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COSINE REPRESENTATIONS OF ABELIAN $*$-SEMIGROUPS AND GENERALIZED COSINE OPERATOR FUNCTIONS

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Introduction. In the following **R** will denote the real numbers, for a Hilbert space H, $B(H)$ and $L(H)$ will denote the collections of bounded linear operators on H and linear, but not necessarily bounded, operators on H respectively. Cosine Operator Functions, namely functions $C : \mathbf{R} \to B(H)$ which satisfy D'Alembert's functional equation

$$(1)\quad 2C(s)C(t) = C(s+t) + C(s-t)$$

and

$$(2)\quad C(0) = I$$

have been extensively studied by several authors, notably Nagy [**4**], Kurepa [**1**; **2**; **3**] and Sova [**6**]. In [**1**] Kurepa considers functions $C : X \to B$ satisfying (1) and (2) where X is a Banach space, and B is a Banach algebra. In the present paper we would like to introduce a generalization of the cosine operator function which we propose to call a *cosine representation of a $*$-semigroup*. That is, if Γ is a $*$-semigroup, with identity ϵ, then $C : \Gamma \to L(H)$ is a cosine representation if C satisfies

$$(3)\quad 2C(s)C(t) = C(st) + C(s^*t)$$

and

$$(4)\quad C(\epsilon) = I.$$

In Theorem 1 of Section I we prove a result which parallels the celebrated "principle theorem" of Professor Sz.-Nagy in [**5**]. That is, we extend a family of operators C to a cosine representation.

In Section II we obtain an integral representation of the family $C : \mathbf{R} \to B(H)$ in terms of a generalized spectral family E whose domain is the Borel sets in **C** [**7**]. That is to say that the following formulas hold: For each $x \in D_{C(t)}$ and $y \in H$,

$$(5)\quad \langle C(t)x, y\rangle = \int \cos(\lambda t) d\langle E_\lambda x, y\rangle$$

and

$$(6)\quad ||C(t)x||^2 \leqq \int \cos^2(\lambda t) d\langle E_\lambda x, x\rangle.$$

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Following this we apply this representation to the Hilbert space of real numbers to obtain a solution to the moment problem of representing a real function f in the form

$$f(t) = \int \cos(st)d\alpha(s)$$

where α is a regular probability measure on **C**.

1. *Definition* 1. A semigroup Γ with identity ϵ will be called a $*$-*semigroup* provided there is a "star" operation $* : \Gamma \to \Gamma$ satisfying

(i) $\epsilon^* = \epsilon$
(ii) $s^{**} = s$
(iii) $(st)^* = t^*s^*$.

Definition 2. Let Γ be an abelian $*$-semigroup with identity ϵ and $L(H)$ the collection of all linear operators on the Hilbert space H. A function $C : \Gamma \to L(H)$ will be called a *cosine representation of* Γ provided C satisfies

(i) $C(\epsilon) = I$
(ii) $2C(s)C(t) = C(st) + C(s^*t)$ for all $s, t \in \Gamma$.

Definition 3. Let A be a linear operator on a Hilbert space H, then an operator $\tilde{A}$ on a Hilbert space $\tilde{H}$ containing H as a subspace will be said to be a *dilation* of A provided

$$A = P_H\tilde{A}|_H$$

where P_H is the orthogonal projection of $\tilde{H}$ onto H. This relationship will be denoted $A = \text{pr } \tilde{A}$.

The study of dilations was initiated in order to generalize the conventional concept of the extension of an operator. It may in fact be thought of as an extension which goes beyond the space on which the original operator was defined.

