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ON INTERSECTING FAMILIES OF FINITE SETS

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Let F be a family of k-element subsets of an n-set, $n > n_0(k)$. Suppose any two members of F have non-empty intersection. Let $\tau(F)$ denote $\min|T|$, T meets every member of F. Erdös, Ko and Rado proved $|F| \le \binom{n-1}{k-1}$ and that if equality holds then $\tau(F) = 1$. Hilton and Milner determined $\max|F|$ for $\tau(F) = 2$. In this paper we solve the problem for $\tau(F) = 3$.

The extremal families look quite complicated which shows the power of the methods used for their determination.

1. Introduction

Let X be a finite set of cardinality n and let F be a family of k-element subsets of it. The family F is called *intersecting* if for any two $F, G \in F$ we have $F \cap G \ne \emptyset$.

The *transversal number* $\tau(F)$ is defined to be the smallest integer t such that there exists a t-element subset Y of X satisfying $F \cap Y \ne \emptyset$ for every $F \in F$.

Clearly, for F intersecting, $\tau(F) \le k$ holds. Erdös, Ko and Rado proved the following

THEOREM 1 (Erdös, Ko and Rado [2]). *If* F *is intersecting and* $n > 2k$ *then* $|F| \le \binom{n-1}{k-1}$. *In the case of equality for some* $x \in X$ *we*

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have $\mathcal{F} = \{F \subset X \mid |F| = k, x \in F\}$; *that is,* $\tau(\mathcal{F}) = 1$.

Hilton and Milner generalized this theorem.

THEOREM 2 (Hilton and Milner [4]). *Suppose* $\mathcal{F}$ *is intersecting,* $\tau(\mathcal{F}) \geq 2$ *and that the cardinality of* $\mathcal{F}$ *is maximal subject to these restrictions. Then there exist* $k + 1$ *different elements* $y, x_1, \ldots, x_k \in X$ *such that setting*

$$Y_0 = \{x_1, \ldots, x_k\},\ Y_1 = \{y, x_1\},\ \ldots,\ Y_k = \{y, x_k\}$$

we have

$$\mathcal{F} = \{F \subset X \mid |F| = k,\ \exists i,\ 0 \leq i \leq k,\ Y_i \subseteq F\} .$$

Clearly in this case $\tau(\mathcal{F}) = 2$.

The aim of this paper is to investigate the case $\tau(\mathcal{F}) > 2$.

2. The statement of the result and some preliminaries

Let $x \in X$, $Y \subset X$, $|Y| = k$, $Z \subset X$, $|Z| = k - 1$, $x \notin (Y \cup Z)$, $Y \cap Z = \emptyset$. Let $Y_0 = \{y_1, y_2\}$ be a 2-element subset of Y . Let us set

$$\mathcal{G} = \{G \subset X \mid |G| = 3, x \in G, G \cap Y \neq \emptyset, G \cap Z \neq \emptyset\} \cup \{x \cup Y_0\} \cup \{Y\} \cup \{y_1 \cup Z\} \cup \{y_2 \cup Z\} .$$

Let us define now

$$\mathcal{F}_{\mathcal{G}} = \{F \subset X \mid |F| = k, \exists G \in \mathcal{G}, G \subseteq F\} .$$

It is easy to see that $\mathcal{F}_{\mathcal{G}}$ is intersecting and that $\tau(\mathcal{F}_{\mathcal{G}}) = 3$. We prove the following

THEOREM 3. *Let* $\mathcal{F}$ *be an intersecting family consisting of* k*-element subsets of* X *such that* $\tau(\mathcal{F}) \geq 3$. *Suppose further* $k \geq 3$, $n > n_0(k)$. *Then* $|\mathcal{F}| \leq |\mathcal{F}_{\mathcal{G}}|$ *and for* $k \geq 4$ *up to isomorphism* $\mathcal{F}_{\mathcal{G}}$ *is the only optimal family.*

Before proceeding with the proof of this theorem we need some preparations. The following definitions and lemmas are from [3].

$$F^* = \Big\{E \subset X \mid E \neq \emptyset,\ \exists F_1, F_2, \ldots, F_{(k+1)^{|E|}} \in F$$
$$\text{such that } F_i \cap F_j = E,\ 1 \le i < j \le (k+1)^{|E|}\Big\} ;$$

$$B(F) = \{B \in F^* \mid \nexists E \in F^*, E \subset B\} .$$

Then obviously $F^* \supseteq F$; consequently for every $F \in F$ there exists $B \in B(F)$ such that $B \subseteq F$. Therefore $B(F)$ is called the Δ-base of F .

