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Determinant of the Laplacian on Tori of Constant Positive Curvature with one Conical Point

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Abstract. We find an explicit expression for the zeta-regularized determinant of (the Friedrichs extensions of) the Laplacians on a compact Riemann surface of genus one with conformal metric of curvature 1 having a single conical singularity of angle 4π.

1 Introduction

Let X be a compact Riemann surface of genus one and let $P \in X$. According to [1, Cor. 3.5.1], there exists at most one conformal metric on X of constant curvature 1 with a (single) conical point of angle 4π at P. The following simple construction shows that such a metric, $m(X,P)$, in fact always exists (and, due to [1], is unique).

Consider the spherical triangle $T = \{(x_1,x_2,x_3) \in S^2 \subset \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ with all three angles equal to $\pi/2$. Gluing two copies of T along their boundaries, we get the Riemann sphere $\mathbb{C}P^1$ with metric m of curvature 1 and three conical points P_1, P_2, P_3 of conical angle π. Consider the two-fold covering

$$\mu\colon X(Q) \longrightarrow \mathbb{C}P^1$$

ramified over P_1, P_2, P_3 and some point $Q \in \mathbb{C}P^1 \setminus \{P_1,P_2,P_3\}$. Lifting the metric m from $\mathbb{C}P^1$ to the compact Riemann surface $X(Q)$ of genus one via μ, one gets the metric μ^*m on $X(Q)$ that has curvature 1 and the unique conical point of angle 4π at the preimage $\mu^{-1}(Q)$ of Q. Clearly, any compact surface of genus one is (biholomorphically equivalent to) $X(Q)$ for some $Q \in \mathbb{C}P^1 \setminus \{P_1,P_2,P_3\}$. Now let X be an arbitrary compact Riemann surface of genus one and let P be any point of X. Take $Q \in \mathbb{C}P^1$ such that $X = X(Q)$ and consider the automorphism $\alpha\colon X \to X$ (the translation) of X sending P to $\mu^{-1}(Q)$. Then

$$m(X,P) = \alpha^*\big(\mu^*(m)\big) = (\mu\circ\alpha)^*(m).$$

Introduce the scalar (Friedrichs) self-adjoint Laplacian $\Delta(X,P) := \Delta^{m(X,P)}$ on X corresponding to the metric $m(X,P)$. For any P and Q from X the operators $\Delta(X,P)$ and $\Delta(X,Q)$ are isospectral and, therefore, the ζ-regularized (modified, *i.e.*, with zero modes excluded) determinant $\det\Delta(X,P)$ is independent of $P \in X$ and, therefore, is

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a function on moduli space $\mathcal{M}_1$ of Riemann surfaces of genus one. The main result of the present work is the following explicit formula for this function:

$$\det\Delta(X,P) = C_1|\Im\sigma||\eta(\sigma)|^4 F(t) = C_2 \det\Delta^{(0)}(X)F(t), \tag{1.1}$$

where σ is the b-period of the Riemann surface X, C_1 and C_2 are absolute constants, η is the Dedekind eta-function, $\Delta^{(0)}$ is the Lapalacian on X corresponding to the flat conformal metric of unit volume, the surface X is represented as the two-fold covering of the Riemann sphere $\mathbb{C}P^1$ ramified over the points $0,1,\infty$ and $t\in\mathbb{C}\smallsetminus\{0,1\}$, and

$$F(t) = \frac{|t|^{\frac{1}{24}}|t-1|^{\frac{1}{24}}}{(|\sqrt{t}-1|+|\sqrt{t}+1|)^{\frac{1}{4}}}.$$

As is well known, the moduli space $\mathcal{M}_1$ coincides with the quotient space

$$\big(\mathbb{C}\smallsetminus\{0,1\}\big)/G,$$

where G is a finite group of order 6, generated by transformations $t\to\frac{1}{t}$ and $t\to 1-t$. A direct check shows that $F(t)=F(\frac{1}{t})$ and $F(t)=F(1-t)$, and, therefore, the right hand side of (1.1) is in fact a function on $\mathcal{M}_1$.

Remark 1.1 Using the classical relation (see, *e.g.*, [2, f-la (3.35)])

$$t = -\Big(\frac{\Theta[^1_0](0\mid\sigma)}{\Theta[^0_1](0\mid\sigma)}\Big)^4,$$

one can rewrite the right-hand side as a function of σ only.

