

The Equivariant Grothendieck Groups of the Russell-Koras Threefolds

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Abstract. The Russell-Koras contractible threefolds are the smooth affine threefolds having a hyperbolic \mathbb{C}^* -action with quotient isomorphic to the corresponding quotient of the linear action on the tangent space at the unique fixed point. Koras and Russell gave a concrete description of all such threefolds and determined many interesting properties they possess. We use this description and these properties to compute the equivariant Grothendieck groups of these threefolds. In addition, we give certain equivariant invariants of these rings.

1 Introduction

In this paper we compute the equivariant Grothendieck groups of the Russell-Koras contractible threefolds. These threefolds are the smooth affine contractible threefolds having a hyperbolic \mathbb{C}^* -action with the quotient isomorphic to the corresponding quotient of the \mathbb{C}^* -action on the tangent space at the unique fixed point. Let us first remark that a \mathbb{C}^* -action on a ring R gives a grading $\bigoplus_{i \in \mathbb{Z}} R_i$. If $v \in R_i$, we say that v is a homogeneous element of weight i . In the paper [7], Koras and Russell give many interesting properties of the threefolds that bear their name. Theorem 4.1 of [7] shows that such a threefold determines a triple of weights a'_1, a'_2 and a'_3 , where a'_1, a'_2, a'_3 are pairwise relatively prime integers satisfying, $-a'_1, a'_2, a'_3 > 0$. With these as weights, a hyperbolic \mathbb{C}^* -action is put on $W = \text{Spec}(B)$, where $B = \mathbb{C}[\eta, \xi, \zeta]$ and η, ξ, ζ have weights a'_1, a'_2 and a'_3 respectively. Moreover B has another homogeneous coordinate system, η, τ, ξ' , in which τ and ξ' have weights a'_2 and a'_3 respectively. We shall write $\tau = G(\eta, \xi, \zeta)$ and $\zeta = F(\eta, \xi', \tau)$.

Finally, the threefold determines a triple of pairwise relative prime natural numbers $\alpha_1, \alpha_2, \alpha_3$, such that $\gcd(\alpha_i, a'_i) = 1$ for $i = 1, 2, 3$. We take x, x', y, z, t satisfying $y^{\alpha_1} = \eta, x = \xi, x' = \xi', z^{\alpha_3} = \zeta$ and $t^{\alpha_2} = \tau$. Then $X = \text{Spec}(A)$, where A is formed by adjoining y, z and t to B . Notice A can then be written as

$$A = \mathbb{C}[x, y, z, t] / (t^{\alpha_2} - G(y^{\alpha_1}, x, z^{\alpha_3}))$$

and

$$A = \mathbb{C}[x', y, z, t] / (z^{\alpha_3} - F(y^{\alpha_1}, x', t^{\alpha_2})).$$

It is the fact that A can be expressed as a hypersurface in these two ways that will allow us to compute a finite set of generators for the Grothendieck ring of A . Now $x, x', y,$

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z and t have weights $a'_2\alpha_1\alpha_2\alpha_3, a'_3\alpha_1\alpha_2\alpha_3, a'_1\alpha_2\alpha_3, a'_3\alpha_1\alpha_2$ and $a'_2\alpha_1\alpha_3$ respectively (see [7], Proposition 2.11). To simplify notation, we shall write

$$(1.1) \quad \begin{aligned} a^* &= a'_1\alpha_2\alpha_3 && \text{(the weight of } y) \\ b^* &= a'_3\alpha_1\alpha_2 && \text{(the weight of } z) \\ c^* &= a'_2\alpha_1\alpha_3 && \text{(the weight of } t) \end{aligned}$$

We always assume that $\alpha_2 > 1$ and $\alpha_3 > 1$ (otherwise $A \cong \mathbb{C}^{[3]}$ with a linear action and the equivariant Grothendieck group is easily determined). Let ρ be the number of irreducible components of $V(z) \cap V(t)$, where $V(f_1, \dots, f_k)$ denotes the subscheme of X defined by the ideal (f_1, \dots, f_k) . Russell and Koras show in Theorem 4.1 of [7] that on the level surface $\eta = 1$, $G(1, \xi, \zeta) = 0$ is a line that is $\omega_{a'_1}$ -homogeneous and intersects the line $\zeta = 0$ normally in r points consisting of the $\omega_{a'_1}$ -orbit $(1, 0, 0)$ and $\rho - 1$ further $\omega_{a'_1}$ -orbits; these orbits are in one to one correspondence with the irreducible components of $V(z) \cap V(t)$. Hence $r = 1 + \frac{\rho-1}{a'_1}$. Let, as in Corollary 4.3.2 of [7],

$$(1.2) \quad \varepsilon = (r - 1)(\alpha_2 - 1)(\alpha_3 - 1) = a'_1(\rho - 1)(\alpha_2 - 1)(\alpha_3 - 1).$$

Kaliman and Makar-Limanov ([5], Theorem 8.5) proved the remarkable result that

$$X \cong \mathbb{C}^3$$

if and only if $\varepsilon = 0$.

