Can. J. Math., Vol. XXIX, No. 2, 1977, pp. 243–254

APPLICATIONS OF DUALITY IN THE THEORY OF FINITELY GENERATED LATTICE-ORDERED ABELIAN GROUPS

W. M. BEYNON

Introduction. In a previous paper by the author [**3**], duality theorems for finitely generated vector lattices and lattice-ordered Abelian groups are described. In particular, the category of finitely generated semi-simple vector lattices is shown to be equivalent to a geometrical category V whose objects are topologically closed cones in Euclidean space, and whose morphisms, called '*l-maps*', form a special subclass of the class of piecewise homogeneous linear maps between such cones. Under this categorical duality, finitely generated projective vector lattices and closed polyhedral cones correspond; indeed, the category of finitely generated projective vector lattices is equivalent to the dual of a category whose objects are Euclidean closed polyhedral cones and whose morphisms consist of all piecewise homogeneous linear maps between such cones. Two Euclidean closed polyhedral cones are then seen to be '*l-equivalent*' (that is to say, are isomorphic in V) if and only if they have polyhedrally equivalent sections. (A polyhedron P is said to be a *section* of a closed polyhedral cone C if every ray of C meets P in a single point).

In this paper, some of the results proved for vector lattices in [**3**] are generalised to the context of lattice-ordered Abelian groups, and a number of applications are described. In particular, it is shown that the finitely generated projective lattice-ordered Abelian groups are the quotients of free finitely generated lattice-ordered Abelian groups by principal ideals. (An analogous characterisation of finitely generated projective vector lattices is described by Kirby Baker in [**1**]). Other applications include a classification of projective lattice-ordered Abelian groups with two generators analogous to that described for vector lattices by Bleier in [**6**], and a characterisation of lattice-ordered Abelian groups freely generated by the elements of a finite partially-ordered set.

Throughout the paper, proofs are geometric in spirit, and the main result is a non-trivial application of ideas of combinatorial topology to algebra.

0. Preliminaries. For background results on lattice-ordered Abelian groups, see Birkhoff [**5**].

A lattice-ordered Abelian group A is *projective* if, whenever $\phi : X \rightarrow Y$ is a surjective l-morphism and $\alpha : A \rightarrow Y$ is an l-morphism, there is an l-morphism $\theta : A \rightarrow X$ such that $\phi\theta = \alpha$.

For elementary geometrical concepts not mentioned below, see Baker [**1**],

Received January 15, 1975 and in revised form, November 1, 1976.

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

Stallings [**9**], Glaser [**7**] and Beynon [**3**]. A subset of $\mathbf{R}^n$ is a *convex closed polyhedral cone* if it is the positive hull of a finite set of points. Such a cone is *simplicial* if it is the positive hull of a linearly independent set of points.

A *closed polyhedral cone* is a finite union of convex closed polyhedral cones.

A *complex of simplicial cones in* $\mathbf{R}^n$ is a finite set $\mathscr{K}$ of simplicial cones in $\mathbf{R}^n$ such that

(i) if A is in $\mathscr{K}$ then every face of A is in $\mathscr{K}$; and

(ii) if A and B are in $\mathscr{K}$ then $A \cap B$ is a face of both A and B.

The set of simplicial cones of $\mathscr{K}$, ordered by the relation: $A \leqq B$ if and only if A is a face of B, is an abstract simplicial complex S. $\mathscr{K}$ is said to be a *realisation of S by simplicial cones*.

If $\mathscr{K}$ is a complex of simplicial cones in $\mathbf{R}^n$, the union of all the simplicial cones of $\mathscr{K}$ is a closed polyhedral cone in $\mathbf{R}^n$, and is denoted by $|\mathscr{K}|$. If $x \neq 0$ is a point of $|\mathscr{K}|$, then x is relatively interior to an unique simplicial cone C of dimension k in $|\mathscr{K}|$, where C is the positive hull of k points $x_1, x_2, \ldots, x_k$, and there is a subdivision of $\mathscr{K}$ canonically associated with x as follows: Let $C_1, \ldots, C_s$ be the simplicial cones of $\mathscr{K}$ which contain C as a face, and $C_{s+1}, \ldots, C_r$ the remaining cones. For each $i \leqq s$ the simplicial cone C_i, of dimension $m = m(i)$, is the positive hull of m points $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_m$. For each $j \leqq k$, let C_{ij} be the simplicial cone of dimension m which is the positive hull of the points $x_1, \ldots, \hat{x}_j, \ldots, x_k, x_{k+1}, \ldots, x_m, x$. If simplicial cones C_{ij} are constructed in this way for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, k$, the set of simplicial cones of the form C_{ij} with $i \leqq s$ and $j \leqq k$ together with the simplicial cones $C_{s+1}, \ldots, C_r$ forms a new complex of simplicial cones $\alpha\mathscr{K}$, which is a subdivision of $\mathscr{K}$. The subdivision $\alpha\mathscr{K}$ is called *the elementary starring of* $\mathscr{K}$ *at* x.

The polyhedron P is a *section* of the closed polyhedral cone C if every ray of C meets P in an unique point.

If $\mathscr{S}$ is a simplicial presentation of P, then the collection of simplicial cones obtained by forming the infinite cone with vertex O on each simplex A of $\mathscr{S}$ is a subdivision of C into simplicial cones, called the subdivision of C *induced by* $\mathscr{S}$.

