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A DUAL CHARACTERISATION OF THE EXISTENCE OF SMALL COMBINATIONS OF SLICES

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We characterise, by a property of roughness, the norms of a Banach space X such that the dual unit ball has no small combination of $w^\star$-slices. Among separable Banach spaces, the existence of an equivalent norm for this new property of roughness characterises spaces which contain an isomorphic copy of $\ell_1(\mathbb{N})$.

1 INTRODUCTION

Throughout this paper, X denotes a Banach space, $B(X)$ its unit ball, $S(X)$ its unit sphere, $B(X^\star)$ the unit ball of its dual and $S(X^\star)$ the unit sphere of its dual. Let us first recall some basic definitions and introduce the notion of "average rough norm":

Definitions.

1. Let C be a closed convex subset of X. We say that C is ε-dentable (respectively ε-$w^\star$-dentable if X is a dual space) if there exists a slice S (respectively $w^\star$-slice S) with $\operatorname{diam}(S) < \varepsilon$ ($\operatorname{diam}(S)$ denotes the diameter of S).

We say that C contains an ε-combination of slices (respectively an ε-combination of $w^\star$-slices) if there are slices (respectively $w^\star$-slices) $S_1, \dots, S_n$ of C with $\operatorname{diam}\left(\frac{1}{n}(S_1 + \cdots + S_n)\right) < \varepsilon$.

2. A one-sided Gâteaux differential of the norm $\|\ \|$ of X at $x \in X$ is a function $d^+\|x\| : X \to \mathbb{R}$ such that, for all $u \in X$, $d^+\|x\|(u) = \lim_{t \to 0^+} \dfrac{\|x + tu\| - \|x\|}{t}$.

3. We say that the norm of X, or merely X where there is no ambiguity, is ε-rough if for all $x \in S(X)$ and for all $\eta > 0$ there exist $y, z \in S(X)$ and $u \in S(X)$ such that

(a) $\|y - x\| < \eta$ and $\|z - x\| < \eta$
(b) $(d^+\|y\| - d^+\|z\|)(u) > \varepsilon - \eta$.

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4. We say that the norm of X, or merely X when there is no ambiguity, is ε-average rough if, for all $x_1, \dots, x_n \in S(X)$ and for all $\eta > 0$, there exist $y_1, \dots, y_n$, $z_1, \dots, z_n \in S(X)$ and $u \in S(X)$ such that:

(a) for all i, $1 \leqslant i \leqslant n$, $\|y_i - x_i\| < \eta$ and $\|z_i - z_i\| < \eta$

(b) $\frac{1}{n} \sum_{i=1}^{n} (d^+\|y_i\| - d^+\|z_i\|)(u) > \varepsilon - \eta$.

We refer the reader to [**2**] and [**7**] for a study of the small combination of slices property.

An obvious observation is that, if X is ε-average rough, then X is ε-rough. More precisely, in the definition of ε-average roughness, for any $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in S(X)$, there exists a common direction of roughness u for many of the x_i, $1 \leqslant i \leqslant n$ (but not necessarily for all of them).

In [**6**], Leach and Whitfield introduced and studied rough norms. In [**5**], John and Zizler have shown that X is ε-rough if and only if $B(X^\star)$ is not ε-$w^\star$-dentable. Moreover, it is shown, for instance in [**3**], that for separable Banach spaces, the existence of an equivalent rough norm characterises the separable Banach spaces with non-separable dual.

We shall show that X is ε-average rough if and only if $B(X^\star)$ does not contain any ε-combination of $w^\star$-slices. Moreover, for separable Banach spaces, the existence of an equivalent ε-average rough norm, for some $\varepsilon > 0$, characterises the spaces which contain $\ell^1(\mathbb{N})$.