Notice r is just the x degree of the polynomial $G(y^{\alpha_1}, x, 0)$. The weight of this polynomial is equal to the weight of x . Since the weight of y is negative, it follows that $G(y^{\alpha_1}, x, 0) = xf(y^{\alpha_1}, x)$, where $f(y^{\alpha_1}, x)$ is a homogeneous polynomial having weight 0. Let

$$(1.3) \quad v = x^{-a'_1} y^{a'_2\alpha_1}.$$

It is easy to see that any weight zero homogeneous polynomial in x and y must be a polynomial in v . Thus we have that

$$(1.4) \quad G(y^{\alpha_1}, x, 0) = xq(v)$$

for some polynomial q . Moreover, the degree of q is equal to $\frac{r-1}{a'_1} = \rho - 1$. The roots of q and zero are in one-to-one correspondence with the irreducible components of $V(z) \cap V(t)$. Hence $q(0) \neq 0$ and q has $\rho - 1$ simple roots $c_1, \dots, c_{\rho-1}$. Moreover height 2 prime ideals of A that contain (z, t) are just (x, z, t) and $\{(v - c_i, z, t) : 1 \leq i < \rho\}$. (We shall show that these prime ideals generate the equivariant Grothendieck ring of A over the subring generated by equivariantly free modules (see Theorem 5.1).) In a similar manner, we have that $F(y^{\alpha_1}, x', 0) = x'Q(v')$, where

$v' = x'^{-a'_i} y^{a'_i \alpha_1}$ and Q is a polynomial that doesn't vanish at zero and has $\rho - 1$ simple roots. Now let $d_1, \dots, d_{\rho-1}$ be the roots of Q . Then we must have

$$\begin{aligned} \{\wp : \wp \supseteq (z, t), \wp \text{ height } 2\} &= \{(x, z, t), (v - c_i, z, t) : 1 \leq i < \rho\} \\ &= \{(x', z, t), (v' - d_j, z, t) : 1 \leq j < \rho\}. \end{aligned}$$

Hence by relabeling if necessary, we may assume that

$$(1.5) \quad (v - c_i, z, t) = (v' - d_i, z, t) \quad \text{for } 1 \leq i < \rho.$$

A final fact that we will need is that A is a UFD and that $A^* = \mathbb{C}^*$ (e.g., see [7], Lemma 1.3). These facts will be useful during our computations. We shall show that the equivariant Grothendieck ring of A is isomorphic to

$$\mathbb{Z}[T, T^{-1}, E_1, \dots, E_{\rho-1}]/\mathcal{J}',$$

where \mathcal{J}' is the ideal generated by

$$\left\{ E_i^2 - (1 - T^{b^*})(1 - T^{c^*})E_i, E_i E_j, \frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} E_i : 1 \leq i < j < \rho \right\}.$$

Our strategy will be to first compute the equivariant Grothendieck ring of $A/(z)$. $A/(z)$ is a regular ring by 3.3 of [7]. We shall show that $A/(z)$ has an interesting cancellation property that we shall exploit in our computation of its equivariant Grothendieck ring. We shall then use our knowledge of the equivariant Grothendieck ring of $A/(z)$ to compute the equivariant Grothendieck ring of A .

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2 Preliminaries

It can be shown that the Chow groups of A are trivial. This implies that any finitely generated projective A -module is stably free ([12], Proposition 1.2).

Let us turn now to the equivariant setting. Suppose k is an algebraically closed field and R is a finitely generated k -algebra having an action of a linearly reductive, algebraic group G . We work in the category of finitely generated R - G -modules. The morphisms in this category are the equivariant homomorphisms. The equivariant Grothendieck groups can be defined analogously to how they are defined in the category of R -modules.

We shall let $[[M]]$ denote the equivariant isomorphism class of M . We define $K(R - G)$ to be the free abelian group on equivariant isomorphism classes of finitely generated R - G -modules, quotiented by the subgroup generated by all relations of the form $[[M'']] + [[M']] - [[M]]$, where

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0$$

is an exact sequence of R - G -modules. We define $K_1(R - G)$ to be the free abelian group on equivariant isomorphism classes of projective R - G -modules, quotiented by the subgroup generated by all relations of the form $[[P]] + [[P']] - [[P \oplus P']]$. We shall let $[N]$ represent the class of a finitely generated R - G -module N in either $K(R - G)$ or $K_1(R - G)$. Also, $K_1(R - G)$ can be given a ring structure by defining $[P] \cdot [Q]$ to be $[P \otimes_R Q]$. When G is a linearly reductive, algebraic group, and B is a representation of G on the Euclidean n -space over k , we can speak of B - G -modules. This is an R - G -module with $R = k[x_1, \dots, x_n]$ and the action of G on R generated by B . It was shown by Bass and Haboush [2] that for a representation B of G , every finitely generated projective B - G -module is equivariantly stably free. This theorem gave evidence supporting the truth of the so-called equivariant Serre conjecture, which speculated that projective $k[x_1, \dots, x_n]$ - G -modules are necessarily equivariantly free if G acts linearly on $k[x_1, \dots, x_n]$ and if the action of G is generated by a representation. The equivariant Serre conjecture was proven to be true in the case that G is abelian by Masuda, Moser-Jauslin and Petrie [10]. On the other hand, Schwarz [14] constructed counter-examples to the equivariant Serre conjecture for $\mathcal{O}(2)$ -actions on $\mathbb{C}^{[4]}$. Knop [6] later gave constructions for any non-abelian reductive group.

In this paper we shall compute the rings $K_1(A - \mathbb{C}^*)$. Our computations show that if $\varepsilon \neq 0$ (see equation (1.2)), then $K_1(A - \mathbb{C}^*) \not\cong K_1(\mathbb{C} - \mathbb{C}^*)$. This shows that in contrast to the situation for representations of \mathbb{C}^* , there exist finitely generated projective A - \mathbb{C}^* -modules that are not equivariantly stably free when $\varepsilon \neq 0$.

We now give some basic facts that will be needed in this paper.