Let $\mathscr{K}$ be a complex of simplicial cones in $\mathbf{R}^n$, and $\mathscr{L}$ a subcomplex of $\mathscr{K}$. If $Ox_1, Ox_2, \ldots, Ox_t$ are 1-dimensional simplicial cones of $\mathscr{K}$, there is an uniquely determined map $f_{\mathscr{L}}: |\mathscr{K}| \rightarrow [0, 1]$ piecewise homogeneous linear with respect to $\mathscr{K}$, such that $f_{\mathscr{L}}(x_i) = 1$ if $x_i \in \mathscr{L}$ and $f_{\mathscr{L}}(x_i) = 0$ otherwise. $\mathscr{L}$ is then a *full subcomplex of* $\mathscr{K}$ if $|\mathscr{L}| = f_{\mathscr{L}}(0)$ (c.f. [**8**] p. 31).

If K is a convex closed polyhedral cone in $\mathbf{R}^n$, then K is *rational* if K can be expressed as the positive hull of a finite set of points with rational coordinates. If C is an arbitrary closed polyhedral cone, C is *rational* if C can be expressed as the union of finitely many rational convex closed polyhedral cones.

Two basic results on subdivisions of rational closed polyhedral cones are required; these are stated in Lemma 0.1 below. The proof of this lemma depends essentially upon a theorem proved by the author in [**4**].

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

LEMMA 0.1. (i) *If C and D are rational closed polyhedral cones which have isomorphic subdivisions into simplicial cones, then they have isomorphic subdivisions into rational simplicial cones.*

(ii) *If C and D are rational closed polyhedral cones such that $C \subseteq D$, there are subdivisions $\mathscr{K}$ and $\mathscr{L}$ of C and D respectively into rational simplicial cones such that $\mathscr{K}$ is a subcomplex of $\mathscr{L}$.*

Proof. (i) Let P and Q be sections of C and D, respectively, which are rational polyhedra in the sense of [**4**]. Since C and D have isomorphic subdivisions into simplicial cones, there exist sections S and T of C and D respectively such that S and T are polyhedrally equivalent. Hence (by [**3**, Corollary 2 to Theorem 4.1], or by direct geometrical argument), P and Q are themselves polyhedrally equivalent. By [**4**, Theorem 1], there are isomorphic simplicial presentations $\mathscr{S}$ and $\mathscr{T}$ of P and Q respectively, both of which have vertices at rational points. Let $\mathscr{K}$ be the subdivision of C into simplicial cones induced by $\mathscr{S}$, and $\mathscr{L}$ the subdivision of P induced by $\mathscr{T}$. Then $\mathscr{K}$ and $\mathscr{L}$ are isomorphic subdivisions of C and P into rational simplicial cones.

(ii) Let Q be a rational polyhedron which is a section of D. Then $P = Q \cap C$ is also a rational polyhedron, and is a section of C. By [**4**, Corollary to Theorem 1], there are simplicial presentations $\mathscr{S}$ and $\mathscr{T}$ of P and Q respectively, with vertices at rational points, such that $\mathscr{S}$ is a subcomplex of $\mathscr{T}$. Let $\mathscr{K}$ be the subdivision of C into simplicial cones induced by $\mathscr{S}$, and $\mathscr{L}$ the subdivision of D induced by $\mathscr{T}$.

1. The category V^*. Following the notation introduced in [**3**], the symbol $Fl\text{-}G(n)$ will be used to denote the free lattice-ordered Abelian group on n generators, and $\mathbf{Z}$ the totally-ordered group of integers under addition. If α is an element of $Fl\text{-}G(n)$, then α can be expressed as $\bigvee_i (\bigwedge_j f_{ij})$ where i and j range over finite index sets, and each f_{ij} is a linear expression with integer coefficients in the free generators $e_1, e_2, \ldots, e_n$ of $Fl\text{-}G(n)$. Accordingly, α determines a map from $\mathbf{Z}^n$ to $\mathbf{Z}$ such that the n-tuple $(x_1, x_2, \ldots, x_n)$ is mapped to the image of α under the unique l-morphism $Fl\text{-}G(n) \rightarrow \mathbf{Z}$ mapping e_i to x_i for $i = 1, 2, \ldots, n$. A map $\theta : \mathbf{Z}^m \rightarrow \mathbf{Z}^n$ is then said to be an *integral l-map* if there exist n elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $Fl\text{-}G(m)$ such that for each element x of $\mathbf{Z}^m$ the relation $\theta(x) = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x))$ holds. If X and Y are subsets of $\mathbf{Z}^m$ and $\mathbf{Z}^n$ respectively, $\theta : X \rightarrow Y$ is said to be an integral l-map if it is the restriction of an integral l-map $\mathbf{Z}^m \rightarrow \mathbf{Z}^n$.

As in [**3**], the symbol V^* will denote the category which has as its typical object the set of integer lattice points lying within a closed cone in an Euclidean space $\mathbf{R}^n$, and whose morphisms are integral l-maps between such cones. The following duality theorem is proved in [**3**].

THEOREM 1.1. *The full subcategory of the category of finitely generated lattice-ordered Abelian groups consisting of subdirect products of copies of the totally-ordered Abelian group* $\mathbf{Z}$ *is equivalent to the dual of* V^*.