2 Characterisation of average rough norms.

THEOREM 1. *Let $0 < \varepsilon < 1$ and let X be a Banach space. The following conditions are equivalent:*

1. *X is ε-average rough;*

2. *For each $x_1, \dots, x_n \in S(X)$,*

$$\limsup_{\|y\| \to 0} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \right) \geqslant \varepsilon;$$

3. *$B(X^\star)$ does not contain any ε-combination of $w^\star$-slices.*

Dually, an analogous result holds:

THEOREM 2. *The following conditions are equivalent:*

1. *$X^\star$ is ε-average rough;*

2. *For each $x_1^\star, \dots, x_n^\star \in S(X^\star)$,*

$$\limsup_{y^\star \in X^\star, \|y^\star\| \to 0} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\|x_i^\star + y^\star\| + \|x_i^\star - y^\star\| - 2}{\|y^\star\|} \right) \geqslant \varepsilon;$$

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3. *$B(X)$ does not contain any ε-combination of slices.*

REMARK:: These two results are isometric, and give dual characterisation of the existence of small combinations of slices $B(X)$ (respectively small combinations of $w^\star$-slices of $B(X^\star)$).

PROOF OF THEOREM 1: Some of the arguments are refinements of ideas in [**5**] and [**8**].

(1) $\Rightarrow$ (3): Suppose X is ε-average rough and let $S_1, \dots, S_n$ be $w^\star$-slices of $B(X^\star)$. Replacing $S_1, \dots, S_n$ by smaller $w^\star$-slices, we can assume that for all $i \in \{1, \dots, n\}$, $S_i = S_i(x_i, B(X^\star), \eta) = \{f \in B(X^\star);\ f(x_i) > 1 - \eta\}$, with $x_i \in S(X)$ and $\eta > 0$.

By hypothesis there exists $y_1, \dots, y_n$, $z_1, \dots, z_n \in S(X)$ and $u \in S(X)$ satisfying:

a. for all i, $\|y_i - x_i\| < \eta$ and $\|z_i - x_i\| < \eta$;
b. $\frac{1}{n} \sum_{i=1}^{n} (d^+\|y_i\| - d^+\|z_i\|)(u) > \varepsilon - \eta$.

Using the Ascoli-Mazur theorem ([4]), choose $f_i, g_i \in S(X^\star)$ satisfying $f_i(y_i) = 1$, $f_i(u) = d^+\|y_i\|(u)$, $g_i(z) = 1$ and $g_i(u) = d^+\|z_i\|(u)$. Condition (b) implies that $\frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)(u) > \varepsilon - \eta$ hence $\|\frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)\| > \varepsilon - \eta$. On the other hand, $f_i(x_i) \geqslant f_i(y_i) - \|x_i - y_i\| > 1 - \eta$ and so $f_i \in S_i$; an analogous calculation shows that $g_i \in S_i$ and we have shown that $\operatorname{diam}\left(\frac{1}{n}(\sum_{i=1}^{n} S_i)\right) > \varepsilon - \eta$.

Since if we replace η by $\eta' \in (0, \eta)$, then the S_i are replaced by $S_i' \subset S_i$ and

$$\operatorname{diam}\left(\frac{1}{n}\left(\sum_{i=1}^{n} S_i\right)\right) \geqslant \operatorname{diam}\left(\frac{1}{n}\left(\sum_{i=1}^{n} S_i'\right)\right) > \varepsilon - \eta'$$

this shows that $\operatorname{diam}\left(\frac{1}{n}(\sum_{i=1}^{n} S_i)\right) \geqslant \varepsilon$.

(3) $\Rightarrow$ (2): Let $x_i, \dots, x_n \in S(X)$, let λ and α be two non-negative real numbers, and let $\delta \in (0, \alpha\lambda\varepsilon)$ be fixed. By hypothesis, $B(X^\star)$ does not contain any ε-combination of $w^\star$-slices, so there exist $f_1, \dots, f_n$, $g_1, \dots, g_n$ so that

a. for all i, $f_i \in S_i$ and $g_i \in S_i$, where $S_i = S(x_i, B(X^\star), \delta)$;
b. $\left\| \frac{1}{n} \sum_{i=1}^{n} f_i - \frac{1}{n} \sum_{i=1}^{n} g_i \right\| > \varepsilon(1 - \alpha)$.

