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COMPACTIFICATIONS

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Every completely regular space has at least one Hausdorff compactification and much research in Topology has been devoted to methods of constructing the compactifications of completely regular spaces. These methods fall into two general categories: internal methods and external methods. External methods are characterized by their reliance on structures which are not topological or outside the immediate topological structure of the space in question; examples of the former are A. Weil [**20**] – uniform structures and Yu. Smirnov [**15**] who uses the proximity structures of Efremovich; using the continuous real-valued functions to produce an embedding of the space serves as an example of the latter. Internal methods, on the other hand, use only the topological properties of the space under investigation and the most notable of these procedures is the Wallman compactification devised by Orrin Frink in [**8**]. Frink provides an internal characterization of complete regularity via the concept of a normal base and shows that each such base yields a compactification of the underlying space.

The object of this paper is to provide a method for constructing compactifications based on a single family of open sets each member of which determines an open set in the compactification. We accomplish this by providing a new internal characterization of Tychonov spaces thereby answering affirmitively a question raised by Frink in [**7**]. The new method is therefore applicable to all Tychonov spaces. Furthermore we show that any compactification of the Wallman type is constructable via the new technique so the new construction is at least as general as the Wallman-Frink construction.

In [**8**] the crucial condition on the family of closed sets is that of "normality"; the work of Alexandrov and Ponomarev [**1**; **2**] indicates that a corresponding condition on open sets is "functional inclusion". Their work relies on the concept of "subordination", an axiomatically defined relation on subsets of a space which is equivalent with the notion of a proximity. We obviate the need for these external considerations by introducing the relation "well-inside" which is defined entirely by a single family of sets.

In what follows all spaces will be assumed Hausdorff.

Definition. Let $\mathscr{B}$ be a set of subsets of a space X. We define a relation $<(\mathrm{rl}\,\mathscr{B})$ on subsets of X as follows: $A_1 < A_2(\mathrm{rl}\,\mathscr{B})$ means there exists $G_1, G_2 \in \mathscr{B}$ so that $A_1 \subset G_1$, $X - A_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

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If $A_1 < A_2(\mathrm{rl}\,\mathscr{B})$ we say "A_1 is well-inside A_2 relative to $\mathscr{B}$". Since we will usually be working with only one family $\mathscr{B}$ at a time we will drop the phrase $(\mathrm{rl}\,\mathscr{B})$ when no confusion can occur. The relation $<$ can be restricted to any family of subsets of X. We shall be interested in the properties the relation has on $\mathscr{B}$ itself and we shall assume that $\mathscr{B}$ is an open base for the topology on X. These conditions are made precise in the next definition.

Definition. $\mathscr{B}$ is an Alexandroff base for X if and only if $\mathscr{B}$ is a base for the open sets of X satisfying:
(1) $\mathscr{B}$ is closed under finite unions and intersections.
(2) $p \in G \in \mathscr{B} \Rightarrow$ there is an $H \in \mathscr{B}$ satisfying $p \in H < G(\mathrm{rl}\,\mathscr{B})$.
(3) $\mathscr{B}$ is densely ordered by the "well-inside" relations it defines.

As examples of Alexandroff bases one can consider the open sets of a normal space or, more specifically, if X is the real line with the usual topology, the set of finite intervals with rational end-points together with the complements of closures of certain of these form an Alexandroff base. Later we shall prove that every completely regular space has an Alexandroff base.

THEOREM 1. *If $\mathscr{B}$ is an Alexandroff base for X and A_i, $(i = 1, 2, 3, 4)$ are subsets of X, then the relation $<(\mathrm{rl}\,\mathscr{B})$ satisfies the following:*
(1) $A_1 < A_2 \Rightarrow \bar{A}_1 \subset A_2$.
(2) $A_1 < A_2 \Rightarrow A_1 \subset \mathrm{int}(\bar{A}_2)$.
(3) $A_1 \subset A_2 < A_3 \subset A_4 \Rightarrow A_1 < A_3, A_2 < A_4, A_1 < A_4$.
(4) $A_1 < A_2 \Rightarrow X - A_2 < X - A_1$.
(5) $A_1 < A_2, A_3 < A_4 \Rightarrow (A_1 \cup A_3) < (A_2 \cup A_4)$ *and*
$A_1 \cap A_3 \neq \emptyset \Rightarrow (A_1 \cap A_3) < (A_2 \cap A_4)$.