The well known Ray–Singer relation $\det\Delta^{(0)} = C|\Im\sigma||\eta(\sigma)|^4$ (see [10–12]) used in (1.1) implies that (1.1) can be considered as a version of Polyakov's formula (relating determinants of the Laplacians corresponding to two smooth metrics in the same conformal class) for the case of two conformally equivalent metrics on a torus: one of them is smooth and flat, another is of curvature one and has exactly one singular point.

2 Metrics on the Base and on the Covering

Here we find an explicit expression for the metric m on the Riemann sphere $\mathbb{C}P^1$ of curvature 1 and with three conical singularities at $P_1=0$, $P_2=1$, and $P_3=\infty$.

The stereographic projection (from the south pole) maps the spherical triangle T onto quarter of the unit disk $\{z\in\mathbb{C}\,;|z|\le 1,\ 0\le\operatorname{Arg}z\le\pi/2\}$. The conformal map

$$z\longmapsto w=\Big(\frac{1+z^2}{1-z^2}\Big)^2 \tag{2.1}$$

sends this quarter of the disk to the upper half-plane H; the corner points $i,0,1$ go to the points $0,1$, and ∞ on the real line. The push forward of the standard round metric

$$\frac{4|dz|^2}{(1+|z|^2)^2}$$

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on the sphere by this map gives rise to the metric

$$m = \frac{|dw|^2}{|w||w-1|(|\sqrt{w}+1| + |\sqrt{w}-1|)^2} \tag{2.2}$$

on H; clearly, the latter metric can be extended (via the same formula) to $\mathbb{C}P^1$. The resulting curvature one metric on $\mathbb{C}P^1$ (also denoted by m) has three conical singularities of angle π: at $w = 0$, $w = 1$, and $w = \infty$.

Consider a two-fold covering of the Riemann sphere by a compact Riemann surface $X(t)$ of genus 1:

$$\mu\colon X(t) \to \mathbb{C}P^1 \tag{2.3}$$

ramified over four points: $0, 1, \infty$, and $t \in \mathbb{C} \setminus \{0, 1\}$. Clearly, the pull back metric $\mu^* m$ on $X(t)$ is a curvature one metric with exactly one conical singularity. The singularity is a conical point of angle 4π located at the point $\mu^{-1}(t)$.

3 Variation of Spectral Zeta-function with Respect to t

The analysis from [5] in particular implies that one can introduce the standard Ray–Singer ζ-regularized determinant

$$\det \Delta^{\mu^* m} := \exp\{-\zeta'_{\Delta^{\mu^* m}}(0)\} \tag{3.1}$$

of the (Friedrichs) self-adjoint Laplacian $\Delta^{\mu^* m}$ in $L_2(X(t), \mu^* m)$, where $\zeta'_{\Delta^{\mu^* m}}$ is the spectral zeta-function. In this section we establish a formula for the variation of $\zeta'_{\Delta^{\mu^* m}}(0)$ with respect to the parameter t (the fourth ramification point of the covering (2.3)). The derivation of this formula coincides almost verbatim with the proof of [5, Proposition 6.1]; therefore, we give only few details.

For the sake of brevity we identify the point t of the base $\mathbb{C}P^1$ with its (unique) preimage $\mu^{-1}(t)$ on $X(t)$.

Let $Y(\lambda; \cdot)$ be the (unique) special solution of the Helmholz equation (here λ is the spectral parameter) $(\Delta^m - \lambda)Y = 0$ on $X \setminus \{t\}$ with asymptotic $Y(\lambda)(x) = \frac{1}{x} + O(x)$ as $x \to 0$, where $x(P) = \sqrt{\mu(P) - t}$ is *the distinguished holomorphic local parameter* in a vicinity of the ramification point $t \in X(t)$ of the covering (2.3). Introduce the complex-valued function $\lambda \mapsto b(\lambda)$ as the coefficient near x in the asymptotic expansion

$$Y(x, \overline{x}; \lambda) = \frac{1}{x} + c(\lambda) + a(\lambda)\overline{x} + b(\lambda)x + O(|x|^{2-\epsilon}) \quad \text{as } x \to 0.$$

The following variational formula is proved in [5, Proposition 6.1]:

$$\partial_t\big(-\zeta'_{\Delta^{\mu^* m}}(0)\big) = \tfrac{1}{2}\big(b(0) - b(-\infty)\big). \tag{3.2}$$

The value $b(0)$ is found in [5, Lemma 4.2]: one has the relation

$$b(0) = -\tfrac{1}{6} S_{Sch}(x)\Big|_{x=0}, \tag{3.3}$$

where S_{Sch} is the Schiffer projective connection on the Riemann surface $X(t)$.