Proposition 2.1 ([1], Corollary 4.2) *Let R be a finitely generated k -algebra having an action of linearly reductive group G . Let P be a finitely R -generated R - G -module. Then P is R -projective if and only if P is R - G -projective.*

Theorem 2.1 ([13], Section 4, Corollary 1) *Suppose R is a finitely generated regular k -algebra having an action of a linearly reductive group G . Then $K(R - G) = K_1(R - G)$.*

An ideal I is an R - \mathbb{C}^* -submodule of R if and only if it is generated by homogeneous elements. We then call I a homogeneous ideal. Note that such an ideal I is prime if for all u, v homogeneous in R with $uv \in I$ implies that $u \in I$ or $v \in I$.

Definition 2.1 Let R be a finitely generated \mathbb{C} -algebra with a \mathbb{C}^* -action. We define:

- $R(j)$ to be the free R - \mathbb{C}^* -module $R \otimes_{\mathbb{C}} \mathbb{C}(j)$, where $\mathbb{C}(j)$ has \mathbb{C}^* -action given by $g \cdot v = g^j v$ for all $g \in \mathbb{C}^*$ and all $v \in \mathbb{C}$;
- $R(j_1, \dots, j_n)$ to be the free R - \mathbb{C}^* -module $\bigoplus_{i=1}^n R(j_i)$;
- $M(j_1, \dots, j_n)$ to be the R - \mathbb{C}^* -module $M \otimes_R R(j_1, \dots, j_n)$, where M is a finitely generated R - \mathbb{C}^* -module.

Proposition 2.2 *Let R be a finitely generated \mathbb{C} -algebra with a \mathbb{C}^* -action. Then the Grothendieck group $K(R - \mathbb{C}^*)$ is generated as a \mathbb{Z} -module by*

$$\{[R/\wp \otimes R(j)] : j \in \mathbb{Z}, \wp \text{ a homogeneous prime}\}.$$

Proof To prove this Proposition it suffices to show that any finitely generated $R\text{-}\mathbb{C}^*$ -module M has a filtration by $R\text{-}\mathbb{C}^*$ submodules

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_k = M$$

such that for each i with $1 \leq i \leq k$, $M_i/M_{i-1} \cong (R/\wp_i) \otimes R(j)$, for some homogeneous prime ideal \wp_i , and some $j \in \mathbb{Z}$. We shall show this now. The proof is almost identical to that of Theorem 1 on page 265 in Bourbaki [3]. Let M be a finitely generated $R\text{-}\mathbb{C}^*$ -module, and let \mathcal{S} be the set of all $R\text{-}\mathbb{C}^*$ submodules of M that have a filtration with the desired property. Certainly $\mathcal{S} \neq \emptyset$. Thus we may choose a maximal element N of \mathcal{S} . Suppose $N \neq M$. Then for a homogeneous element $x \in M - N$, let $I^x = \{a \in R : ax \in N\}$. Certainly I^x is a homogeneous ideal. Now since R is noetherian, there exists a z such that I^z is a maximal element (with respect to ordering by inclusion) of the collection $\{I^x\}$. Let $N' = N + Rz$. We claim $N'/N \cong R/\wp \otimes R(j)$ for some prime \wp and some integer j . To see this, let j denote the weight of z . Note that $N'/N \cong R(j)/I^z(j) \cong R/I^z \otimes R(j)$. Hence it suffices to prove that I^z is prime. Suppose there exist homogeneous elements $u, v \in R$ such that $u, v \notin I^z$, but $uv \in I^z$. Then the ideal $I^{uz} = \{a \in R : a(uz) \in N\}$ will strictly contain I^z , contradicting the maximality of I^z . And so we see that M has a desired filtration. ■

Finally, we prove some facts about ideals in $\mathbb{Z}[X]$ that we will need later on.

Lemma 2.1 Suppose m, n, l, m', n' are positive integers and $\gcd(m, n) = \gcd(m, l) = 1$. Then in $\mathbb{Z}[X]$:

- a. The ideal $(X^{nl} - 1, \frac{X^{mm} - 1}{X^m - 1})$ is equal to the principal ideal $(\frac{X^n - 1}{X - 1})$;
- b. $((X^{mm'} - 1)(X^{nn'} - 1), \frac{(1 - X^{mm})(1 - X^{nn})}{(1 - X^n)(1 - X^m)}) = (1)$ if and only if $(m - 1)(n - 1) = 0$.

Proof (a.) Since m and l are relatively prime, we can choose positive integers μ and λ such that $m\mu = 1 + l\lambda$. Notice

$$\begin{aligned} (X^{nl} - 1, X^{mm} - 1) &\supseteq (X^{nl\lambda} - 1, X^{nm\mu} - 1) \\ &\supseteq (X^{nm\mu - n} - 1, X^{nm\mu} - 1) \\ &\supseteq (X^n - 1). \end{aligned}$$

Therefore

$$\left(\frac{X^{nm} - 1}{X^m - 1}, X^{nl} - 1 \right) = \left(X^n - 1, \frac{X^{nm} - 1}{X^m - 1} \right).$$

Now any root of $X^n - 1$ is a root of $X^{nm} - 1$. Moreover the only root of $X^n - 1$ that is a root of $X^m - 1$ is 1, since m and n are relatively prime. Hence there exists a polynomial $p(X) \in \mathbb{Z}[X]$ such that

$$\frac{X^{nm} - 1}{X^m - 1} = \left(\frac{X^n - 1}{X - 1} \right) p(X).$$

Setting $X = 1$ in both sides of this equation shows that $p(1) = 1$. Now

$$\begin{aligned} \left(\frac{X^{mn} - 1}{X^m - 1}, X^n - 1\right) &= \left(\frac{X^n - 1}{X - 1}\right) (X - 1, p(X)) \\ &= \left(\frac{X^n - 1}{X - 1}\right) (X - 1, p(1)) \\ &= \left(\frac{X^n - 1}{X - 1}\right). \end{aligned}$$