Obviously if $B_1, B_2 \in B(F)$ then $B_1 \cap B_2 \neq \emptyset$. Hence for any $B \in B$ we have

(1) $$|B| \ge \tau(F) .$$

By a Δ-system of cardinality s we mean a family $C = \{C_1, \ldots, C_s\}$ such that for some $K \subset C_1$ we have $C_i \cap C_j = K$ for any $1 \le i < j \le s$ (*cf.* Erdös and Rado [1]).

The next lemma is a consequence of Lemma 1 in [3].

LEMMA 1. *Among the members of* $B(F)$ *we cannot find* $B_1, \ldots, B_{(k+1)^i}$ *forming a* Δ*-system of cardinality* $(k+1)^i$ *and satisfying further* $|B_j| = i + 1$ *for* $1 \le j \le (k+1)^i$.

Now a result of Erdös and Rado [1] implies that $|B(F)| \le k_0$ where k_0 is a constant depending only on k .

We infer

(2) $$|F| \le \sum_{B \in B} \binom{n-|B|}{k-|B|} .$$

3. Some reductions

From now on we suppose that F is an intersecting k-family satisfying $\tau(F) \ge 3$, and of maximal size.

Let $D_1, D_2, \ldots, D_t$ be the 3-sets in $B(F)$. Then using (1) and (2) we conclude

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(3) $$|F| \le t\binom{n-3}{k-3} + O\left(\binom{n-4}{k-4}\right) .$$

Comparing the right-hand side of (3) to the cardinality of F_G in Theorem 3, for $n > n_0(k)$ we infer $t \ge k^2 - k + 1$.

In the case $k = 3$, $|F| \le 10 = |F_G|$ is folklore. So we see that we can assume that $k \ge 4$.

We investigate $\mathcal{D} = \{D_1, \ldots, D_t\}$.

As $t \ge 4^2 - 4 + 1 = 13$ and $\mathcal{D}$ is intersecting we infer from the case $k = 3$ that $\tau(\mathcal{D}) \le 2$.

Our next aim is to prove $\tau(\mathcal{D}) = 1$.

Let $C = \{u_1, u_2\}$ be a 2-element set satisfying $D_i \cap C \ne \emptyset$ for $1 \le i \le t$.

We need a lemma.

LEMMA 2. *Among the members of* $\mathcal{D}$ *we cannot find* $k + 1$ *forming a* Δ*-system.*

Proof. Let us suppose on the contrary that $B_1, \ldots, B_{k+1} \in \mathcal{D}$ form a Δ-system with kernel K . Then $|K| \le 2$. Hence there exists an $F \in F$ such that $F \cap K = \emptyset$, implying $F \cap (B_i - K) \ne \emptyset$ for $i = 1, 2, \ldots, k+1$. But the sets $B_i - K$, $i = 1, \ldots, k+1$, are pairwise disjoint and we come to a contradiction with $|F| = k$.

Using Lemma 2 we infer that in $\mathcal{D}$ at most k sets contain C .

Let $D_1, \ldots, D_v$ be the remaining sets. Then $v \ge t-k \ge (k-1)^2$. These remaining sets contain exactly one of u_1, u_2 .

Let us suppose $D_1, \ldots, D_s$ are the sets in $\mathcal{D}$ containing u_1 but not u_2 . By symmetry reasons we may assume $s \ge t/2$.