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

The principal object of this paper is to examine the nature of the category V^*, and to give an interpretation of the subcategory of V^* corresponding (under the duality of Theorem 1) to the category of finitely generated projective lattice-ordered Abelian groups. Note first that if C is a closed cone in $\mathbf{R}^n$, and C' denotes the set of integer lattice points in C, then C is uniquely determined by C', as the closure of the set of rational points which lie on rays passing through points of C'. If now C and D are closed cones in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and if C' and D' denote the set of integer lattices points in C and D respectively, then an integral l-map $\theta : C' \to D'$ can be extended in an unique way to a piecewise homogeneous linear map $\bar{\theta} : C \to D$, for the image under θ of an integer lattice point q determines the image under θ of each rational point on the ray Oq, and the set of rational points is dense in C. Indeed, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are elements of Fl-$G(m)$ such that $\theta(q) = (\alpha_1(q), \alpha_2(q), \ldots, \alpha_n(q)$ for all q in C', then $\alpha_1, \alpha_2, \ldots, \alpha_n$ determine l-maps $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ respectively, mapping C to D, and $\bar{\theta}(y) = (\bar{\alpha}_1(y), \bar{\alpha}_2(y), \ldots, \bar{\alpha}_n(y))$ for all y in C. A map $\bar{\theta} : C \to P$ defined in this way will be called an integral l-map between closed cones. It is not difficult to show that if $\bar{V}^*$ denotes the category whose objects are closed cones and whose morphisms are integral l-maps between closed cones, then there is an equivalence between V^* and $\bar{V}^*$ under which the set of integer lattice points C' corresponds to C, the closed cone which envelops C', and a morphism $\theta : C' \to D'$ in V^* corresponds to $\bar{\theta}$ the associated integral l-map $C \to D$. It will be convenient to identify V^* with $\bar{V}^*$ under the canonical equivalence defined in this way.

In [**3**, § 3], it is shown that a map $\mathbf{R}^n \to \mathbf{R}$ is an l-map if and only if it is piecewise homogeneous linear. Let a piecewise homogeneous linear map $\mathbf{R}^n \to \mathbf{R}$ be said to have integer coefficients when there exists a finite set of linear functions $f_1, f_2, \ldots, f_m$ with integer coefficients such that given x there is an index $i \leqq m$ for which $f(x) = f_i(x)$. A theorem proved by the author in [**2**] and quoted as Theorem 3.1 in [**3**] shows that a piecewise homogeneous linear map $\mathbf{R}^n \to \mathbf{R}$ is an integral l-map if and only if it has integer coefficients. This characterisation of integral l-maps will be used subsequently.

2. Subdivisions of rational closed polyhedral cones. Let C be a rational simplicial cone of dimension k in $\mathbf{R}^n$. Then C is the positive hull of k linearly independent rational points $g_1, g_2, \ldots, g_k$ and on each ray Og_i there is an unique non-zero integer lattice point a_i such that the open line segment $(0, a_i)$ contains no integer lattice point. The point a_i will be called the *initial integer lattice point* on Og_i. If the simplicial cone C of dimension k is the positive hull of the k integer points $a_1, a_2, \ldots, a_k$ where a_i the initial lattice point on the ray Oa_i for each i, it will be convenient to denote C by $C(a_1, a_2, \ldots, a_k)$. This notation is unambiguous, for C uniquely determines $a_1, a_2, \ldots, a_k$. By the same token, the number of integer lattice points of the form $\sum_{i=1}^{k} \epsilon_i a_i$ with $0 \leqq \epsilon_i < 1$ is uniquely determined by C, and will be called *the modulus of* $C(a_1, a_2, \ldots, a_k)$. The rational simplicial cone $C(a_1, a_2, \ldots, a_k)$ is then *primitive*

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

if its modulus is 1. It is easy to verify that the modulus of C is the index of the subgroup generated by $a_1, a_2, \ldots, a_k$ in the additive group of all integer lattice points in the linear subspace spanned by $a_1, a_2, \ldots, a_k$. In particular, $C(a_1, a_2, \ldots, a_k)$ is primitive if and only if $a_1, a_2, \ldots, a_k$ considered as elements of the additive group $\mathbf{R}^n$, generate the lattice of all integer lattice points in the linear subspace spanned by $a_1, a_2, \ldots, a_k$. Consequently the n-dimensional rational simplicial cone $C(a_1, a_2, \ldots, a_n)$ is primitive if and only if the determinant of the matrix $[a_1, a_2, \ldots, a_n]$, whose ith column is a_i, has modulus 1.

In this section, some properties of subdivisions of rational closed polyhedral cones into primitive simplicial cones are established; these are applied to the study of integral l-maps in Section 3.

LEMMA 2.1. *Let $C = C(a_1, a_2, \ldots, a_n)$ be a rational simplicial cone of dimension n in $\mathbf{R}^m$, and let z be an integer lattice point of the form $\sum_{i=1}^{k} \epsilon_i a_i$ where $0 < \epsilon_i < 1$ for $i = 1, 2, \ldots, k$, and $k \leqq n$. Suppose further that z is initial on the ray Oz.*

Then, for $j = 1, 2, \ldots, k$, the rational simplicial cone $C(a_1, \ldots, \hat{a}_j, \ldots, a_k, a_{k+1}, \ldots, a_n, z)$ is of modulus strictly smaller than the modulus of $C(a_1, a_2, \ldots, a_n)$.

Proof. Consider the set X of all integer lattice points of the form

$$\sum_{\substack{i \neq j \\ i=1}}^{n} \alpha_i a_i + \alpha z, \quad \text{where} \quad 0 \leqq \alpha < 1 \quad \text{and} \quad 0 \leqq \alpha_i < 1 \text{ for } i \neq j.$$

Let Γ denote the lattice of all integer lattice points lying in the linear subspace spanned by $a_1, a_2, \ldots, a_n$ and Λ the sublattice of Γ generated by $a_1, a_2, \ldots, a_n$. Let Π denote the canonical projection from Γ to Γ/Λ.