This completes the proof of *a*. To prove *b*., let us first suppose that $m \neq 1$ and $n \neq 1$. We clearly have the containment

$$\left((X^{mm'} - 1)(X^{nn'} - 1), \frac{(X^{mm} - 1)(X - 1)}{(X^m - 1)(X^n - 1)}\right) \subseteq \left(X^m - 1, \frac{(X^{mm} - 1)(X - 1)}{(X^m - 1)(X^n - 1)}\right).$$

Since m and n are relatively prime, we can choose $s > 0$ such that ns is congruent to 1 mod m . Hence $(1 - X) = (1 - X^{ns}) + h(X)(1 - X^m)$ for some polynomial $h(X)$. It follows that

$$\begin{aligned} \left(X^m - 1, \frac{(X^{mm} - 1)(X - 1)}{(X^m - 1)(X^n - 1)}\right) &= \left(X^m - 1, \frac{X^{mn} - 1}{X^m - 1} \cdot \frac{X^{ns} - 1}{X^n - 1} + h(X)\frac{X^{mn} - 1}{X^n - 1}\right) \\ &\subseteq \left(X^m - 1, \frac{X^{mn} - 1}{X^m - 1}, \frac{X^{mn} - 1}{X^n - 1}\right) \\ &= \left(X^m - 1, 1 + X^m + \dots + X^{m(n-1)}, \frac{X^{mn} - 1}{X^n - 1}\right) \\ &\subseteq \left(X^m - 1, n, \frac{X^{mn} - 1}{X^n - 1}\right) \\ &= \left(\frac{X^m - 1}{X - 1}, n\right) \quad (\text{by } a). \\ &\neq (1). \end{aligned}$$

Hence the result is true when $(m - 1)(n - 1) \neq 1$. The result is clearly true when either m or n is equal to one. This proves part *b*. ■

Finally, we introduce some notation that shall be used throughout this paper.

Notation 2.1 Given a commutative ring R , $x \in R$ and an R -module M , we let M_x denote the localization of M with respect to the multiplicative system $\{x^n \mid n \geq 0\}$.

3 First Computation

Consider the subring $\mathbb{C}[x, y]$ of A . We shall compute the equivariant Grothendieck ring of $C := \mathbb{C}[x, y]/(v - \mu)$ where μ is a non-zero complex number and v is as in equation (1.3). Notice that

$$C \cong \mathbb{C}[x, y]/(x^{-a_1'} y^{a_2' \alpha_1} - 1).$$

Moreover $\gcd(-a'_1, a'_2\alpha_1) = 1$ and so we can find nonnegative integers β_1, β_2 such that $\beta_1 a'_1 + \beta_2 a'_2\alpha_1 = 1$. It is not difficult to see that

$$C[w, w^{-1}] \cong C[x, y]/(v - 1)$$

under the mapping which sends w to $x^{\beta_2} y^{\beta_1}$. As x and y have weights $a'_2\alpha_1\alpha_2\alpha_3$ and $a'_1\alpha_2\alpha_3$ respectively, we see that w has weight

$$(\beta_2 a'_2\alpha_1 + \beta_1 a'_1)\alpha_2\alpha_3 = \alpha_2\alpha_3.$$

We shall show that

$$K_1(C - \mathbb{C}^*) \cong \mathbb{Z}[T, T^{-1}]/(T^{\alpha_2\alpha_3} - 1).$$

From the remarks made, this isomorphism follows from the following proposition.

Proposition 3.1 *Suppose $C = \mathbb{C}[w, w^{-1}]$ is given the \mathbb{C}^* -action $g \cdot w = g^m w$ for all $g \in \mathbb{C}^*$. Then*

$$K_1(C - \mathbb{C}^*) \cong \mathbb{Z}[T, T^{-1}]/(T^m - 1),$$

where the isomorphism is given by the map sending $[C(1)]$ to w .

Proof Notice C is a regular ring. Moreover, it is a dimension one ring and hence all of its non-zero prime ideals must be maximal. None of its maximal ideals are homogeneous, and hence (0) is its only homogeneous prime ideal. Therefore by Proposition 2.2, $K(C - \mathbb{C}^*)$ is generated by $\{[C(n)] : n \in \mathbb{Z}\}$. Since C is regular, we have that $K_1(C - \mathbb{C}^*)$ is generated as a \mathbb{Z} -algebra by $[C(1)]$ and $[C(-1)]$. Consider the surjective homomorphism from $\mathbb{Z}[T, T^{-1}]$ onto $K_1(C - \mathbb{C}^*)$ which sends T and T^{-1} to $[C(1)]$ and $[C(-1)]$ respectively. Let \mathcal{K} be the kernel of this map. Notice w is invertible in C and thus $C(m) = (w) = (1) \cong C(0)$. Hence $T^m - 1 \in \mathcal{K}$. We claim that this element generates \mathcal{K} . To see this, suppose that there is an element in \mathcal{K} that is not in the ideal generated by $T^m - 1$. By multiplying it by a suitable power of T and then reducing it modulo the ideal $T^m - 1$, we can assume that our element is a non-zero polynomial in T having degree less than m . This is equivalent to saying that there exist nonnegative integers $A_0, B_0, A_1, B_1, \dots, A_{m-1}, B_{m-1}$ such that $A_i \neq B_i$ for some i and

$$\bigoplus_i C(i)^{A_i} \cong \bigoplus_i C(i)^{B_i}.$$

However, any homomorphism from $C(i)$ into $C(j)$ is necessarily the zero homomorphism if i and j aren't congruent to one another mod m since any homogeneous element in C has weight that is a multiple of m . It follows by a simple rank argument that $A_i = B_i$ for all i . This is a contradiction, and so we see that $\mathcal{K} = (T^m - 1)$. ■