Let us set $\mathcal{D}_1 = \{D_i - C \mid i = 1, \ldots, s\}$, $\mathcal{D}_2 = \{D_i - C \mid s < i \le t\}$. $\mathcal{D}_1$ and $\mathcal{D}_2$ are families of 2-element subsets such that for $D \in \mathcal{D}_1$,

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$D^* \in \mathcal{D}_2$ we have $D \cap D^* \neq \emptyset$. Suppose first $\mathcal{D}_2 \neq \emptyset$.

If $\tau(\mathcal{D}_2) > 1$ then $|\mathcal{D}_1| \leq 4$ follows yielding $t \leq 2s \leq 8 < 9 \leq (k-1)^2$, a contradiction. Hence $\tau(\mathcal{D}_2) = 1$. Let v be an element satisfying $v \in D$ for every $D \in \mathcal{D}_2$.

If $|\mathcal{D}_2| \geq 3$ then we conclude that v is contained in every set D_i , $1 \leq i \leq s$. Hence the sets $D_1, \ldots, D_s$ form a Δ-system of cardinality $s \geq (k-1)^2/2$, contradicting Lemma 2.

If $|\mathcal{D}_2| = 2$ then we conclude that at most one of $D_1, \ldots, D_s$ does not contain v , and we obtain again a Δ-system of cardinality at least $t-3 \geq (k-1)^2-3 \geq k+1$, contradicting Lemma 2.

If $|\mathcal{D}_2| = 1$ then let $\mathcal{D}_2 = \{\{u_2, u_3, u_4\}\}$.

Then every member of $\mathcal{D} - \mathcal{D}_2$ contains u_1 and has non-empty intersection with $\{u_2, u_3, u_4\}$. Hence for $(k^2-k)/3 > k$ we come to a contradiction with Lemma 2. The only remaining possibility is $k = 4$, $|\mathcal{D}| = 13$. It follows further from Lemma 2, that $|D \cap \{u_2, u_3, u_4\}| = 1$ and that exactly four of the D's intersect $\{u_2, u_3, u_4\}$ in $\{u_2\}$ - otherwise we could find a Δ-system of cardinality 5 .

Let these sets be $\{u_1, u_2, v_j\}$ where $j = 1, 2, 3, 4$. As $\tau(F) > 2$, there must be an $F \in F$ such that $F \cap \{u_1, u_2\} = \emptyset$. As $\{u_1, u_2, v_j\} \in B(F)$, we infer $F \cap \{u_1, u_2, v_j\} \neq \emptyset$. Hence we conclude $F = \{v_1, v_2, v_3, v_4\}$. However it is a contradiction as $\{u_2, u_3, u_4\} \in B(F)$ and $\{u_2, u_3, u_4\} \cap F = \emptyset$.

Now we have proved that $|\mathcal{D}_2| = 0$, that is every set in $\mathcal{D}$ contains u_1 ; thus $\tau(\mathcal{D}) = 1$.

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4. The structure of $\mathcal{D}$

In this paragraph we determine the exact structure of $\mathcal{D} = \{D_1, \ldots, D_t\}$. We know already that $u_1 \in D_i$ for every $i = 1, \ldots, t$ and that $t \geq k^2-k+1$. As $\tau(F) > 1$, there exists a set, say $F = \{f_1, \ldots, f_k\} \in \mathcal{F}$ such that $u_1 \notin F$.

As $D_i \in \mathcal{B}(\mathcal{F})$, $D_i \cap F \neq \emptyset$ for $i = 1, \ldots, t$.

Let us set $E_i = D_i - \{u_1\}$ for $i = 1, \ldots, t$. Then the E_i's are the edges of a simple 2-graph, which we denote by $\mathcal{E}$. Let c_i (d_i) be the number of edges adjacent to f_i and having their other extremity in F (not in F), respectively.

Then we have

$$t = \sum_{i=1}^{k} \left(d_i+\tfrac{1}{2}c_i\right) . \tag{4}$$

Now we prove

$$d_i + c_i \leq k \quad (i = 1, \ldots, k) . \tag{5}$$

Suppose that (5) fails for some i. It means that we can find $k + 1$ edges, say $E_1^i, \ldots, E_{k+1}^i$ which are adjacent to f_i.