Let x and y be elements of X (a subset of Γ), which are equivalent in Γ/Λ. Suppose that

$$x = \sum_{\substack{i \neq j \\ i=1}}^{n} \alpha_i a_i + \alpha z \quad \text{and} \quad y = \sum_{\substack{i \neq j \\ i=1}}^{n} \beta_i a_i + \beta z$$

where $0 \leqq \alpha, \beta < 1$ and $0 \leqq \alpha_i, \beta_i < 1$ for $i \neq j$. Then

$$x - y = \sum_{\substack{i \neq j \\ i=1}}^{n} (\alpha_i - \beta_i) a_i + (\alpha - \beta) z$$

has integer coefficients relative to the basis $a_1, a_2, \ldots, a_n$ of Γ. Now, the coefficient of a_j in $x - y$ is $(\alpha - \beta)\epsilon_j$. Since $\epsilon_j \neq 0$, and $1 > |\alpha - \beta|\epsilon_j \geqq 0$, it follows that $\alpha = \beta$. Thus for $i \neq j$ the coefficient of a_i in $x - y$ is $\alpha_i - \beta_i$. Since $1 > |\alpha_i - \beta_i| \geqq 0$ for $i \neq j$ it follows that $\alpha_i = \beta_i$ for $i \neq j$, and $x = y$. Hence Π maps the elements of X to distinct equivalence classes in Γ/Λ, and the modulus of $C(a_1, \ldots, \hat{a}_j, \ldots, a_k, a_{k+1}, \ldots, a_n, z)$, which is the cardinality of X, cannot exceed the modulus of $C(a_1, a_2, \ldots, a_n)$, that is, the cardinality of Γ/Λ. Indeed, the cardinality of X is strictly smaller than the cardinality of

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

Γ/Λ, for if $x = \sum_{j \neq i=1}^{n} \alpha_i a_i + \alpha z$, where $0 \leqq \alpha < 1$ and $0 \leqq \alpha_i < 1$ for $i \neq j$, were an element of X congruent to z, then $x - z = \sum_{j \neq i=1}^{n} \alpha_i a_i + (\alpha - 1)z$ would have integral coefficients relative to the basis $a_1, a_2, \ldots, a_n$ of Γ. But the coefficient of a_j in $x - z$ is $(\alpha - 1)\epsilon_j$, which cannot be an integer, since the relations $0 \leqq \alpha < 1$ and $1 > \epsilon_j > 0$ together imply $1 > |(\alpha - 1)\epsilon_j| > 0$.

PROPOSITION 2.1. *Let $\mathscr{K}$ be a complex of rational simplicial cones in $\mathbf{R}^m$. There is a sequence of subdivisions $\alpha_0\mathscr{K} = \mathscr{K}, \alpha_1\mathscr{K}, \ldots, \alpha_t\mathscr{K}$ such that*

(i) *for $1 \leqq i \leqq t$, $\alpha_i\mathscr{K}$ is an elementary starring of $\alpha_{i-1}\mathscr{K}$ at a rational point;*
(ii) *all the simplicial cones in $\alpha_t\mathscr{K}$ are primitive; and*
(iii) *any primitive simplicial cone in $\mathscr{K}$ is also in $\alpha_t\mathscr{K}$.*

Proof. Let $\alpha_0\mathscr{K} = \mathscr{K}$, and suppose that $\alpha_0\mathscr{K}, \ldots, \alpha_{k-1}\mathscr{K}$ have been defined, such that $\alpha_i\mathscr{K}$ is an elementary starring of $\alpha_{i-1}\mathscr{K}$ at a rational point for $i = 1, 2, \ldots, k - 1$, and such that any primitive simplicial cone in $\mathscr{K}$ is also in $\alpha_{k-1}\mathscr{K}$. Let $\alpha_{k-1}\mathscr{K}$ consist of the set of simplicial cones $K_1, K_2, \ldots, K_n$, and let ρ be the maximum value attained by the modulus K_i as i ranges over $1, 2, \ldots, n$.

If $\rho = 1$, then $\alpha_{k-1}\mathscr{K}$ is a complex of simplicial cones satisfying the required conditions. Otherwise $\rho > 1$, and there is an index r for which the simplicial cone K_r has modulus ρ. Let $K_r = K(a_1, a_2, \ldots, a_s)$. Since K_r is not primitive there is a non-zero integer lattice point $z = \sum_{i=1}^{s} \epsilon_i a_i$, which may be supposed initial on the ray Oz, such that for $i = 1, 2, \ldots, s$ the relation $0 \leqq \epsilon_i < 1$ holds. Let $\alpha_k\mathscr{K}$ be that subdivision of $\alpha_{k-1}\mathscr{K}$ obtained by an elementary starring at the point z. If F is that face of K_r which contains z as a relatively interior point, then the simplicial cones of $\alpha_{k-1}\mathscr{K}$ which are subdivided in this way are precisely those which contain F as a face, and none of these is primitive. Hence, every primitive simplicial cone of $\mathscr{K}$ is in $\alpha_k\mathscr{K}$. Finally, applying Lemma 2.1, the maximum value of the modulus of simplicial cones in $\alpha_k\mathscr{K}$ does not exceed ρ, whilst the subdivision $\alpha_k\mathscr{K}$ of $\mathscr{K}$ necessarily has at least one fewer simplicial cone of modulus ρ then $\alpha_{k-1}\mathscr{K}$. Thus the result follows by induction.