Since $\lambda = -\infty$ is a local regime, in order to find $b(-\infty)$, the solution Y can be replaced by a local solution with the same asymptotic as $x \to 0$. A local solution $\widehat{Y}$

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with asymptotic

$$\widehat{Y}(u,\overline{u};\lambda) = \tfrac{1}{u} + \widehat{c}(\lambda) + \widehat{a}(\lambda)\overline{u} + \widehat{b}(\lambda)u + O(|u|^{2-\epsilon}) \quad \text{as } u \to 0$$

in the local parameter $u^2 = z - s$ was constructed in [5, Lemma 4.1] by separation of variables; here z and $w = \mu(P)$ (resp. s and t) are related by (2.1) (resp. by (2.1) with $z = s$ and $w = t$) and $\widehat{b}(-\infty) = \frac{1}{2}\frac{\overline{s}}{1+|s|^2}$. One can easily find the coefficients $A(t)$ and $B(t)$ of the Taylor series $u = A(t)x + B(t)x^3 + O(x^5)$. As a local solution replacing Y, we can take $A(t)\widehat{Y}$. This immediately implies that $b(-\infty) = A^2(t)\widehat{b}(-\infty) - B(t)/A(t)$. A straightforward calculation verifies that

$$b(-\infty) = \partial_t \log\Big(|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2\Big)^{1/4}. \tag{3.4}$$

Observe that the right-hand side in (3.4) is actually the value of $\partial_w \log \rho(w,\overline{w})^{-1/4}$ at $w = t$, where $\rho(w,\overline{w})$ is the conformal factor of the metric (2.2); this is also a direct consequence of [4, Lemma 4].

Substituting (3.3) and (3.4) into (3.2), we obtain the desired formula for the variation of $\zeta'_{\Delta^{\mu^* m}}(0)$ with respect to the parameter t.

4 Explicit Formula for the Determinant

Equations (3.2), (3.3), and (3.4) imply that the determinant (3.1) can be represented as a product

$$\det \Delta^{\mu^* m} = C|\Im\sigma||\tau(t)|^2\Big|\frac{1}{|t||t-1|(|\sqrt{t}+1| + |\sqrt{t}-1|)^2}\Big|^{1/8}, \tag{4.1}$$

where $\tau(t)$ is the value of the Bergman tau-function (see [7–9]) on the Hurwitz space $H_{1,2}(2)$ of two-fold genus one coverings of the Riemann sphere, having ∞ as a ramification point at the covering, ramified over $1, 0, \infty$, and t. More specifically, τ is a solution of the equation

$$\partial_t \log \tau = -\tfrac{1}{12} S_B(x)|_{x=0},$$

where S_B is the Bergman projective connection on the covering Riemann surface $X(t)$ of genus one and x is the distinguished holomorphic parameter in a vicinity of the ramification point t of $X(t)$. We remind the reader that the Bergman and the Schiffer projective connections are related via the equation

$$S_{Sch}(x) = S_B(x) - 6\pi(\Im\sigma)^{-1}v^2(x)$$

where v is the normalized holomorphic differential on $X(t)$ and that the Rauch variational formula (see, *e.g.*, [7]) implies the relation

$$\partial_t \log \Im\sigma = \tfrac{\pi}{2}(\Im\sigma)^{-1}v^2(x)|_{x=0}.$$

The needed explicit expression for τ can be found *e.g.*, in [9, f-la (18)] (it is a very special case of the explicit formula for the Bergman tau-function on general coverings of arbitrary genus and degree found in [8] as well as of a much earlier formula of Kitaev and Korotkin for hyperelliptic coverings [6]). Namely, [9, f-la (18)] implies that

$$\tau = \eta^2(\sigma)\Big[\frac{v(\infty)^3}{v(P_1)v(P_2)v(Q)}\Big]^{\frac{1}{12}}, \tag{4.2}$$