As $\tau(F) > 2$, there exists a $G \in \mathcal{F}$ such that $G \cap \{u_1, f_i\} = \emptyset$. But $\mathcal{F}$ is intersecting and the D_i's belong to its Δ-base; consequently, for $j = 1, \ldots, k+1$, $\left(E_j^i-\{f_i\}\right) \in G$ holds. However this is impossible since $|G| = k < k + 1$. Now (5) is proved.

Next we prove

$$d_i \leq k-1 \quad (i = 1, \ldots, k) . \tag{6}$$

Suppose that, on the contrary, (6) fails for a given i. Then by (5) we have $d_i = k$.

Let $g_1, \ldots, g_k$ be the other endpoints of the edges adjacent to

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f_i . If G is an edge of F , which necessarily exists since $\tau(F) > 2$, disjoint to $\{u_1, f_i\}$ then we conclude in the above way $G = \{g_1, \ldots, g_k\}$. However $G \cap F = \emptyset$, a contradiction proving (6).

As $t \geq k^2-k+1$, we conclude from (4), taking into account (5) and (6), that there are at least two of the f_i's , say f_1, f_2 , such that $d_i = k - 1$, $c_i = 1$.

We distinguish two cases.

(a) $\{u_1, f_1, f_2\} \in B(F)$.

This means that $\{f_1, f_2\}$ is an edge in E . Let $\{g_1, \ldots, g_{k-1}\}$ be the set of points different to f_2 and connected in E to f_1 . Then for $G \in F$, $G \cap \{u_1, f_1\} = \emptyset$ we infer $G = \{f_2, g_1, \ldots, g_{k-1}\}$. As $\tau(F) > 2$, $G \in F$. Similarly if $f_1, g_1', \ldots, g_{k-1}'$ are the points adjacent to f_2 , then $G' = \{f_1, g_1', \ldots, g_{k-1}'\} \in F$.

Let $3 \leq i \leq k$, and let h be a point which is adjacent to f_i . Then $\{u_1, f_i, h\} \in B(F)$ implies

(7) $$h \in (G \cap G') .$$

If $|G \cap G'| \leq k-2$ we infer $t \leq 2k - 1 + (k-2)(k-2) < k^2 - k + 1$, a contradiction.

Hence $|G \cap G'| = k - 1$; that is,

$$\{g_1, g_2, \ldots, g_{k-1}\} = \{g_1', \ldots, g_{k-1}'\} .$$

Now $t \geq k^2-k+1$ and (7) imply $\{u_1, f_i, g_j\} \in B(F)$ for every $1 \leq i \leq k$, $1 \leq j < k$. Thus D has the same structure as it has in F_G .

(b) $\{u_1, f_1, f_2\} \notin B(F)$.

This means $\{f_1, f_2\} \notin E$.

Let $f_3, g_1, \ldots, g_{k-1}$ be the points adjacent to f_1 in E . As

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$\tau(F) > 2$, there exists $F \in F$ such that $F \cap \{u_1, f_1\} = \emptyset$. From the intersecting property and $|F| = k$ we deduce $F = \{f_3, g_1, \ldots, g_{k-1}\}$. Now if $\{u_1, f_2, h\} \in B(F)$ then it follows $\{u_1, f_2, h\} \cap F \neq \emptyset$; that is, $h \in F$. Hence we conclude that f_2 is adjacent in E to the same points as f_1 .

It follows in the same way for $4 \le i \le k$ and any h such that $\{u_1, f_i\ h\} \in B(F)$: $h \in F$. Hence we have

(8) $$t = \sum_{i=1, i\neq 3}^{k} d_i + d_3 + c_3 .$$

From (8) using (4) and $t \ge k^2-k+1$ we deduce $d_i = k - 1$ for $i \neq 3$ and $d_3 + c_3 = k$.

Let $h_1, \ldots, h_k$ be the neighbours of f_3 in E . As $\tau(F) > 2$ there exists $H \in F$ such that $H \cap \{u_1, f_3\} = \emptyset$. We infer $H = \{h_1, \ldots, h_k\}$.