So there exists $u \in S(X)$ such that $\frac{1}{n}\sum_{i=1}^{n}(f_i - g_i)(u) > \varepsilon(1-\alpha)$, hence

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(\|x_i+\lambda u\| + \|x_i-\lambda u\|) &\geqslant \frac{1}{n}\sum_{i=1}^{n}(f_i(x_i+\lambda u) + g_i(x_i-\lambda u)) \\
&\geqslant \frac{1}{n}\sum_{i=1}^{n} f_i(x_i) + \frac{1}{n}\sum_{i=1}^{n} g_i(x_i) + \lambda\left(\frac{1}{n}\sum_{i=1}^{n}(f_i-g_i)(u)\right) \\
&\geqslant 1-\delta+1-\delta+\lambda\varepsilon(1-\alpha) \\
&\geqslant 2-2\lambda\alpha\varepsilon+\lambda\varepsilon(1-\alpha) \\
&\geqslant 2+\lambda\varepsilon(1-3\alpha).
\end{aligned}$$

So we have:

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\|x_i+\lambda u\|+\|x_i-\lambda u\|-2}{\lambda}\right) \geqslant \varepsilon(1-3\alpha).$$

This shows that, for every $\lambda > 0$:

$$\sup_{y\in X,\,\|y\|=\lambda} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\|x_i+y\|+\|x_i-y\|-2}{\|y\|}\right) \geqslant \varepsilon$$

whence

$$\limsup_{y\in X,\,\|y\|\to 0} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\|x_i+y\|+\|x_i-y\|-2}{\|y\|}\right) \geqslant \varepsilon.$$

(2) $\Rightarrow$ (1): Let $x_i,\dots,x_n \in S(X)$ and $\eta \in (0,1)$. By hypothesis, there exists $u \in S(X)$ and $t \in (0,\eta/3)$ such that

$$\text{(1)} \qquad \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\|x_i+tu\|+\|x_i-tu\|-2}{t}\right) \geqslant \varepsilon - \frac{\eta}{3}$$

Since the real functions $\varphi_i : t \to \|x_i + tu\|$ are convex, if φ_i' denotes the right derivative of φ_i, we have, for $t > 0$:

$$\varphi_i'(t) \geqslant \frac{1}{t}(\varphi_i(t)-\varphi_i(0)) \text{ and } \varphi_i'(-t) \leqslant \frac{1}{t}(\varphi_i(0)-\varphi_i(-t)).$$

So

$$d^+\|x_i+tu\|(u) \geqslant \frac{1}{t}(\|x_i+tu\|-1)$$

and

$$-d^+\|x_i-tu\|(u) \geqslant \frac{1}{t}(\|x_i-tu\|-1).$$

Summing these inequalities for $1 \leqslant i \leqslant n$:

$$\sum_{i=1}^{n} \left(d^{+}\|x_i + tu\| - d^{+}\|x_i - tu\|\right)(u) \geqslant \sum_{i=1}^{n} \frac{\|x_i + tu\| + \|x_i - tu\| - 2}{t}.$$

Therefore, using (1):

$$(2) \qquad \frac{1}{n}\sum_{i=1}^{n} \left(d^{+}\|x_i + tu\| - d^{+}\|x_i - tu\|(u)\right) \geqslant \varepsilon - \frac{\eta}{3}.$$

Putting $y_i = \dfrac{x_i + tu}{\|x_i + tu\|}$, $z_i = \dfrac{x_i - tu}{\|x_i - tu\|}$ we have that

(a) For all i, $y_i, z_i \in S(X)$.

(b) For all i, $\|x_i - y_i\| \leqslant \left|\dfrac{\|x_i + tu\| - 1}{\|x_i - tu\|}\right| + \dfrac{\|tu\|}{\|x_i + tu\|} \leqslant \dfrac{2t}{\|x_i + tu\|} \leqslant 3t$ so $\|x_i - y_i\| < \eta$ and similarly $\|x_i - z_i\| < \eta$.

(c) $\frac{1}{n}\sum_{i=1}^{n}\left(d^{+}\|y_i\| - d^{+}\|z_i\|\right)(u) \geqslant \varepsilon - \eta$.