COROLLARY 1. *Let C be a primitive rational simplicial cone of dimension k in $\mathbf{R}^m$. There is a primitive simplicial cone of dimension m in $\mathbf{R}^m$ which contains C as a face.*

Proof. Let K be a rational simplicial cone of dimension m containing B as a face, and apply Proposition 2.1 to the complex $\mathscr{K}$ consisting of K and all its faces.

COROLLARY 2. *Let $C = C(a_1, a_2, \ldots, a_k)$ be a primitive rational simplicial cone of dimension k in $\mathbf{R}^m$, and let $y_1, y_2, \ldots, y_k$ be any integer lattice points in $\mathbf{R}^m$. There is a linear map $\mathbf{R}^m \to \mathbf{R}^m$ with integer coefficients mapping a_i to y_i for $i = 1, 2, \ldots, k$.*

Proof. In view of Corollary 2, it suffices to consider the case $k = m$. In that

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

case let $q_1, \ldots, q_m$ be the elements of the standard basis for $\mathbf{R}^m$. Since C is primitive, then the matrix $[a_1, a_2, \ldots, a_m]$, whose ith column is a_i for $i = 1, 2, \ldots, m$, has an inverse which also has integer coefficients. That is to say the unique linear map $\mathbf{R}^m \to \mathbf{R}^m$ which maps a_i to q_i for $i = 1, 2, \ldots, m$ has integer coefficients, and the result follows immediately.

COROLLARY 3. *Suppose that K and L are rational closed polyhedral cones in $\mathbf{R}^m$ which have isomorphic subdivisions into simplicial cones. Then K and L have isomorphic subdivisions into primitive rational simplicial cones.*

Proof. By Lemma 0.1, there are isomorphic subdivisions $\mathscr{K}$ and $\mathscr{L}$ of K and L respectively into rational simplicial cones. Let $\alpha_0\mathscr{K} = \mathscr{K}, \alpha\mathscr{K}, \ldots, \alpha_t\mathscr{K}$ be a sequence of subdivisions of the complex $\mathscr{K}$ which satisfy the conditions prescribed in Proposition 2.1. Suppose that rational subdivisions $\beta_0\mathscr{L} = \mathscr{L}, \beta_1\mathscr{L}, \ldots, \beta_{k-1}\mathscr{L}$ have been defined such that for $i = 0, 1, \ldots, k-1$ there is an isomorphism between the complexes $\alpha_i\mathscr{K}$ and $\beta_i\mathscr{L}$.

Suppose that the complex $\alpha_k\mathscr{K}$ is obtained from $\alpha_{k-1}\mathscr{K}$ by an elementary starring at the rational point z, and that z is relatively interior to the simplicial cone A of $\alpha_{k-1}\mathscr{K}$. Let θ be an isomorphism between the complexes $\alpha_{k-1}\mathscr{K}$ and $\beta_{k-1}\mathscr{L}$, and let B be the simplicial cone associated with A under θ. Suppose that y is a relatively interior rational point of B, and let the complex $\beta_k\mathscr{L}$ be obtained from $\beta_{k-1}\mathscr{L}$ by an elementary starring at the point y. Then $\beta_k\mathscr{L}$ and $\alpha_k\mathscr{K}$ are isomorphic, and by induction there is a subdivision $\beta_t\mathscr{L}$ of $\mathscr{L}$ isomorphic with the subdivision $\alpha_t\mathscr{K}$ of $\mathscr{K}$.

The above argument shows that there are isomorphic subdivisions $\mathscr{K}^*$ and $\mathscr{L}^*$ of K and L respectively into rational simplicial cones, such that $\mathscr{K}^*$ consists entirely of primitive simplicial cones. As in Proposition 2.1, there is a sequence $\gamma_0\mathscr{L}^* = \mathscr{L}^*, \gamma_1\mathscr{L}^*, \ldots, \gamma_r\mathscr{L}^*$ of subdivisions of $\mathscr{L}^*$ such that $\gamma_i\mathscr{L}^*$ is an elementary starring of $\gamma_{i-1}\mathscr{L}^*$ for $i = 1, 2, \ldots, r$, and $\gamma_r\mathscr{L}^*$ consists entirely of primitive simplicial cones. Suppose that for some $k < r$ rational subdivisions $\delta_0\mathscr{K}^* = \mathscr{K}^*, \delta_1\mathscr{K}^*, \ldots, \delta_{k-1}\mathscr{K}^*$ of $\mathscr{K}^*$ have been defined such that for $i = 1, 2, \ldots, k$ the complex $\delta_{i-1}\mathscr{K}^*$ consists entirely of primitive simplicial cones, and is isomorphic with $\gamma_{i-1}\mathscr{L}^*$.

Let $\gamma_i\mathscr{L}^*$ be the complex obtained from $\gamma_{i-1}\mathscr{L}^*$ by an elementary starring at the point q. Let C be the unique simplicial cone of $\gamma_{i-1}\mathscr{L}^*$ which contains q in its relative interior, and suppose that D is the simplicial cone of $\delta_{i-1}\mathscr{K}^*$ associated with C under an isomorphism ϕ between the complexes $\delta_{i-1}\mathscr{K}^*$ and γ_{i-1}. Suppose that $D = D(a_1, a_2, \ldots, a_m)$, and let $x = \sum_{i=1}^m a_i$. If $\delta_i\mathscr{K}^*$ is the complex obtained from $\delta_{i-1}\mathscr{K}^*$ by elementary starring at x, then certainly $\delta_i\mathscr{K}^*$ and $\gamma_i\mathscr{L}^*$ are isomorphic. If I is a simplicial cone in $\delta_i\mathscr{K}^*$, then either I belongs to $\delta_{i-1}\mathscr{K}^*$ and is primitive by the inductive hypothesis, or else I is the result of starring a primitive simplicial cone J of $\delta_{i-1}\mathscr{K}^*$ at the point x. In the latter case J necessarily contains D as a face, so that

$$J = J(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_s)$$

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

and $I = I(a_1, a_2, \ldots, \hat{a}_j, \ldots, a_m, b_1, \ldots, b_s, x)$ for some j, where $1 \leqq j \leqq m$. Since the vectors $a_1, a_2, \ldots, a_n$ generate under addition the same subgroup of $\mathbf{R}^n$ as is generated by $a_1, a_2, \ldots, \hat{a}_j, \ldots, a_m$ and x, it follows at once that I is primitive.