Definition 4. Let E be an abstract set, H a Hilbert space and $K : E \times E \to B(H)$. Then the kernel K is said to be of *positive type* if and only if for finite subsets $\{s_1, s_2, \ldots, s_n\} \subseteq E$ and $\{x_1, x_2, \ldots, x_n\} \subseteq H$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle K(s_i, s_j)x_j, x_i \rangle \geqq 0.$$

THEOREM 1. *Let* Γ *be an Abelian* $*$-*semigroup with identity* ϵ. *Suppose* $C : \Gamma \to B(H)$ *is a family of operators satisfying*

(a) $C(\epsilon) = I$, $C(s) = C(s^*)$, *and*
(b) *the kernel* $K(s, t) = \frac{1}{2}\{C(st) + C(s^*t)\}$ *is of positive type.*

Then there exists a dilation space $\tilde{H}$ *containing* H *and a family of operators* $\tilde{C} : \Gamma \to L(\tilde{H})$ *satisfying*

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(a′) *$\tilde{C}$ is a cosine representation of* Γ.
(b′) pr $\tilde{C}(t) = C(t)$ *for all* $t \in \Gamma$.
(c′) *$\tilde{C}(t)$ is symmetric for all* $t \in \Gamma$.

Proof. Let H_0 be the linear space formed by the collection of all linear combinations of functions of the form $s \mapsto \phi(s) = K(s, t)x$, where $x \in H$. We may define a bilinear form $(\cdot, \cdot)$ on H_0 as follows: if $\phi(s) = \sum_{i=1}^{n} K(s, t_i)x_i$ and $\nu(s) = \sum_{j=1}^{m} K(s, t_j')x_j'$ are functions in H_0, we define $(\phi(s), \nu(s))$ by

$$(\phi, \nu) = \sum_{i,j=1}^{n} \langle K(t_j', t_i)x_i, x_j' \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner-product on H. We observe that

$$(\phi(s), \nu(s)) = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle K(t_j', t_i)x_i, x_j' \rangle = \sum_{j=1}^{m} \langle \phi(t_j'), x_j' \rangle = \sum_{i=1}^{n} \langle x_i, \nu(t_i) \rangle,$$

where the last equation follows from the relation $\langle K(s, t)x, y \rangle = \langle x, K(s, t)y \rangle$, which is a consequence of K being of positive type. In any case, this implies that $(\phi(s), \nu(s))$ is independent of the representation of either ϕ or ν. It follows from this that $(\cdot, \cdot)$ is well-defined.

We now show that $(\cdot, \cdot)$ satisfies the axions of an inner-product. It is clear that $(\cdot, \cdot)$ is both bilinear and symmetric. In addition by the positivity of K, $(\cdot, \cdot)$ is non-negative. Suppose now that $(\phi(s), \phi(s)) = 0$. Since the Cauchy-Schwarz inequality is valid for $(\cdot, \cdot)$, we have for any $x \in H$ and any $t \in \Gamma$

$$0 \leqq |\langle \phi(t), x \rangle|^2 = |(\phi(s), K(s, t)x)|^2 \leqq (\phi(s), \phi(s))(K(s, t)x, K(s, t)x) = 0$$

from which it follows that $(\cdot, \cdot)$ is positive definite. Thus with $(\cdot, \cdot)$, H_0 is a pre-Hilbert space. Let $\tilde{H}$ denote the completion of H_0.

We may embed H into $\tilde{H}$ by identifying $x \rightarrow K(s, \epsilon)x = C(s)x$. Since $(K(s, \epsilon)x, K(s, \epsilon)x) = \langle K(\epsilon, \epsilon)x, x \rangle = \langle x, x \rangle$, the embedding is an isometry. Henceforth H will denote the image of this identification.

Define an operator $P : H_0 \rightarrow H$ by

$$PK(s, t)x = K(s, \epsilon)K(\epsilon, t)x = K(s, \epsilon)C(t)x.$$

We have $(P\nu - \nu, \phi) = 0$ for $\phi \in H_0$, $\nu \in H$ so that P is the orthogonal projection of H_0 onto H. By the same letter we denote its extension to the space $\tilde{H}$.