Proof. The verification of these properties is an easy consequence of the definition of "well-inside" and the standard definitions of topology.

Definition. If $\mathscr{B}$ is an Alexandroff base for X then δ, a non-empty collection of non-empty members of $\mathscr{B}$, is called a $\mathscr{B}$-filter (on X) provided δ satisfies:
(1) $G_1, G_2 \in \delta = G_1 \cap G_2 \in \delta$;
(2) $G_1 \in \delta$ and $G_2 \in \mathscr{B}$ with $G_1 \subset G_2 \Rightarrow G_2 \in \delta$;
(3) $G_1 \in \delta \Rightarrow$ there exists $G_2 \in \delta$ with $G_2 < G_1(\mathrm{rl}\,\mathscr{B})$.

A $\mathscr{B}$-filter is *fixed* if it has non-void intersection, otherwise it is *free*. A maximal $\mathscr{B}$-filter will be called a cluster. The following lemmas will be useful in constructing the compactification determined by $\mathscr{B}$.

As an immediate consequence of a standard Zorn's lemma argument we get:

LEMMA 1. *Every $\mathscr{B}$-filter is contained in a cluster.*

LEMMA 2. *If $\mathscr{B}$ is an Alexandroff base for X and $x \in X$, then*

$$x^* = \{G \in \mathscr{B} : x \in G\}$$

is a cluster on X.

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Proof. x^* is easily seen to be a $\mathscr{B}$-filter by the definition of an Alexandroff base. Now suppose δ is a $\mathscr{B}$-filter such that $x^* \subsetneq \delta$; then there is an $H \in \delta$ with $x \notin H$. Since δ is a $\mathscr{B}$-filter, there is a $G \in \delta$ with $G < H$; in other words there are $G_1, G_2 \in \mathscr{B}$ satisfying: $G \subset G_1$, $X - H \subset G_2$, and $G_1 \cap G_2 = \emptyset$. Since δ is a $\mathscr{B}$-filter containing x^* the above conditions yield $G_1, G_2 \in \delta$, but $G_1 \cap G_2 = \emptyset$. This contradiction establishes the maximality of x^*.

LEMMA 3. *If δ is a cluster and β is a $\mathscr{B}$-filter such that $\beta \neq \delta$, then $\beta \subset \delta$ or β and δ contain disjoint members.*

Proof.† Assume that every two members of β and δ meet. Let ξ be the filter in $\mathscr{B}$ generated by $\{G \cap H : G \in \beta \text{ and } H \in \delta\}$. By Theorem 1, ξ is a $\mathscr{B}$-filter and the maximality of δ yields $\delta = \xi \supset \beta$.

COROLLARY. *If δ is a cluster and $G_0, H_0 \in \mathscr{B}$ with $\emptyset \neq G_0 < H_0$, then either $H_0 \in \delta$ or some member of δ is disjoint from G_0.*

Proof. Assume $H_0 \notin \delta$ and that every member of δ meets G_0. Define

$$\beta = \{H \in \mathscr{B} : G \cap G_0 < H \text{ for some } G \in \delta\};$$

it is easily shown that β is a $\mathscr{B}$-filter. Now apply the above lemma.

For a given Alexandroff base $\mathscr{B}$ we denote by $\alpha_{\mathscr{B}}X$ the set of clusters defined by $\mathscr{B}$; whenever no confusion can result, we simply use the symbol αX. Following Stone [**16**], we define, for each $G \in \mathscr{B}$, the set

$$G^* = \{\delta \in \alpha X : G \in \delta\}.$$

The family $\{G^* : G \in \mathscr{B}\}$ is a base for a topology on αX. The next four theorems establish the fundamental properties of the space αX.

THEOREM 2. *αX is a Hausdorff extension of X.*

Proof. If $\delta_1, \delta_2 \in \alpha X$ with $\delta_1 \neq \delta_2$ then by Lemma 3 there are $G_1 \in \delta_1$, $G_2 \in \delta_2$ so that $G_1 \cap G_2 = \emptyset$. It is easily shown that $(G_1 \cap G_2)^* = G_1^* \cap G_2^*$ for any $G_1, G_2 \in \mathscr{B}$, thus αX is Hausdorff.