Thus, by induction, there are isomorphic subdivisions $\delta_r\mathscr{K}^*$ and $\gamma_r\mathscr{L}^*$ of K and L respectively, both of which are complexes of primitive simplicial cones.

3. Applications.

THEOREM 3.1. *The projective lattice-ordered Abelian groups which are generated by n elements are the quotients of Fl-$G(n)$ by its principal ideals.*

Proof. For the proof that a projective lattice-ordered Abelian group generated by n elements is the quotient of Fl-$G(n)$ by a principal ideal, see Baker [**1**, Theorem 5.1]. (Baker's proof generalises directly to the context of lattice-ordered Abelian groups).

For the converse, let A be the quotient of Fl-$G(n)$ by a principal ideal. Under the categorical duality of Theorem 1, A is associated with a rational closed polyhedral cone C in $\mathbf{R}^n$ (see Baker [**1**] and Beynon [**3**, § 1]), and there are subdivisions $\mathscr{K}$ and $\mathscr{L}_0$ of C and $\mathbf{R}^n$ respectively into rational simplicial cones, such that $\mathscr{K}$ is a subcomplex of $\mathscr{L}_0$ (see Lemma 0.1). Form the subdivision $\mathscr{L}$ of $\mathscr{L}_0$ by starring each simplicial cone A of $\mathscr{L}_0 \neq \mathscr{K}$ which meets $\mathscr{K}$ in its whole boundary at a rational interior point. Then $\mathscr{K}$ and $\mathscr{L}$ are subdivisions of C and $\mathbf{R}^n$ respectively into rational simplicial cones such that $\mathscr{K}$ is a full subcomplex of $\mathscr{L}$. (cf. [**8**, Lemma 3.3]).

By Proposition 2.1, $\mathscr{L}$ has a subdivision $\mathscr{L}'$ consisting entirely of primitive rational simplicial cones. The associated subdivision, $\mathscr{K}'$ of $\mathscr{K}$ is then a full subcomplex of $\mathscr{L}'$ consisting of primitive simplicial cones. (See [**8**, Lemma 3.3]).

Suppose that the 1-dimensional simplicial cones in $\mathscr{L}'$ are $Ox_1, Ox_2, \ldots, Ox_k$, and let r be the uniquely determined map $\mathbf{R}^n \to \mathbf{R}^n$ which is piecewise homogeneous linear with respect to $\mathscr{L}'$, and maps x_i to itself if Ox_i is a 1-dimensional simplicial cone of $\mathscr{K}'$, and to zero otherwise. Since $\mathscr{K}'$ is a full subcomplex of $\mathscr{L}'$, r is a piecewise homogeneous linear retract from $\mathbf{R}^n$ onto C. By Corollary 2 to Proposition 2.1, the retract r acts as the restriction of a linear map with integer coefficients $\mathbf{R}^n \to \mathbf{R}^n$ on each simplicial cone of $\mathscr{L}'$. As remarked in § 1, this shows that each of the maps $\mathbf{R}^n \to \mathbf{R}$ obtained by composing r with a canonical projection $\mathbf{R}^n \to \mathbf{R}$ is an integral l-map, so that r is an integral l-map $\mathbf{R}^n \to C$, by definition.

Consider the l-morphism $r^* : A \to Fl$-$G(n)$ associated with r under the categorical duality of Theorem 1.1. Then the composite map $\Pi r^* : A \to A$ is the identity on A (here Π denotes the canonical projection Fl-$G(n) \to A$), and A is the retract of a free algebra. Thus A is projective.

If C is a rational closed polyhedral cone in $\mathbf{R}^m$, a map $f : C \to \mathbf{R}^n$ will be called *piecewise homogeneous linear with integer coefficients* if and only if it is the restriction of a piecewise homogeneous linear map with integer coeffi-

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

cients $\mathbf{R}^m \to \mathbf{R}^n$. Theorem 3.1 makes it possible to characterise integral l-maps between rational closed polyhedral cones.

COROLLARY 1 (to Theorem 3.1). *Let C be a rational closed polyhedral cone in $\mathbf{R}^m$ and $f : C \to \mathbf{R}^n$ a piecewise homogeneous linear map with integer coefficients. Then f is an integral l-map.*

Proof. It suffices to consider the case $n = 1$, for if $n > 1$, then f can be regarded as the product of n maps $C \to \mathbf{R}$, each of which is a piecewise homogeneous linear map with integer coefficients.

If $n = 1$, let r be a retract $\mathbf{R}^n \to C$ constructed as in Theorem 3.1 and consider the composite map $fr : \mathbf{R}^n \to \mathbf{R}$. Since fr is a piecewise homogeneous linear map with integer coefficients $\mathbf{R}^n \to \mathbf{R}$ it is an integral l-map $\mathbf{R}^n \to \mathbf{R}$, as remarked in § 1. Hence f, which is the restriction of fr to C, is also an integral l-map.