We now define $\tilde{C} : \Gamma \rightarrow L(H_0)$ by

$$\tilde{C}(s')(K(s, t)x) = \tfrac{1}{2}(K(s, s't)x + K(s, s'^{*}t)x).$$

Direct calculation shows that for $\phi, \nu \in H_0$

(1) $\quad (\tilde{C}(t)\phi, \nu) = (\phi, \tilde{C}(t)\nu)$

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so that, since H_0 is dense in $\tilde{H}$, if $\phi = 0$, it follows that $\tilde{C}(t)\phi = 0$. That is to say that $\tilde{C}(t)$ is well-defined for each $t \in \Gamma$.

We note that

$$\tilde{C}(\epsilon)(K(s, t)x) = \tfrac{1}{2}K(s, \epsilon t)x + \tfrac{1}{2}K(s, \epsilon^* t)x = K(s, t)x$$

so that $\tilde{C}(\epsilon) = I$.

Now for a function $\phi(s) = K(s, t)x$ we obtain

$$2\tilde{C}(\nu)\tilde{C}(\mu)(K(s, t)x) = \tfrac{1}{2}(K(s, \nu\mu t)x + K(s, \nu^*\mu t)x + K(s, \nu\mu^* t)x + K(s, \nu^*\mu^* t)x) = (\tilde{C}(\nu\mu) + \tilde{C}(\nu^*\mu))(K(s, t)x)$$

so that $\tilde{C}$ satisfies (i) and (ii) of Definition 2. We observe that at this point we make essential use of the commutativity of Γ.

To see that pr $\tilde{C}(t) = C(t)$ we observe that

$$\text{pr } \tilde{C}(t)(K(s, \epsilon)x) = P_H(\tfrac{1}{2}K(s, t)x + \tfrac{1}{2}K(s, t^*)x) = K(s, \epsilon)(\tfrac{1}{2}K(\epsilon, t)x + \tfrac{1}{2}K(\epsilon, t^*)x) = K(s, \epsilon)C(t)x.$$

The symmetry of $\tilde{C}(t)$ is given in (1), since $D(\tilde{C}(t)) = H_0$.

Definition 5. Let X be a Banach space and $C : \mathbf{R} \to L(X)$. C is said to be a *cosine operator* function provided C satisfies

(a) $2C(s)C(t) = C(s + t) + C(s - t)$
(b) $C(0) = I$.

Cosine operator functions have been extensively studied as was mentioned in the introduction.

Definition 6. Let $C(t)$ be a family of operators in $L(H)$ on a Hilbert space H. If there is a Hilbert space $\tilde{H}$ containing H and a cosine operator function $\tilde{C}(t)$ in $L(\tilde{H})$ satisfying

$$C(t) = \text{pr } \tilde{C}(t)$$

then $C(t)$ is said to be a *generalized cosine operator function.*

As a consequence of Theorem 1 we have the following:

THEOREM 2. *Let* $C : \mathbf{R} \to B(H)$ *satisfy*

(a) $C(0) = I$, $C(s) = C(-s)$, *and*
(b) $K(s, t) = \frac{1}{2}(C(s + t) + C(s - t))$ *is of positive type.*

Then $C(t)$ *is a generalized cosine operator function.*

Proof. We need only take $\Gamma = \mathbf{R}$ and "$*$" as the additive inverse in Theorem 1. We note that it is possible to have unbounded cosine operator functions so that without further hypotheses we cannot assume that $\tilde{C}(t) \in B(\tilde{H})$. As an example of an unbounded cosine operator function, let P be an unbounded projection on the Hilbert space H and $P_c = I - P$. Direct calculation shows that $C(t) = P_c + (\cos t)P$ is an unbounded cosine operator function.

Suppose that H is a subspace of the Hilbert space $\tilde{H}$ and $\tilde{C}(t) \in L(\tilde{H})$ is a self-adjoint cosine representation of Γ. Let $C(t) \in L(H)$ be given by

$$C(t) = \text{pr } \tilde{C}(t).$$

Let $K(s, t)$ be given as before by

$$K(s, t) = \tfrac{1}{2}\{C(st) + C(s^*t)\}.$$

Then for an element of $\tilde{H}$ of the form $g = \sum_{i=1}^{n} \tilde{C}(t_i)x_i$, where $\{t_1, \ldots, t_n\} \subseteq \Gamma$ and $\{x_1, \ldots, x_n\} \subseteq H$ we obtain