4 Second Computation

Let $R = A/(z) \cong \mathbb{C}[x, y, t]/(t^{\alpha_2} - xq(v))$ (see equations (1.3), (1.4)). We give R the \mathbb{C}^* -action that is induced by the \mathbb{C}^* -action on A . Recall that q is a polynomial having $\rho - 1$ simple roots $c_1, \dots, c_{\rho-1}$ and that $q(0) \neq 0$. Let I denote the ideal (x, t) and J_i denote the ideal $(v - c_i, t)$ for $1 \leq i < \rho$. Our aim in this section is to compute the equivariant Grothendieck ring of R . A necessary first step in doing this is to show that any projective $R\text{-}\mathbb{C}^*$ -module that is equivariantly stably free is in fact equivariantly free. We shall do this now.

Lemma 4.1 *Any homogeneous element of R can be written as $Dp(v)$, where p is a polynomial with coefficients in \mathbb{C} , $p(0) \neq 0$ and D is a monomial of the form $t^i x^d y^e$, where d, e are nonnegative integers and $0 \leq i < \alpha_2$.*

Proof Let h be a homogeneous element of R . Since $t^{\alpha_2} = xq(v)$ in R , h can be expressed as $\sum_{j=0}^{\alpha_2-1} t^j h_j(x, y)$, where the h_j are polynomials in x and y . Since the weights of x and y are both $0 \pmod{\alpha_2}$, and the weight of t is $a'_2 \alpha_1 \alpha_3$, which is relatively prime to α_2 , it follows that a homogeneous element of R can in fact be expressed as $t^i u(x, y)$ for some polynomial u and some integer i with $0 \leq i < \alpha_2$. Let x^d be the highest power of x that divides u . Similarly, let y^e be the highest power of y that divides u . We can write $u(x, y)$ as $x^d y^e f(x, y)$, where f is a homogeneous polynomial with $f(0, 0) \neq 0$.

Since $f(0, 0)$ is homogeneous with a non-zero constant term, it follows that f has weight zero and is therefore a polynomial in v , say $p(v)$. Note that $p(0) = f(0, 0) \neq 0$. The lemma now follows. ■

We shall now generalize the concept of the completion of unimodular rows to the equivariant setting.

Definition 4.1 Let L be a ring with a \mathbb{C}^* -action. We say that a homogeneous unimodular row $\mathbf{r} \in \bigoplus_{j=1}^n L(a_j)$ is equivariantly completable, if there exists a matrix $M = (m_{i,j}) \in \text{GL}_n(L)$ such that

1. The first row of M is \mathbf{r} ;
2. The entries of M are homogeneous elements of L ;
3. $m_{i,j} m_{i',j'} - m_{i,j'} m_{i',j}$ is homogeneous for $1 \leq i, i', j, j' \leq n$.

Remark 4.1 Suppose (a_1, \dots, a_n) is a homogeneous unimodular row in $L^{1 \times n}$ and that $M \in \text{GL}_n(L) = (m_{i,j})$ is a matrix whose entries are homogeneous and has the property that $m_{i,j} a_{i'} - m_{i',j} a_i$ is homogeneous for each j and all i, i' . Then (a_1, \dots, a_n) is equivariantly completable if and only if $(a_1, \dots, a_n)M$ is equivariantly completable.

Proposition 4.1 *Let L be a ring with a \mathbb{C}^* -action. Suppose*

$$\mathbf{r} = (r_1, \dots, r_n) \in \bigoplus_{j=1}^n L(a_j)$$

is a homogeneous unimodular row that is equivariantly completable. Then the module

$$\bigoplus_{j=1}^n L(a_j)/\langle \mathbf{r} \rangle$$

is equivariantly free.

Proof Suppose $M \in GL_n(L)$ satisfies (1), (2) and (3) of Definition 4.1. Let \mathbf{m}_i denote the i -th row of M for $2 \leq i \leq n$. Conditions (2) and (3) of the definition say that \mathbf{m}_i is homogeneous in $\bigoplus_{j=1}^n L(a_j)$. Moreover $\mathbf{m}_2, \dots, \mathbf{m}_n$ are linearly independent over L . Hence they generate an equivariantly free submodule N of $\bigoplus_{j=1}^n L(a_j)$. We have a short exact sequence of G -maps

$$0 \longrightarrow L \cdot \mathbf{r} \xrightarrow{i} \bigoplus_{j=1}^n L(a_j) \xrightarrow{g} N \longrightarrow 0,$$

where the map i is the inclusion, and g is the map that takes \mathbf{r} to zero and maps \mathbf{m}_j to itself for $2 \leq j \leq n$. Hence

$$\bigoplus_{j=1}^n L(a_j)/\langle \mathbf{r} \rangle \cong N.$$

This proves the Proposition. (This result is a straightforward generalization of Proposition 4.8 of [8].) ■