COROLLARY 2. *The full subcategory of the category of finitely generated lattice-ordered Abelian groups consisting of projective lattice-ordered Abelian groups is equivalent to the dual of the category whose objects are rational Euclidean closed polyhedral cones, and whose morphisms are piecewise homogeneous linear maps with integer coefficients.*

Proof. It is enough to observe that every integral l-map between rational closed polyhedral cones is piecewise homogeneous linear with integer coefficients, and that rational closed polyhedral cones are in bijective correspondence with quotients of $Fl\text{-}G(n)$ by its principal ideals. (See Beynon [**3**, § 1], or Baker [**1**]).

COROLLARY 3. *Let V and W be rational closed polyhedral cones in $\mathbf{R}^m$. Then V and W are integrally l-equivalent if and only if they are l-equivalent.*

Proof. It is sufficient to show that if V and W have polyhedrally equivalent sections then there is an integral l-equivalence mapping V to W. (See [**3**, Corollary 2 to Theorem 4.1]).

Let P and Q be polyhedral sections of V and W respectively, and suppose that $\mathscr{S}$ and $\mathscr{T}$ are isomorphic simplicial subdivisions of P and Q respectively. Let $\mathscr{K}$ be the subdivision of V into simplicial cones induced by $\mathscr{S}$, and $\mathscr{L}$ the subdivision of W into simplicial cones induced by $\mathscr{T}$; then $\mathscr{K}$ and $\mathscr{L}$ are isomorphic subdivisions of V and W into simplicial cones.

By Corollary 3 to Proposition 2.1 there exist isomorphic subdivisions $\mathscr{K}^*$ and $\mathscr{L}^*$ of V and W into primitive rational simplicial cones. Let $Ox_1, Ox_2, \ldots, Ox_k$ be the 1-dimensional simplicial cones of $\mathscr{K}^*$ and $Oy_1, Oy_2, \ldots, Oy_k$ the corresponding 1-dimensional simplicial cone of $\mathscr{L}^*$, where for each i the points x_i and y_i are the initial integer lattice points on their respective rays. There is an uniquely defined bijective map $\theta : V \to W$, piecewise homogeneous linear with respect to $\mathscr{K}^*$, mapping x_i to y_i for $i = 1, 2, \ldots, k$ and θ is an integral l-map in view of Corollary 2 to Proposition 2.1 and Corollary 1 to Theorem 3.1.

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

Corollary 3 makes it possible to give simple proofs of non-trivial algebraic results about finitely generated projective lattice-ordered Abelian groups; this is illustrated by the examples described below. It is convenient to introduce the symbol $Fl\text{-}G^+(n)$ to denote the finitely generated projective lattice-ordered Abelian group which is associated with a rational closed cone in $\mathbf{R}^n$ having as its section a polyhedral $(n - 1)$-ball. It is easy to see that if $e_1, e_2, \ldots, e_n$ are free generators of $Fl\text{-}G(n)$, then $Fl\text{-}G^+(n)$ is the quotient of $Fl\text{-}G(n)$ by the ideal generated by the relations $e_i \geqq 0$ for $i = 1, 2, \ldots, n$.

Example 1. *A classification of projective lattice-ordered Abelian groups with* 2 *generators.* (cf. Bleier [**6**] where an analogous characterisation is described for projective vector lattices with 2 generators).

By Corollary 3, there is a bijective correspondence between isomorphism types of projective lattice-ordered Abelian groups with 2 generators and l-equivalence classes of rational closed polyhedral cones in $\mathbf{R}^2$. Each rational closed polyhedral cone C in $\mathbf{R}^2$ can be expressed uniquely as a union of finitely many rational closed polyhedral cones $C_1, C_2, \ldots, C_k$, where $C_i \cap C_j = \{0\}$ if $i \neq j$, such that each cone is either a single ray or has its section a polyhedral 1-simplex. Following the notation of [**3**], let $\Gamma(C)$ be the projective lattice-ordered Abelian group associated with C under the duality of Theorem 1.1. Then $\Gamma(C)$ is isomorphic with the algebra of all integral l-maps C to R under pointwise operations of addition, supremum and infimum (See [**3**, Lemma 2.1] and § 1 of this paper), and is identified with the direct product $\Gamma(C_1) \times \Gamma(C_2) \times \ldots \times \Gamma(C_n)$ by the map sending the integral l-map $f : C \rightarrow R$ to $(f|_{C_1}, f_{C_2}, \ldots, f|_{C_n})$. Since each of the lattice-ordered Abelian groups $\Gamma(C_i)$ is isomorphic to $\mathbf{Z}$ or to $Fl\text{-}G^+(2)$, this shows that every projective lattice-ordered Abelian group with 2 generators other than $Fl\text{-}G(2)$ itself is a direct product of finitely many copies of Z and copies of $Fl\text{-}G^+(2)$, and has an unique representation of this form.

Example 2. *A characterisation of lattice-ordered Abelian groups freely generated by finite partially-ordered sets.*

Let X be a finite partially-ordered set, and let $F(X)$ denote the lattice-ordered Abelian group freely generated by X. That is to say, let μ be an order-preserving map from X into a lattice-ordered Abelian group $F(X)$ with the universal property that if θ is any order-preserving map from X into a lattice-ordered Abelian group A there is an unique l-morphism $\phi : F(X) \rightarrow A$ such that $\phi\mu = \theta$. A presentation for $F(X)$ can be described as follows: let $x_1, x_2, \ldots, x_r$ be the elements of X and let P be the subset of $X \times X$ consisting of pairs (x_i, x_j) such that $x_i \leqq x_j$. Let $Fl\text{-}G(r)$ be freely generated by $e_1, e_2, \ldots, e_r$ and let $F(X)$ be the quotient of $Fl\text{-}G(r)$ by the ideal generated by all relations of the form $e_i \vee e_j = e_j$, where $(x_i, x_j) \in P$. The map $\mu : X \rightarrow F(X)$ mapping x_i to e_i is then order-preserving, and has the stated universal property.