For each $x \in X$, let $\varphi(x) = x^*$; by Lemma 2 and the assumption that X is Hausdorff, φ is a 1-1 function from X into αX. If $G \in \mathscr{B}$, then G^* is a basic open set in αX and the identity $\varphi[G] = G^* \cap \varphi[X]$ shows that φ is an embedding of X in αX. Furthermore, for any $G \in \mathscr{B}$, $(G \neq \emptyset) x^* \in G^*$ for any $x \in G$, thus $\varphi[X]$ is dense in αX.

LEMMA 4. *If $\emptyset \neq G < H$, then $\bar{G}^* \subset H^* \subset \operatorname{int}(\bar{H}^*)$ where the closure and interior operators are those in αX.*

†The author wishes to thank the referee for suggesting this proof in place of the more involved one originally given.

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Proof. Let $\delta \in \bar{G}^*$, then every member of δ meets G, thus by the corollary to Lemma 3, $H \in \delta$ therefore $\delta \in H^*$. The second containment is obvious.

THEOREM 3. *αX is completely regular.*

Proof. We need only show that if $G \in \delta \in \alpha X$, then there is a continuous function $f : \alpha X \to [0, 1]$ so that $f(\delta) = 0$ and $f[\alpha X - G^*] = \{1\}$.

If $G \in \delta$, then there is an $H \in \delta$ with $H < G$; since $\mathscr{B}$ is densely ordered by $<$, we can assign to each rational $r \in [0, 1]$, a set $G_r \in \delta$ so that $H = G_0$, $G_r < G_s$ if $r < s$, and $G_1 = G$. Now, for $\beta \in \alpha X$ define

$$f(\beta) = \inf\{r : \beta \in G_r^*\};$$

Lemma 4 and the standard Urysohn argument prove the continuity of f and the theorem is established.

THEOREM 4. *A completely regular space is compact if and only if every cluster in any base is fixed.*

Proof. If X has a base which admits a free cluster δ, then $\{x^* : x \in X\} \cup \{\delta\}$ with the Stone topology is a proper Hausdorff extension of X; therefore X is not H-closed so it is not compact.

If X is not compact, let Y be a compactification of X and $p \in Y - X$. Since Y is normal it has a local base which is a cluster; the trace of this is a cluster in X which is free by the Hausdorffness of Y.

THEOREM 5. *αX is compact.*

Proof. Let C be a cluster in $\{G^* : G \in \mathscr{B}\}$; then $\{G : G^* \in C\}$ is a $\mathscr{B}$-filter and is therefore contained in a cluster δ – we claim that $\delta \in \bigcap C$, for $G^* \in C$ implies that $G \in \delta$, thus $\delta \in G^*$.

Theorems 2 and 5 establish that whenever a Hausdorff space X admits an Alexandroff base, then the space αX is a compactification of X; it follows therefore that any Hausdorff space which admits an Alexandroff base is completely regular. In the next theorem we show that every completely regular space has such a base; thus we will have established a new internal characterization of complete regularity. In addition, the theorem yields an immediate corollary showing that every Wallman-Frink compactification is an Alexandroff base compactification.

THEOREM 6. *Every completely regular space has an Alexandroff base.*

Proof. Let N be a normal base for X. (See O. Frink [**8**] for the results used in this theorem.) Then the base $\mathscr{B} = \{G : X - G \in N\}$ is an Alexandroff base. $\mathscr{B}$ is a ring of open sets since N is and $\mathscr{B}$ is a base for the same reason. If $p \in G \in \mathscr{B}$, then $p \notin X - G$ and the disjunctive property of normal bases yields a set $F \in N$ satisfying $p \in F$ and $F \cap (X - G) = \emptyset$; but disjoint members of N are separated by disjoint complements, i.e. there are $H_1, H_2 \in \mathscr{B}$ satisfying: $F \subset H_1$, $X - G \subset H_2$, and $G_1 \cap G_2 = \emptyset$. This yields $p \in H_1 < G$.