Let us check condition (c). Indeed, for each i,

$$\|x_i + tu\| d^{+}\|y_i\|(u) = d^{+}\|x_i + tu\|(u)$$

and

$$\|x_i - tu\| d^{+}\|z_i\|(u) = d^{+}\|x_i - tu\|(u)$$

therefore

$$\begin{aligned} d^{+}\|y_i\|(u) - d^{+}\|z_i\|(u) &\geqslant d^{+}\|x_i + tu\|(u) - d^{+}\|x_i - tu\|(u) \\ &\quad - \left|(\|x_i + tu\| - 1) d^{+}\|x_i + tu\|(u)\right| \\ &\quad - \left|(\|x_i - tu\| - 1) d^{+}\|x_i - tu\|(u)\right| \\ &\geqslant d^{+}\|x_i + tu\|(u) - d^{+}\|x_i - tu\|(u) - 2t. \end{aligned}$$

Condition (c) is obtained by summing these inequalities and applying (2). This shows that the norm of X is ε-average rough and completes the proof of Theorem 1. ∎

The proof of Theorem 2 is similar and left to the interested reader. Note that in the proof of Theorem 2, (1) $\Rightarrow$ (3), it is enough to choose $f_i, g_i \in S(X)$ satisfying $f_i(y_i) > 1 - (\eta - \|x_i - y_i\|)$, $g_i(z_i) > 1 - (\eta - \|z_i - x_i\|)$ and analogous conditions for $f_i(u)$ and $g_i(u)$, and to apply the local reflexivity principle ([**10**, Theorem 3.1, p.33]).

3 Examples and applications

In [**3**], Godefroy and Maurey define a norm $\| \quad \|$ on X to be *everywhere octahedral* if, for every finite dimensional subspace Y of X and every $\varepsilon > 0$, there is an $x \in X \backslash \{0\}$ (depending on Y and ε) such that for all $t \in Y$, $\|t + x\| \geqslant (1-\varepsilon)(\|t\| + \|x\|)$.

Example:: The usual norm in $\ell_1(\mathbb{N})$ is everywhere octahedral. Indeed, let Y be a finite dimensional supspace of $\ell_1(\mathbb{N})$ and $\varepsilon > 0$. By compactness of $S(Y)$, we can find an $n \in \mathbb{N}$ such that, if Z is the subspace of $\ell_1(\mathbb{N})$ whose elements are supported by the n first coordinates, then for all $y \in S(Y)$,

$$d(y, Z) = \inf\{\|y - z\|; z \in Z\} < \varepsilon/2.$$

Let $x \in \ell^1(\mathbb{N})\backslash\{0\}$ have its first n coordinates equal to 0. If $y \in Y$, choose $z \in Z$ such that $\|y - z\| < \frac{\varepsilon}{2}\|y\|$, then

$$\begin{aligned} \|y + x\| &\geqslant \|z + x\| - \|y - z\| \\ &\geqslant \|z\| + \|x\| - \|y - z\| \\ &\geqslant \|y\| + \|x\| - 2\|y - z\| \\ &\geqslant (1-\varepsilon)(\|y\| + \|x\|) \end{aligned}$$

as required.

Proposition 3. *An everywhere octahedral norm is* 2*-average rough.*

Proof: Observe that by homogeneity, a norm on X is everywhere octahedral if and only if, for every finite dimensional subspace Y of X and every $\varepsilon > 0$, there is an $x \in X$, $\|x\| = 1$ such that, for all $t \in Y$ and $\alpha \in \mathbb{R}$, $\|t + \alpha x\| \geqslant (1-\varepsilon)(\|t\| + |\alpha|)$.

Let $\varepsilon > 0$ and let $x_1, \dots, x_n \in X$ be of norm 1. Denote by Y the linear space spanned by $x_1, \dots, x_n$. By the previous remark, there is an $x \in S(X)$ such that for all $y \in Y$,

$$\|t + \sqrt{\varepsilon} x\| \geqslant (1-\varepsilon)\big(\|t\| + \sqrt{\varepsilon}\big)$$

and

$$\|t - \sqrt{\varepsilon} x\| \geqslant (1-\varepsilon)\big(\|t\| + \sqrt{\varepsilon}\big).$$