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

To examine the isomorphism type of $F(X)$, consider the rational closed polyhedral cone in $\mathbf{R}^n$ canonically associated with $F(X)$. (See Baker [**1**], or [**3**, § 1]). Then C is the set of points $(a_1, a_2, \ldots, a_r)$ such that $a_i \vee a_j = a_j$ whenever $(x_i, x_j) \in P$. Since $a_i \vee a_j = a_j$ if and only if $a_i - a_j \leqq 0$, it follows that C is an intersection (possibly empty) of closed half-spaces, so that C is a convex closed polyhedral cone in $\mathbf{R}^n$. Thus, C has as section either a polyhedral n-sphere for some $n \leqq r - 1$ or a polyhedral n-ball for $n \leqq r - 1$. Hence, by Corollary 3, $F(X)$ has the isomorphism type of $Fl\text{-}G(n)$ or $Fl\text{-}G^+(n)$ for some $n \leqq r$, and each of these possibilities is realised for some partially-ordered set X.

Example 3. Consider the lattice-ordered Abelian group Σ_n consisting of elements of $Fl\text{-}G(n)$ fixed under all l-automorphisms of $Fl\text{-}G(n)$ which are obtained by mapping the free generators $e_1, e_2, \ldots, e_n$ into themselves. For $r = 1, 2, \ldots, n$ let s_r denote the element $\bigwedge(e_{i_1} \vee e_{i_2} \vee \ldots \vee e_{i_r})$ where the intersection is taken over all distinct r-subsets $\{i_1, i_2, \ldots, i_r\}$ of $\{1, 2, \ldots, n\}$. Each s_r is an element of Σ_n, and $s_1 \leqq s_2 \leqq \ldots \leqq s_n$. Moreover, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are integers such that $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$ then for $r = 1, 2, \ldots, n$, the identity $s_r(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_r$ holds. Since each element of Σ_n is determined by the integral l-map $\mathbf{Z}^n \rightarrow \mathbf{Z}$ which it defines, it follows that if $w(e_1, e_2, \ldots, e_n)$ is an element of Σ_n, then $w(e_1, e_2, \ldots, e_n) = w(s_1, s_2, \ldots, s_n)$. Hence the elements $s_1, s_2, \ldots, s_n$ generate Σ_n.

Consider the kernel K of the map $P : Fl\text{-}G(n) \rightarrow \Sigma_n$ defined by setting $P(e_i) = s_i$ for $i = 1, 2, \ldots, n$. Since every element of Σ_n is fixed by P it is easy to show that K consists of those elements of $Fl\text{-}G(n)$ which can be expressed in the form $w(e_1, e_2, \ldots, e_n) - w(s_1, s_2, \ldots, s_n)$ where $w(e_1, e_2, \ldots, e_n)$ is an arbitrary element of $Fl\text{-}G(n)$. The quotient $Fl\text{-}G(n)/K$ is then associated with the closed cone C in $\mathbf{R}^n$ consisting of all points $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for which $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ for all f in K (see [**3**, § 1]). Trivially C contains the rational closed polyhedral cone C' consisting of all points α such that $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$. On the other hand, if $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ lies outside C', then there is an index r for which β_r is strictly greater than β_j for at least r distinct values of the index j, and the element $e_r - s_r$, which lies in K, is non-zero at the point β. Hence C and C' coincide.

Thus the elements $s_1, s_2, \ldots, s_n$ generate Σ_n freely subject to the relations $s_1 \leqq s_2 \leqq \ldots \leqq s_n$, and in particular Σ_n is isomorphic with $Fl\text{-}G^+(n)$.

Acknowledgements. The author wishes to thank Professor Philip Higgins for helpful discussions on the topic considered in this paper, to thank Professor Kirby Baker for incidental comments and advice, and to thank the S.R.C. for its financial support. He is also grateful to the referee for his most conscientious report, which contained valuable suggestions regarding presentation.

References

1. K. A. Baker, *Free vector lattices*, Can. J. Math. *20* (1968), 58–66.

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press

2. W. M. Beynon, *Combinatorial aspects of piecewise linear maps*, J. London Math. Soc. (2) (1974), 719–727.

3. ——— *Duality theorems for finitely generated vector lattices*, Proc. London Math. Soc. (3) *31* (1975), 114–128.

4. ——— *On rational subdivisions of polyhedra with rational vertices*, Can. J. Math. *29* (1977), 238–242.

5. G. A. Birkhoff, *Lattice theory*, A. M. S. Colloquium Publications Vol. XXV 3rd edition (1967).

6. R. J. Bleier, *Archimedean vector lattices generated by 2 elements*, Proc. Amer. Math. Soc. *39* (1973), 1–9.

7. L. C. Glaser, *Geometrical combinatorial topology, Vol. 1*, Van Nostrand Reinhold Mathematical Studies *27*.

8. C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete Band *69* (Springer-Verlag, 1972).

9. J. R. Stallings, *Lectures on polyhedral topology*, Tata Institute of Fundamental Research, Bombay (1968).

University of Warwick,
Coventry, England

https://doi.org/10.4153/CJM-1977-026-4 Published online by Cambridge University Press