$$\sum_{i,j=1}^{n} \langle K(t_j, t_i)x_i, x_j \rangle = \sum_{i,j=1}^{n} \langle \tilde{C}(t_j)\tilde{C}(t_i)x_i, x_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle \tilde{C}(t_i)x_i, \tilde{C}(t_j)x_j \rangle = ||g||^2 \geqq 0$$

so that condition (b) is necessary in this case. In addition it is trivial to see that

$$C(0) = \text{pr } \tilde{C}(0) = \text{pr } I_{\tilde{H}} = I_H$$

and since

$$2\tilde{C}(s)\tilde{C}(\epsilon) = \tilde{C}(s\epsilon) + \tilde{C}(s^*\epsilon)$$

we have

$$C(s) = \text{pr } \tilde{C}(s) = \text{pr } \tilde{C}(s^*) = C(s^*)$$

so that condition (a) is necessary. It follows that we have

THEOREM 3. *Let $\tilde{C}(t)$ be a self-adjoint cosine representation of the $*$-semigroup Γ. If $\tilde{C}(t)$ is a dilation of $C(t)$ then*

(a) $C(\epsilon) = I$, $C(s) = C(s^*)$, *and*
(b) *the kernel $K(s, t) = \frac{1}{2}\{C(st) + C(s^*t)\}$ is of positive type.*

We will now show that, without loss of generality, we may take $\tilde{C}(t)$ to be self-adjoint.

THEOREM 4. *Let $\tilde{C}(t)$ be as in Theorem 1. Then $\tilde{C}(t)$ admits a self-adjoint extension $\hat{C}(t)$ for each t and the family so obtained is again a cosine representation.*

Proof. Let $\{e_\alpha\}$ be an orthonormal base for H. We define $J : H_0 \to H_0$ by

$$J[K(s, t) \sum_{\alpha} b_\alpha e_\alpha] = K(s, t) \sum_{\alpha} \bar{b}_\alpha e_\alpha.$$

J extends to a conjugation of $\tilde{H}$ which commutes with $\tilde{C}(t)$ for each t. Thus each of $\tilde{C}(t)$ admits a self-adjoint extension in $\tilde{H}$, say $\hat{C}(t)$. Now $\hat{C}(t) \subseteq C^*(t)$ which is easily seen to be a cosine representation of Γ, thus $\hat{C}(t)$ is a cosine representation of Γ.

We will now investigate conditions under which $C(t)$ can be dilated to a bounded cosine representation. In the following Γ will be assumed to be an abelian group with "$*$" being "inverse". As before let us consider an element of $\tilde{H}$ of the form $g = \sum \tilde{C}(t_i)x_i$, for $\{t_1, \ldots, t_n\} \subseteq \Gamma$ and $\{x_1, \ldots, x_n\} \subseteq H$. Now, however, we assume that $\tilde{C}(t) \in B(\tilde{H})$. Let us define the operation Δ_s, for $s \in \Gamma$, by

$$\Delta_s K(t_1, t_2)x = K(t_1 + s, t_2 + s) + K(t_1 - s, t_2 + s) + K(t_1 + s, t_2 - s) + K(t_1 - s, t_2 - s).$$

We calculate

$$\sum_{i,j=1}^{n} \langle \Delta_s K(t_j, t_i)x_i, x_j \rangle = \sum_{i,j=1}^{n} \langle (I + \tilde{C}(2t))(\tilde{C}(t_i + t_j) + \tilde{C}(t_i - t_j))x_i, x_j \rangle$$
$$= 2 \sum_{i,j=1}^{n} \langle \tilde{C}^2(t)\tilde{C}(t_j)\tilde{C}(t_i)x_i, x_j \rangle = 2 \sum_{i,j=1}^{n} \langle \tilde{C}(t)\tilde{C}(t_i)x_i, \tilde{C}(t)\tilde{C}(t_j)x_j \rangle$$
$$\leqq 2||\tilde{C}(t)||^2||g||.$$