We know $\{f_1, f_2\} \subset H$, whence for some j , $1 \le j \le k-1$, $g_j \notin H$. If for some i , $4 \le i \le k$, $f_i \notin H$, then $\{u_1, f_i, g_j\} \in B(F)$ and $\{u_1, f_i, g_j\} \cap H = \emptyset$ gives a contradiction.

Hence $f_1, f_2, f_4, f_5, \ldots, f_k$ are all neighbours of f_3 in E .

As $F \cap H \neq \emptyset$, we conclude that the remaining neighbour of f_3 is one of the g_j's , say g_1 .

Now setting $Y = \{g_1, g_2, \ldots, g_{k-1}, f_3\}$, $Z = \{f_1, f_2, f_4, \ldots, f_k\}$, $x = u_1$, we see that again D has the same structure as it has in F_G .

5. The deduction of Theorem 3 and some remarks

For optimal families we have now proved the existence of

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$$x \in X\ ,\quad Z \subset X\ ,\quad Y \subset X\ ,\quad Y_0 \subset Y\ ,$$
$$|Z| = k - 1\ ,\quad |Y| = k\ ,\quad |Y_0| = 2\ ,\quad x \notin (Y \cup Z)\ ,\quad Y \cap Z = \emptyset\ ,$$

such that

$$B_3 = \{B \in B(F) \mid |B| = 3\}$$
$$= \{B \subset X \mid |B| = 3,\ x \in B,\ B \cap Y \neq \emptyset,\ B \cap Z \neq \emptyset\} \cup \{x \cup Y_0\}\ .$$

Moreover we proved $Y \in F$. Let $Y_0 = \{y_1, y_2\}$ and let $F_1, F_2 \in F$ such that $F_i \cap \{x, y_i\} = \emptyset$. Such sets exist as $\tau(F) > 2$. We infer $F_i = \{y_{3-i} \cup Z\}$ for $i = 1, 2$.

As for every subset of cardinality at most k of X which does not contain any of the sets in $B = B_3 \cup \{F_1, F_2, Y\}$ we can find a set $B \in B$ which is disjoint to it, we infer $B = B(F) = G$. Hence the maximality of $|F|$ implies $F = F_G$. //

Now the next problem would be to determine $\max|F|$ for F intersecting, $\tau(F) > 3$. Or more generally $\tau(F) \geq \tau$.

We could only prove $|F| \leq (1+o(1))k^{\tau-1}\binom{n-\tau}{k-\tau}$.

To obtain a lower estimate let $x \in X$ and let $Y_1, Y_2, \ldots, Y_{\tau-1}$ be disjoint subsets of $X - x$. Let further $Z_i \subset Y_i$, $|Z_i| = \tau - i$, $|Y_i| = k - i + 1$.

Let us define

$$B_\tau = \{B \subset X \mid |B| = \tau,\ x \in B,\ \exists j,\ 1 \leq j \leq \tau,$$
$$\text{such that } B \cap Y_i \neq \emptyset \text{ for } 1 \leq i < j,\ Z_j \subseteq B\}\ .$$

Let us set further

$$B_k = \{B \subset X \mid |B| = k,\ \exists j$$
$$\text{such that } 1 \leq j < \tau,\ Y_j \subseteq B,\ B \cap Z_i \neq \emptyset \text{ for } 1 \leq i < j\}\ .$$

Now we define $B(F_\tau) = B_\tau \cup B_k$; that is,

$$F_\tau = \{F \subset X \mid |F| = k,\ \exists B \in B(F_\tau) \text{ such that } B \subseteq F\}\ .$$

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It is not hard to see that F is intersecting, $\tau(F) = \tau$, and

$$|F| = \left(\sum_{0 \le i \le \tau-1, i \ne \tau-2} (k)_i\right)\binom{n-\tau}{k-\tau}\left(1+o(1)\right) .$$

$\left((k)_i = k(k-1) \ldots (k-i+1), (k)_0 = 1\right.$.

Let us conclude this paper with a conjecture.

CONJECTURE. Suppose F is an intersecting family of k-subsets of X, $\tau(F) \ge \tau$. Suppose further $k > k_0(\tau)$, $n > n_0(k)$. Then $|F| \le |F_\tau|$.

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