-MV algebra. Since $A \neq M(A)$, by Proposition 4.9 we get that **A** is not locally finite. We also have that $\mathbf{M}(\mathbf{A})$ is not perfect, hence $\mathbf{A}$ is not perfect. Applying Proposition 4.11, we get that $\mathbf{A}$ is peculiar. $\square$

5. Bipartite pseudo-BL algebras

In this section, we shall define (strongly) bipartite pseudo-BL algebra and we shall prove some properties of them, following [17, 8].

A pseudo-BL algebra $\mathbf{A}$ is called *bipartite* if $U \cup U^*_{\sim} = U \cup U^*_{-} = A$ for some ultrafilter U of $\mathbf{A}$. $\mathbf{A}$ is called *strongly bipartite* if $U \cup U^*_{\sim} = U \cup U^*_{-} = A$ for any ultrafilter U of $\mathbf{A}$. Obviously, any strongly bipartite pseudo-BL algebra is bipartite.

A filter F of $\mathbf{A}$ is called *Boolean* if for all $a \in A$, $a \vee a^{\sim} \in F$ and $a \vee a^{-} \in F$. It is obvious that if $F \subseteq G$ are two filters of $\mathbf{A}$ and F is Boolean, then G is also Boolean.

PROPOSITION 5.1. *Let* $\mathbf{A}$ *be a pseudo-*BL *algebra and* F *be a filter of* $\mathbf{A}$. *The following are equivalent*:

(i) *F is a Boolean ultrafilter of* $\mathbf{A}$;
(ii) *F is a Boolean prime filter of* $\mathbf{A}$;
(iii) *F is proper and for all $a \in A$, $a \in F$ or ($a^{\sim} \in F$ and $a^{-} \in F$).*

PROOF. (i) implies (ii). It is obvious, since, by Proposition 1.3, any ultrafilter of $\mathbf{A}$ is a prime filter of $\mathbf{A}$.

(ii) implies (iii). Let $a \in A$. Since F is Boolean, we have that $a \vee a^{\sim} \in F$ and $a \vee a^{-} \in F$. Apply now the fact that F is prime to get (iii).

(iii) implies (ii). Let G be a proper filter of $\mathbf{A}$ such that $F \subseteq G$ and suppose that $F \neq G$. Then there is $a \in G$ such that $a \notin F$. By (iii), it follows that $a^{\sim}, a^{-} \in F \subseteq G$, so by (8), $0 = a^{\sim} \odot a \in G$, hence G is not proper, that is a contradiction. Hence, $G = F$. Thus, F is an ultrafilter of $\mathbf{A}$. Let us prove now that F is Boolean. Let $a \in A$. If $a \in F$, since $a \leq a \vee a^{\sim}$ and $a \leq a \vee a^{-}$, we get that $a \vee a^{\sim}, a \vee a^{-} \in F$. If $a \notin F$, then $a^{\sim}, a^{-} \in F$ and from $a^{\sim} \leq a \vee a^{\sim}$, $a^{-} \leq a \vee a^{-}$ we also get that $a \vee a^{\sim}, a \vee a^{-} \in F$. $\square$

LEMMA 5.2. *Let* $\mathbf{A}$ *be a pseudo-*BL *algebra and* U *be an ultrafilter of* $\mathbf{A}$. *The following are equivalent*:

(i) $U \cup U^*_{\sim} = U \cup U^*_{-} = A$;
(ii) *U is Boolean.*

PROOF. Applying Proposition 5.1 (iii) and Remark 1.13 (ii) and (ii′), we get that U is Boolean if and only if for all $a \in A$, $a \in U$ or ($a^{\sim} \in U$ and $a^{-} \in U$) if and only if for all $a \in A$, $a \in U$ or ($a \in U^*_{\sim}$ and $a \in U^*_{-}$) if and only if $U \cup U^*_{\sim} = U \cup U^*_{-} = A$. $\square$

https://doi.org/10.1017/S144678870000851X Published online by Cambridge University Press

PROPOSITION 5.3. *Let* **A** *be a pseudo*-BL *algebra* **A**. *The following are equivalent*:

(i) **A** *is bipartite*;
(ii) **A** *has a Boolean proper filter.*

PROOF. (i) implies (ii). Apply the above lemma.

(ii) implies (i). Suppose that **A** has a Boolean proper filter F. By Proposition 1.4, we can extend F to an ultrafilter U and U is also Boolean. Applying again Lemma 5.2, we get that **A** is bipartite. □

Let **A** be a pseudo-BL algebra. Following [17], we define

$$\mathscr{B}(A) = \cap\{F \mid F \text{ is a Boolean filter of } \mathbf{A}\},$$

and

$$\sup(A) = \{a \vee a^{\sim} \mid a \in A\} \cup \{a \vee a^{-} \mid a \in A\}.$$

The following remark is obvious.

REMARK 5.4. Let **A** be a pseudo-BL algebra. Then

(i) $\mathscr{B}(A)$ is the smallest Boolean filter of **A**;
(ii) if $\sup(A)$ is a filter of **A**, then it is a Boolean filter;
(iii) $\sup(A) \subseteq \mathscr{B}(A)$.

PROPOSITION 5.5. *Let* **A** *be a pseudo*-BL *algebra. Then*

(i) $\mathscr{B}(A) = \langle \sup(A) \rangle$;
(ii) $\sup(A) = \{a \in A \mid a \geq a^{\sim}$ *or* $a \geq a^{-}\}$.

PROOF. (i) By the above remark, we have that $\sup(A) \subseteq \mathscr{B}(A)$ and $\mathscr{B}(A)$ is a filter of **A**. Hence, $\langle \sup(A) \rangle \subseteq \mathscr{B}(A)$. Obviously, $\langle \sup(A) \rangle$ is a Boolean filter of **A**, so $\mathscr{B}(A) \subseteq \langle \sup(A) \rangle$.

(ii) Let $a \in \sup(A)$. If $a = x \vee x^{\sim}$ for some $x \in A$ then, by (18), $a = x \vee x^{\sim} \geq x^{\sim} \geq x^{\sim} \wedge x^{\approx} = (x \vee x^{\sim})^{\sim} = a^{\sim}$. We prove similarly that if $a = x \vee x^{-}$ for some $x \in A$, then $a \geq a^{-}$. Conversely, if $a \in A$ such that $a \geq a^{\sim}$, then $a = a \vee a^{\sim}$, hence $a \in \sup(A)$. Similarly, if $a \geq a^{-}$, then $a = a \vee a^{-}$, that is, $a \in \sup(A)$. □

PROPOSITION 5.6. *Let* **A** *be a pseudo*-BL *algebra* **A**. *The following are equivalent*:

(i) **A** *is strongly bipartite*;
(ii) *any ultrafilter of* **A** *is Boolean*;
(iii) $\mathscr{B}(A) \subseteq \mathscr{M}(A)$, *where we remind that* $\mathscr{M}(A)$ *denotes the intersection of all ultrafilters of* **A**.

PROOF. (i) if and only if (ii). Apply Lemma 5.2.

(ii) implies (iii). If U is an ultrafilter of **A** then, by (ii), U is Boolean. Applying Remark 5.4 (i), we get that $\mathscr{B}(A) \subseteq U$.

(iii) implies (ii). Let U be an ultrafilter of **A**. Then $\mathscr{B}(A) \subseteq U$ and $\mathscr{B}(A)$ is a Boolean filter of **A**. It follows that U is also Boolean. □

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