Proposition 4.2 *Suppose D is a domain with \mathbb{C}^* -action having the property that for any homogeneous unimodular row $\mathbf{a} \in \bigoplus_{i=1}^m D(n_i)$ of length at least two there exist coordinates a, b in \mathbf{a} such that the ideal (a, b) is principal. Then any homogeneous unimodular row in $\bigoplus_{i=1}^m D(n_i)$ is equivariantly completable.*

Proof Suppose that the proposition were false. Choose a $1 \times n$ equivariant unimodular row $\mathbf{a} = (A_1, \dots, A_n)$ that is not equivariantly completable, with n minimal. We may assume without loss of generality that (A_1, A_2) is a principal ideal. Choose a generator B of the ideal (A_1, A_2) . Write $A_1 = BA'_1$ and $A_2 = BA'_2$. There exist $\lambda_1, \lambda_2 \in D$ such that

$$\lambda_1 A_1 + \lambda_2 A_2 = B.$$

Since D is a domain, we have

$$\lambda_1 A'_1 + \lambda_2 A'_2 = 1.$$

Let M be the matrix

$$\begin{pmatrix} \lambda_1 & -A'_2 & 0 & \cdots & 0 \\ \lambda_2 & A'_1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{n-2} & \\ 0 & 0 & & & \end{pmatrix}$$

Notice M satisfies the properties of Remark 4.1, and hence \mathbf{a} is equivariantly completable if and only if

$$\mathbf{a}M = (B, 0, A_3, \dots, A_n)$$

is equivariantly completable. Note that

$$(B, A_3, \dots, A_n)$$

is equivariantly completable, by the minimality of n . Hence $\mathbf{a}M$ is equivariantly completable, a contradiction. ■

Corollary 4.1 *Any homogeneous unimodular row in $\bigoplus_{i=1}^m R(n_i)$ is equivariantly completable.*

Proof Let \mathbf{a} be a unimodular row of length at least two. By the preceding proposition it suffices to show that there exist coordinates a, b in \mathbf{a} such that the ideal (a, b) is principal. We shall now show this. By Lemma 4.1, there exist polynomials p_1, \dots, p_n not vanishing at zero, and A_1, A_2, \dots, A_n , where A_k is a monomial of the form $t^i x^d y^e$, such that $\mathbf{a} = (A_1 p_1(v), \dots, A_n p_n(v))$. Notice that at least one of the A_i 's must be 1 since otherwise all coordinates of row \mathbf{a} vanish at the origin $x = y = t = 0$, contradicting the unimodularity of \mathbf{a} . Thus we may assume without loss of generality that $A_1 = 1$. We divide the proof into two cases.

Case 1 $t|A_i$ for $2 \leq i \leq n$.

Notice that $q(v)$ and $p_1(v)$ are relatively prime, since otherwise there would exist some root of q , say c_i , such that the ideal

$$(p_1(v), A_2 p_2(v), \dots, A_n p_n(v)) \subseteq (v - c_i, t) \neq (1).$$

We have that $A_2 = t^i x^d y^e$, where $i < \alpha_2$. We may choose positive integers d' and e' such that $x^{d+d'+1} y^{e+e'} = v^\ell$ for some $\ell > 0$. We have that $t^{\alpha_2-i} x^{d'} y^{e'} A_2 p_2(v) = q(v) v^\ell p_2(v)$. As $p_1(0) \neq 0$ and $\gcd(p_1, q) = 1$, we have that $\gcd(p_1(v), v^\ell q(v) p_2(v)) = \gcd(p_1(v), p_2(v))$. Let d denote the greatest common divisor of p_1 and p_2 . We have $(A_1 p_1(v), A_2 p_2(v)) = (d)$. Thus, in this case, there exist coordinates a, b of \mathbf{a} such that (a, b) is principal.

Case 2 t doesn't divide A_i for some $i, 2 \leq i \leq n$.

Without loss of generality we can assume that t doesn't divide A_2 . Hence $A_2 = x^d y^e$ for some nonnegative integers d and e . We can find d' and e' such that $x^{d+d'} y^{e+e'} = v^\ell$ for some nonnegative integer ℓ . Notice $x^{d'} y^{e'} A_2 p_2(v) = v^\ell p_2(v)$. Since $p_1(0) \neq 0$, we necessarily have that the greatest common divisor of $p_1(v)$ and $v^\ell p_2(v)$ is the same as the greatest common divisor of $p_1(v)$ and $p_2(v)$. Proceeding as in the first case, we see that there exist homogeneous coordinates a, b in \mathbf{a} such that (a, b) is principal.

It follows that the conclusion of the corollary is true. ■

Corollary 4.2 *Suppose N is a projective $R\text{-}\mathbb{C}^*$ -module that is equivariantly stably free. Then N is equivariantly free.*

Proof Suppose N is not equivariantly free. Choose integers n_1, n_2, \dots, n_m such that $N \oplus (\bigoplus_{j=1}^m R(n_j))$ is equivariantly isomorphic to an equivariantly free module F , with $m \geq 1$ minimal. There exists an equivariant isomorphism $\Phi: N \oplus (\bigoplus_{j=1}^m R(n_j)) \rightarrow F$. Notice

$$N \oplus \left(\bigoplus_{j=1}^{m-1} R(n_j) \right) \cong F / \langle \Phi(0, (0, 0, \dots, 1)) \rangle.$$

Since Φ is an equivariant isomorphism, $\Phi(0, (0, 0, \dots, 1))$ is an equivariant unimodular row. By Corollary 4.1 we see that it is equivariantly completable. Hence $N \oplus (\bigoplus_{j=1}^{m-1} R(n_j))$ is equivariantly free. This contradicts the minimality of m . Thus N is equivariantly free. ■

We shall now compute the equivariant Grothendieck ring of R . The first step in this computation is to analyze the homogeneous prime ideals of R . This analysis is done in the following proposition.