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To see that $\mathscr{B}$ is densely ordered by $<$, let $G_1 < G_2$. Then there are H_1, $H_2 \in \mathscr{B}$ so that $G_1 \subset H_1$, $X - G_2 \subset H_2$ and $H_1 \cap H_2 = \emptyset$. Now $X - H_2$ and $X - G_2 \in N$ and $(X - H_2) \cap (X - G_2) = \emptyset$, therefore there exists A and $B \in \mathscr{B}$ such that $X - H_2 \subset A$, $X - G_2 \subset B$ and $A \cap B = \emptyset$. We claim $G_1 < A < G_2$, for $A \cap B = \emptyset$ and $X - G_2 \subset B$ implies $A < G_2$. Since $X - A \subset H_2$, $G_1 \subset H_1$, and $H_1 \cap H_2 = \emptyset$, we have $G_1 < A$ and the theorem holds.

COROLLARY. *If N is a normal base for X and $\mathscr{B} = \{G : X - G \in N\}$, then αX and ωX are equal compactifications of X.*

Proof. For $\beta \in \omega X$, let $h(\beta)$ = the cluster containing $\{G \in \mathscr{B} : F \subset G$ for some $F \in \beta\}$. h is easily seen to be a 1-1 function from ωX into αX. If $G \in \mathscr{B}$, then

$$h^{-1}[G^*] = \{\beta \in \omega X : F \subset G \text{ for some } F \in \beta\};$$

this is open in ωX, thus h is continuous. Finally, if $x \in \bigcap \beta$, then $x \in \bigcap h(\beta)$ so, in effect, h is "the identity on X". It now follows that h is a homeomorphism onto αX.

A problem of primary interest in the theory of compactifications is that of continuously entending a function from the space X to one of its compactifications. It is usually the case that a function must be "uniformly" continuous in some sense in order for it to be extendible (see for example [**15**; **20**; **8**]). We define here the notion of $\mathscr{B}$-uniform continuity which is our version of Z-uniform continuity defined by Frink in [**8**].

Definition. If $\mathscr{B}$ is an Alexandroff base for X and f is a continuous real-valued function on X, then f is $\mathscr{B}$-uniformly continuous if and only if for positive epsilon, there is a finite cover of X by members of $\mathscr{B}$ on each member of which f oscillates less than epsilon.

THEOREM 7. *If $\mathscr{B}$ is an Alexandroff base for X then the continuous, real-valued function f is continuously extendible to αX if and only if f is $\mathscr{B}$-uniformly continuous.*

Proof. The proof of this theorem is essentially the same as [**8**] and can be found in [**19**].

Theorem 7 does not depend on Theorem 6 for its validity and can be used together with Theorem 6 to provide and immediate proof of the preceding corollary since compactifications admitting the same class of extendible functions are equal as compactifications.

Conclusion. We have characterized Tychonov spaces as precisely those spaces whose topology is generated by an Alexandroff base. With each such base, we have constructed a compactification whose points are the clusters in the base and whose topology is determined by the members of the base. This gives an affirmative answer to a question raised by Frink in [**7**] who suggests that such a construction might be possible.

Since every Wallman-Frink compactification is an Alexandroff base compactification (corollary to Theorem 6), it follows that the Stone-Čech compactification is always obtainable in this way and, for locally compact Hausdorff spaces, the one-point compactification is an Alexandroff base compactification. More generally, we can appeal to the results of Njastad [**12**] and Alo and Shapiro [**3**] to conclude that the compactifications of Freudenthal [**10**], Fan-Gottesman [**6**] and Gould are all Alexandroff-base compactifications.

Shanin [**13**; **14**] and Banaschewski [**5**] have also constructed internal compactifications; their constructions are essentially the same (for completely regular spaces) as Wallman-Frink, thus they are also obtainable via the construction given here. It is likely that the Hausdorff compactification given by Banaschewski in [**4**] can also be obtained as an Alexandroff base compactification although we have not yet verified this.

Finally we observe that the compactification of Fomin [**9**] can also be obtained via the Alexandroff-base construction since, in an algebraically closed base, "completely regular" containment and "well inside" agree and generate the same systems.

Thus we have shown that virtually all previous methods of constructing Hausdorff compactifications which rely only on some base for the topology of the underlying space are special cases of the Alexandroff-base construction. We have not yet been able to decide whether every Hausdorff compactification of an arbitrary completely regular space can be so constructed nor have we been able to answer this question for the Wallman-Frink procedure. Some partial results along these lines will be presented in a future paper.

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