So

$$\frac{\|t + \sqrt{\varepsilon}\, x\| + \|t - \sqrt{\varepsilon}\, x\| - 2\|t\|}{\sqrt{\varepsilon}} \geqslant 2\big(1 - \varepsilon - \sqrt{\varepsilon}\,\|t\|\big).$$

Applying this inequality successively for $t = x_i$, $1 \leqslant i \leqslant n$, and summing:

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\|x_i + h_\varepsilon\| + \|x_i - h_\varepsilon\| - 2}{\|h_\varepsilon\|} \geqslant 2\big(1 - \varepsilon - \sqrt{\varepsilon}\big),$$

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where $h_\varepsilon = \sqrt{\varepsilon}x$. This inequality holds for arbitary $\varepsilon > 0$, so the proposition follows from Theorem 2(2). ∎

REMARKS: (a) The usual norm in $\ell_1(\mathbb{N})$ is everywhere octahedral, hence, by Proposition 3, it is 2-average rough.

(b) Remark (a) is false if we replace the usual norm of $\ell^1(\mathbb{N})$ by an equivalent norm. Indeed, there exists on $c_0(\mathbb{N})$ an equivalent locally uniformly rotund norm, hence its unit ball is dentable and the dual norm in $\ell^1(\mathbb{N})$ is not even rough.

(c) The author does not know if the converse of Proposition 3 holds.

PROPOSITION 4. *Let X be a separable Banach space. The following are equivalent:*

1. *X has an equivalent ε-average rough norm for some $\varepsilon > 0$;*

2. *X has an equivalent 2-average rough norm;*

3. *X contains $\ell^1(\mathbb{N})$.*

PROOF: (1) $\Rightarrow$ (3): If there exists an ε-average norm on X, then, by Theorem 1, $B(X^\star)$ does not contain any ε-combination of $w^\star$-slices, and so, by a result of Bourgain ([**1**, lemme 3–7]), X contains $\ell^1(\mathbb{N})$.

(3) $\Rightarrow$ (2): If X is separable and contains $\ell^1(N)$, then Godefroy and Maurey ([**3**, corollaire III.13]) showed that there exists on X an everywhere octahedral norm and by Proposition 3, this norm is 2-average rough.

(2) $\Rightarrow$ (1) is obvious. ∎

PROPOSITION 5. *If a Banach space X is ε-average rough, then there is a separable closed subspace $Y \subset X$ such that Y, with the restricted norm of X, is also average rough.*

PROOF: Let Y_0 be a one-dimensional subspace of X and let us define a sequence of finite dimensional subspaces of X in the following way: suppose $Y_0, Y_1, \dots, Y_p$ have been defined and choose a $\frac{1}{p}$-net $\mathcal{A}_p = \{x_1, x_2, \dots, x_{k_p}\}$ in $S(Y_p)$ and for each subset A of $\mathcal{A}_p$, if we denote $A = \{t_1, \dots, t_n\}$, choose $y_1^A, \dots, y_n^A$, $z_1^A, \dots, z_n^A$ and $u^A \in S(X)$ satisfying:

a. for all $i \in \{1, \dots, n\}$, $\|y_i^A - t_i\| < \frac{1}{p}$ and $\|z_i^A - t_i\| < \frac{1}{p}$;

b. $\frac{1}{n}\sum_{i=1}^n \left(d^+\|y_i^A\| - d^+\|z_i^A\|\right)\left(u^A\right) > \varepsilon - \frac{1}{p}$.

Let Y_{p+1} be the subspace of X spanned by Y_p and all the y_i^A, z_i^A, u^A (for all $A \subset \mathcal{A}_p$). Let $Y = \overline{\bigcup_{p\in\mathbb{N}} Y_p}$. Then Y is a separable subspace of X and Y is ε-average rough. ∎

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COROLLARY 6. *Let X be a Banach space (not necessarily separable) which is ε-average rough for some $\varepsilon > 0$. Then X contains $\ell^1(\mathbb{N})$.*

PROOF: By Proposition 5, there exists a separable subspace Y of X which is ε-average rough. By Proposition 4, Y contains $\ell^1(\mathbb{N})$ and the corollary follows. ∎

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