In view of this we have the following:

THEOREM 5. *Let Γ be an abelian group. In order that $C : \Gamma \rightarrow B(H)$ admit a dilation to a bounded self-adjoint cosine representation of Γ it is necessary and sufficient that*

(1) $C(0) = I$ *and* $C(s) = C(-s)$

(2) $K(s, t) = \frac{1}{2}\{C(s + t) + C(s - t)\}$ *is of positive type, and*

(3) $\sum_{i,j=1}^{n} \langle \Delta_s K(t_j, t_i)x_i, x_j \rangle \leqq M_s \sum_{i,j=1}^{n} \langle K(t_j, t_i)x_i, x_j \rangle.$

Proof. In light of previous results we need only show that $\tilde{C}(t)$ of Theorem 1 is bounded under the assumption (3). But this follows from

$$||\tilde{C}(t) \sum_{i=1}^{n} K(s, t_i)x_i||^2 = \frac{1}{4} \sum_{i,j=1}^{n} \langle \Delta_s K(t_j, t_i)x_i, x_j \rangle$$
$$\leqq \frac{M_s}{4} \sum_{i,j=1}^{n} \langle K(t_j, t_i)x_i, x_j \rangle = \frac{M_s}{4} \left\| \sum_{i=1}^{n} K(s, t_i)x_i \right\|^2.$$

Thus the theorem is proved.

2. We now pass to the problem of integral representations of generalized cosine operator functions. In all that follows we will assume that $\Gamma = \mathbf{R}$ with addition. We will need the following.

LEMMA 1. *Let $C(t)$ and $\tilde{C}(t)$ be as in Theorem 1 with $t \in \mathbf{R}$. If $C(t)$ is weakly continuous and $\tilde{C}(t)$ is uniformly bounded, then $\tilde{C}(t)$ is weakly continuous.*

Proof. Let us first consider $\lim_{t \to t_0} (\tilde{C}(t)K(s, u)x, K(s, v)y)$. By direct

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calculation we have

$$(\tilde{C}(t)K(s, u)x, K(s, v)y) = \tfrac{1}{2}(K(s, u + t)x + \tfrac{1}{2}K(s, u - t)x, K(s, v)y)$$
$$= \tfrac{1}{2}\langle\{K(v, u + t) + K(v, u - t)\}x, y\rangle$$
$$= \tfrac{1}{4}\langle C(v + u + t)x + C(v - u - t)x + C(v + u - t)x + C(v - u + t)x, y\rangle$$

so

$$\lim_{t\to t_0} (\tilde{C}(t)K(s, u)x, K(x, v)y) = \tfrac{1}{4}\langle C(v + u + t_0)x + C(v - u - t_0)x + C(v + u - t_0)x + C(v - u + t_0)x, y\rangle = (\tilde{C}(t_0)K(s, u)x, K(s, v)y).$$

Thus, for $\phi, \nu \in H_0$, $(\tilde{C}(t)\phi, \nu) \to (\tilde{C}(t_0)\phi, \nu)$ as $t \to t_0$. Since H_0 is strongly dense and $\tilde{C}(t)$ is uniformly bounded, the above convergence holds if ν be replaced by any element of $\tilde{H}$. Now for arbitrary $\nu, \phi \in \tilde{H}$ we have, for any $u \in H_0$,

$$|(\tilde{C}(t)\phi - \tilde{C}(t_0)\phi, \nu)| \leqq ||C(t)|| \cdot ||\phi - u|| \cdot ||\nu|| + |(\tilde{C}(t)u - \tilde{C}(t_0)u, \nu)| + ||\tilde{C}(t_0)|| \, ||\phi - u|| \, ||\nu||.$$

Since $\tilde{C}(t)$ is uniformly bounded and H_0 strongly dense it follows that $\tilde{C}(t)$ is weakly continuous.