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where P_1 and P_2 are the points of the $X(t)$ lying over 0 and 1, Q is a point of $X(t)$ lying over t and ∞ denotes the point of the covering curve $X(t)$ lying over the point at infinity of the base $\mathbb{C}P^1$; v is an arbitrary nonzero holomorphic differential on $X(t)$; and, say, $v(P_1)$ is the value of this differential in the distinguished holomorphic parameter at P_1. (One has to take into account that $\tau = \tau_I^{-2}$, where τ_I is from [9].) Taking

$$v = \frac{dw}{\sqrt{w(w-1)(w-t)}},$$

and using the following expressions for the distinguished local parameters at P_1, P_2, Q, and ∞

$$x = \sqrt{w}; \quad x = \sqrt{w-1}; \quad x = \sqrt{w-t}; \quad x = \frac{1}{\sqrt{w}}$$

one arrives at the relations (where $\sim$ means $=$ up to insignificant constants like ± 2, etc.)

$$v(P_1) \sim \frac{1}{\sqrt{t}}; \quad v(P_2) \sim \frac{1}{\sqrt{t-1}}; \quad v(Q) \sim \frac{1}{\sqrt{t(t-1)}}; \quad v(\infty) \sim 1.$$

These relations together with (4.2) and (4.1) imply (1.1).

Remark 4.1 The result of this paper can be generalized to hyperelliptic surfaces of genus $g \geq 2$. Indeed, choose $2g-1$ distinct points $Q_1, Q_2, \ldots, Q_{2g-1}$ in $\mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$ and consider the two-fold covering

$$\mu_g : X(Q_1, Q_2, \ldots, Q_{2g-1}) \to \mathbb{C}P^1$$

ramified over $Q_1, \ldots, Q_{2g-1}$ and P_1, P_2, P_3. The pullback $\mu_g^* m$ of the metric m in (2.2) by μ_g is a metric of constant curvature 1 with conical points of angle 4π at $2g-1$ Weierstrass points of the hyperelliptic curve $X(Q_1, Q_2, \ldots, Q_{2g-1})$ (three remaining Weierstrass points are nonsingular points of the metric). Using the same methods as in the genus 1 case, one can derive an explicit expression for the determinant of the Laplacian in the metric $\mu_g^* m$ as a function on moduli space of hyperelliptic curves of genus g. For instance, in genus two one gets the following explicit expression

$$\det \Delta^{\mu_2^* m} = C \mathcal{F}^{2/5} \Phi(t_1, t_2, t_3),$$

where

$$\mathcal{F} = (\det \Im \mathbb{B})^{5/2} \prod_{\beta} |\Theta[\beta](0|\mathbb{B})|$$

is the Petersson norm $\|\Delta_2\|$ of the Siegel cusp form $\Delta_2 = \prod_\beta \Theta[\beta](0|\mathbb{B})$ (β runs through the set of 10 even characteristics) and

$$\Phi(t_1, t_2, t_3) = \frac{|t_1 t_2 t_3 (t_1 - 1)(t_2 - 1)(t_3 - 1)|^{-\frac{1}{40}} |t_1 - t_2|^{\frac{1}{10}} |t_1 - t_3|^{\frac{1}{10}} |t_2 - t_3|^{\frac{1}{10}}}{\prod_{k=1}^3 (|\sqrt{t_k} + 1| + |\sqrt{t_k} - 1|)^{\frac{1}{4}}},$$

where the points $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are identified with the points $t_1, t_2, t_3, 0, 1, \infty$ of $\mathbb{C}P^1$. It is straightforward to check that the right-hand side of (4.1) is a function on the moduli space $\mathcal{M}_2$ of compact Riemann surfaces of genus 2 (it suffices to check that $\Phi(t_1, t_2, t_3) = \Phi(t_1^{-1}, t_2^{-1}, t_3^{-1}) = \Phi(1 - t_1, 1 - t_2, 1 - t_3)$).

Remark 4.2 [In response to referee comments] The necessary and sufficient condition on a triple of positive numbers θ_1, θ_2, θ_3 for the existence of a conformal curvature one metric on the Riemann sphere $\mathbb{C}P^1$, with three conic singularities of angles $2\pi\theta_1$, $2\pi\theta_2$, $2\pi\theta_3$ at the points 0, 1, and ∞, respectively, was obtained in [3, 13]. Let $m = \rho(w, \overline{w})|dw|^2$ stand for the corresponding metric on $\mathbb{C}P^1$. Then the pull back metric $\mu^* m$ on $X(t)$ (here μ is the same as in (2.3)) is a curvature one metric with conical singularity of angle 4π located at the point $\mu^{-1}(t)$ and three conical singularities of angles $4\pi\theta_1$, $4\pi\theta_2$, $4\pi\theta_3$ at the points $\mu^{-1}(0)$, $\mu^{-1}(1)$, and $\mu^{-1}(\infty)$, respectively. It turns out that the formula (3.2) (for the spectral zeta function of the Friedrichs self-adjoint extension of Laplacian $\Delta^{\mu^* m}$) is still valid, where $b(0)$ is the same as before and $b(-\infty) = \partial_w \log \rho(w, \overline{w})^{-1/4}|_{w=t}$. For details, we refer the reader to [4]. As a generalization of (1.1), we thus obtain