Proposition 4.3 *A non-zero homogeneous prime ideal of R is either principal, or is one of the ideals:*

1. $(x, t) = I$;
2. $J_i = (v - c_i, t)$ for $1 \leq i < \rho$, where $c_1, \dots, c_{\rho-1}$ are the roots of $q(v)$;
3. (y, t) .

Proof We prove the proposition by looking at two cases. Let \wp be a homogeneous prime ideal in R .

Case 1 $t \in \wp$.

If $t \in \wp$, then $xq(v) = t^{\alpha_2} \in \wp$ and hence either $x \in \wp$ or one of $v - c_1, \dots, v - c_{\rho-1}$ is in \wp . Both (x, t) and $(v - c_i, t)$ are homogeneous prime ideals. Suppose that \wp is neither (x, t) nor $(v - c_i, t)$ for some i with $1 \leq i < \rho$. As both (x, t) and each of the ideals $(v - c_i, t)$ are height 1 primes, it follows that \wp must be maximal. The only homogeneous maximal ideal is $(x, y, t) = (y, t)$.

Case 2 $t \notin \wp$.

Suppose $f \in \wp$ is homogeneous. Then by Lemma 4.1, f can be expressed as $t^j x^d y^e p(v)$, for some nonnegative integers j, d, e and some polynomial p . As $t \notin \wp$, we have that $x^d y^e p(v) \in \wp$. Now $x \notin \wp$, because if it were, then t would necessarily be an element of \wp . As p factors into linear polynomials, we may assume that either $y \in \wp$ or $v - \beta \in \wp$, for some $\beta \in \mathbb{C}^*$. If $y \in \wp$, then it is necessarily true that $\wp = (y)$. If $y \notin \wp$, then \wp has an element of the form $v - \beta$. Since $t \notin \wp$ by assumption, we must have that $q(\beta) \neq 0$. Notice

$$R/(v - \beta) \cong \mathbb{C}[x, y, t] / (t^{\alpha_2} - xq(\beta), v - \beta) \cong \mathbb{C}[y, t] / (t^{-a'_1 \alpha_2} y^{a'_2 \alpha_1} - \beta q(\beta)^{-a'_1}).$$

Notice that $R/(v - \beta)$ is therefore a domain, as $a'_1\alpha_2$ and $a'_2\alpha_1$ are relatively prime. This completes the proof of the proposition. ■

Notice that the ideal $I = (x, t)$ is a projective R -module. To see this, observe that $I_x = (1)$ and $I_{q(v)} = (t)$, since $t^{\alpha_2}/q(v) = x$. Hence I becomes free upon localization at x and $q(v)$. Since these elements generate the unit ideal, I is projective (see [3] Theorem 1, page 109). Moreover the ideal $J_i = (v - c_i, t)$ is a projective R -module for $1 \leq i < \rho$, as $(J_i)_{v-c_i} = (1)$ and $(J_i)_{\frac{xq(v)}{v-c_i}} = (t)$ are free, and $(v - c_i, \frac{xq(v)}{v-c_i}) = (1)$. Hence by Proposition 2.1 they are projective $R\text{-}\mathbb{C}^*$ -modules. We now use these facts to give a set of generators for $K_1(R - \mathbb{C}^*)$.

Proposition 4.4 $K_1(R - \mathbb{C}^*)$ is generated as a \mathbb{Z} -module by $[R(1)]$, $[R(-1)]$, and $[J_1], \dots, [J_{\rho-1}]$.

Proof We define $I_j = (t, x \prod_{k=1}^j (v - c_k))$ for $0 \leq j < \rho$. Notice that $I_0 = I$. We then have the filtration

$$I = I_0 \supseteq I_1 \supseteq \dots \supseteq I_{\rho-1} = (t) \supseteq (0).$$

Hence we have the relation

$$[I] = [R(c^*)] + \sum_{j=0}^{\rho-2} [I_j/I_{j+1}]$$

in $K(R - \mathbb{C}^*)$.

We also have the short exact sequences

$$0 \longrightarrow I_{\mu+1} \longrightarrow I_{\mu} \longrightarrow (R/J_{\mu+1})(c^*\alpha_2) \longrightarrow 0.$$

Combining these facts, we see that

$$[I] = [R(c^*)] + \sum_{j=1}^{\rho-1} ([R(c^*\alpha_2)] - [J_j(c^*\alpha_2)])$$

in $K(R - \mathbb{C}^*)$. It can be deduced from this identity that

$$I \oplus \left(\bigoplus_{j=1}^{\rho-1} J_j(c^*\alpha_2) \right)$$

is equivariantly stably free. Using Corollary 4.2, we have that

$$(4.1) \quad I \oplus \left(\bigoplus_{j=1}^{\rho-1} J_j(c^*\alpha_2) \right) \cong R(c^*) \oplus R(c^*\alpha_2)^{\rho-1}.$$

Notice R is a regular ring, and hence $K(R - \mathbb{C}^*) \cong K_1(R - \mathbb{C}^*)$ as groups. It follows from Proposition 2.2 and formula (4.1) that $K_1(R - \mathbb{C}^*)$ is generated as a \mathbb{Z} -algebra by $[R(1)]$, $[R(-1)]$, and $[J_1], \dots, [J_{\rho-1}]$. ■