THEOREM 2. *Let $C : \Gamma \to B(H)$ be a weakly continuous family of operators satisfying*

(i) $C(0) = I$, $C(t) = C(-t)$

(ii) $K(s, t) = \frac{1}{2}C(s + t) + \frac{1}{2}C(s - t)$ *is of positive type, and*

(iii) $\sum_{i,j=1}^{n} \Delta_s K(t_j, t_i)x_i, x_j \leqq M \sum_{i,j=1}^{n} \langle K(t_j, t_i)x_i, x_j\rangle$.

Then there exists a positive operator valued measure E on H so that for $x, y \in H$

$$(5) \quad \langle C(t)x, y\rangle = \int \cos \lambda t d\langle E(\lambda)x, y\rangle$$

and

$$(6) \quad ||C(t)x||^2 \leqq \int \cos^2 \lambda t d\langle E(\lambda)x, x\rangle.$$

Conversely, if (5) *and* (6) *hold, and* supp $(E) \cap i\mathbf{R}$ *is compact then the conditions* (i)–(iii) *follow.*

Proof. By Theorems 1 and 5, if (i)–(iii) hold then there exists a Hilbert space $\tilde{H}$ and a cosine operator family $\tilde{C}(t)$ on H so that pr $\tilde{C}(t) = C(t)$. In addition $\tilde{C}(t)$ is uniformly bounded, since M is independent of s, and self-adjoint for each $t \in \mathbf{R}$. Since $\tilde{C}(t)$ is weakly continuous (Lemma 1) it follows

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from [**2**] that

$$\tilde{C}(t) = \int \cos \lambda t d\, \tilde{E}(\lambda)$$

where $\tilde{E}(\lambda)$ is the spectral family of a normal operator (that is integration is over **C**). Thus $E = \text{pr}\, \tilde{E}$ is a positive operator valued measure with

$$\langle \tilde{C}(t)x, y \rangle = \langle C(t)x, y \rangle = \int \cos \lambda t d\, \langle E(\lambda)x, y \rangle$$

for each $x, y \in H$. Also

$$||C(t)x||^2 \leqq ||\tilde{C}(t)x||^2 = \int \cos \lambda t d\, \langle E(\lambda)x, x \rangle.$$

Thus the representations (5) and (6) hold.

Suppose now that (5) and (6) hold and supp $(E) \cap i\mathbf{R}$ is compact. From a theorem by Naimark [**7**] there exists a Hilbert space $\tilde{H} \supseteq H$ and an orthogonal spectral family $\tilde{E}$ on $\tilde{H}$ for which pr $\tilde{E} = E$. In addition $\tilde{E}$ can be chosen so that $\tilde{E}(\Delta) = 0 \leftrightarrow E(\Delta) = 0$. Since $\langle C(t)x, x \rangle$ is real, this implies that the operator family

$$\tilde{C}(t) = \int \cos \lambda t d\, \tilde{E}(\lambda)$$

is a bounded self-adjoint cosine operator family satisfying pr $\tilde{C}(t) = C(t)$. From Theorem 5 we conclude that (i)–(iii) follow.

Application. Let $f : \mathbf{R} \to \mathbf{R}$ *be continuous and satisfy*

(1) $f(0) \neq 0, \quad f(t) = f(-t)$

(2) $K(s, t) = \frac{1}{2}\{f(s + t) + f(s - t)\}$ *is of positive type, and*

(3) $\sum_{i,j=1}^{n} \Delta_s K(t_j, t_i) a_i \bar{a}_j \leqq M \sum_{i,j=1}^{n} K(t_j, t_i) a_i a_j.$

Then

$$(7) \quad f(t) = \int_C \cos \lambda t d\, \alpha(t)$$

where α *is a bounded Borel measure in* C *whose support is in* $\mathbf{R} \cup i\mathbf{R}$. *In addition, if* (7) *holds for* supp $(\alpha) \cap i\mathbf{R}$ *compact, so do* (1)–(3).

Proof. Let $H = C$ with $\langle a, b \rangle = a\bar{b}$. Define $C(t)a = f(t) \cdot a$; then the theorem follows by setting $\alpha(M) = \langle E(M)1, 1 \rangle$.

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