$$\det \Delta^{\mu^* m} = C_1 \Im\sigma \|\eta(\sigma)|^4 \sqrt[12]{|t^2 - t|} \sqrt[8]{\rho(t, \overline{t})} \tag{4.3}$$
$$= C_2 \det \Delta^{(0)}(X) \sqrt[12]{|t^2 - t|} \sqrt[8]{\rho(t, \overline{t})},$$

where C_1 and C_2 are absolute constants and t can be expressed as a function of σ; see Remark 1.1. Having at hand an explicit expression for the conformal factor $\rho(w, \overline{w})$ (in the case $\theta_1 = \theta_2 = \theta_3 = 1/2$ we use (2.2)), one immediately gets the corresponding explicit formula for $\det \Delta^{\mu^* m}$. Let us also note that (4.3) remains valid if $m = \rho(w, \overline{w})|dw|^2$ is any conical metric on $\mathbb{C}P^1$ and t stays outside of the conical singularities of m.

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References

[1] C.-L. Chai, C.-S. Lin, C.-L. Wang, *Mean field equation, hyperelliptic curves and modular forms: I.* Camb. J. Math. 3(2015), no. 1–2, 127–274. http://dx.doi.org/10.4310/CJM.2015.v3.n1.a3

[2] C. H. Clemens, *A scrapbook of complex curve theory.* Second ed., Graduate Studies in Mathematics, 55, American Mathematical Society, Providence, RI, 2003.

[3] A. Eremenko, *Metrics of positive curvature with conic singularities on the sphere.* Proc. Amer. Math. Soc. **132**(2004), no. 11, 3349–3355. http://dx.doi.org/10.1090/S0002-9939-04-07439-8

[4] V. Kalvin, *On determinants of Laplacians on compact Riemann surfaces equipped with pullbacks of conical metrics by meromorphic functions.* J. Geom. Anal., to appear. http://dx.doi.org/10.1007/s12220-018-0018-2

[5] V. Kalvin and A. Kokotov, *Metrics of constant positive curvature, Hurwitz spaces and* det Δ. Int. Math. Res. Not. IMRN, to appear. http://dx.doi.org/10.1093/imrn/rnx224

[6] V. Kitaev and D. Korotkin, *On solutions of the Schlesinger equations in terms of theta-functions.* Int. Math. Res. Not. IMRN **1998** no. 17, 877–905. http://dx.doi.org/10.1155/S1073792898000543

[7] A. Kokotov and D. Korotkin, *Tau-functions on Hurwitz spaces.* Math. Phys. Anal. Geom. 7(2004), no. 1, 47–96. http://dx.doi.org/10.1023/B:MPAG.0000022835.68838.56

[8] ———, *Isomonodromic tau-function of Hurwitz Frobenius manifolds.* Int. Math. Res. Not. IMRN **2006**, Art. ID 18746. http://dx.doi.org/10.1155/IMRN/2006/18746

[9] A. Kokotov and I. Strachan, *On the isomonodromic tau-function for the Hurwitz spaces of branched coverings of genus zero and one.* Math. Res. Lett. **12**(2005), no. 5–6, 857–875. http://dx.doi.org/10.4310/MRL.2005.v12.n6.a7

[10] J. Polchinski, *Evaluation of the one loop string path integral.* Comm. Math. Phys. **104**(1986), no. 1, 37–47.
[11] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds.* Ann. of Math. **98**(1973), 154–177. http://dx.doi.org/10.2307/1970909
[12] P. Sarnak, *Some applications of modular forms.* Cambridge Tracts in Mathematics, 99, Cambridge University Press, 1990. http://dx.doi.org/10.1017/CBO9780511895593
[13] M. Umehara and K. Yamada, *Metrics of constant curvature 1 with three conical singularities on 2-sphere.* Illinois J. Math. **44**(2000), no. 1, 72–94.

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