It follows from this proposition that there is a surjective ring homomorphism from $\mathbb{Z}[W, W^{-1}, U_1, \dots, U_{\rho-1}]$ to $K_1(R - \mathbb{C}^*)$ sending $W, U_1, U_2, \dots, U_{\rho-1}$ to $[R(1)], [J_1], [J_2], \dots, [J_{\rho-1}]$ respectively. Let \mathcal{J} denote the kernel of this epimorphism. That is to say

$$(4.2) \quad \mathbb{Z}[W, W^{-1}, U_1, \dots, U_{\rho-1}]/\mathcal{J} \cong K_1(R - \mathbb{C}^*),$$

where the isomorphism maps $W, U_1, \dots, U_{\rho-1}$ to $[R(1)], [J_1], \dots, [J_{\rho-1}]$ respectively. We shall now find generators for \mathcal{J} .

Lemma 4.2 *Let \mathcal{J} be as in equation (4.2). Then $\frac{1-W^{\alpha_2\alpha_3}}{1-W^{\alpha_3}}(U_i - 1) \in \mathcal{J}$ for $1 \leq i < \rho$.*

Proof First notice that if we let $J_{i,\mu} = (v - c_i, t^\mu)$ for $0 \leq \mu \leq \alpha_2$, ($J_{i,1} = J_i$ and $J_{i,0} = R(0)$), then we have the exact sequence

$$0 \longrightarrow J_{i,\mu+1} \longrightarrow J_{i,\mu} \longrightarrow R(\mu c^*)/J_i(\mu c^*) \longrightarrow 0.$$

From this sequence we derive the relation

$$[J_{i,\mu+1}] + [R(\mu c^*)] - [J_i(\mu c^*)] = [J_{i,\mu}]$$

in $K(R - \mathbb{C}^*)$. Thus by induction we see that for $0 \leq k \leq \alpha_2$

$$[J_{i,k}] = [R(0)] + \sum_{j=0}^{k-1} ([J_i(jc^*)] - [R(jc^*)])$$

in $K(R - \mathbb{C}^*)$. From these remarks we deduce that

$$\begin{aligned} [R(0)] + \sum_{j=0}^{\alpha_2-1} ([J_i(c^*j)] - [R(c^*j)]) &= [J_{i,\alpha_2}] \\ &= [(v - c_i)] \\ &= [R(0)] \end{aligned}$$

in $K(R - \mathbb{C}^*)$.

By the regularity of R , we have that this relation holds in $K_1(R - \mathbb{C}^*)$. This is equivalent to stating that

$$(4.3) \quad \sum_{j=0}^{\alpha_2-1} (U_i W^{c^*j} - W^{c^*j}) = \frac{1 - W^{\alpha_2 c^*}}{1 - W^{c^*}} (U_i - 1)$$

is an element of \mathcal{J} . Notice that $\alpha_2\alpha_3$ is the greatest common divisor of the weight of x and the weight of y . Hence there exist positive integers γ and δ such that $x^\gamma y^\delta$ has weight $\alpha_2\alpha_3$. We define a map $f: J_i(\alpha_2\alpha_3) \oplus R(0) \rightarrow J_i(0)$ by $f(j, r) = jx^\gamma y^\delta + r(v - c_i)$. Now there exist positive integers γ', δ' such that $x^{\gamma+\gamma'} y^{\delta+\delta'}$ is a power of v . Hence we have $f(0, 1) = (v - c_i)$ and $f(tx^{\gamma'} y^{\delta'}, 0) = tv^m$ for some positive integer m . Notice that $(tv^m, v - c_i) = (t, v - c_i) = J_i$. Thus f is surjective. The kernel of f is the submodule of $J_i(\alpha_2\alpha_3) \oplus R(0)$ generated by $(v - c_i) \oplus (-x^\gamma y^\delta) \cong R(\alpha_2\alpha_3)$. Thus we have

$$(4.4) \quad J_i(0) \oplus R(\alpha_2\alpha_3) \cong J_i(\alpha_2\alpha_3) \oplus R(0).$$

Equivalently, $(W^{\alpha_2\alpha_3} - 1)(U_i - 1) \in \mathcal{J}$. Now we have shown that

$$(W^{\alpha_2\alpha_3} - 1)(U_i - 1) \in \mathcal{J} \text{ and } \frac{W^{\alpha_2c^*} - 1}{W^{c^*} - 1}(U_i - 1) \in \mathcal{J}.$$

Using Lemma 2.1 *a*, taking $X = W^{\alpha_3}$ and l, m and n to be $1, c^*/\alpha_3$ and α_2 respectively, we find that

$$\left(W^{\alpha_2\alpha_3} - 1, \frac{W^{\alpha_2c^*} - 1}{W^{c^*} - 1} \right) = \left(\frac{W^{\alpha_2\alpha_3} - 1}{W^{\alpha_3} - 1} \right).$$

The result follows. ■

We continue our search for generators for \mathcal{J} .

Lemma 4.3 *Let \mathcal{J} be as in equation (4.2). Then $(U_i - 1)(U_j - 1) \in \mathcal{J}$ for $i \neq j$ and $(U_i - 1)(U_i - W^{c^*})$ for $1 \leq i < \rho$.*

Proof Using equations (4.4) and (4.1) along with Corollary 4.2 we see that

$$(4.5) \quad I(0) \oplus R(\alpha_2\alpha_3 + c^*) \cong I(\alpha_2\alpha_3) \oplus R(c^*).$$

Next, notice that if $1 \leq i < j < \rho$, then, since J_i is projective, and therefore, flat, we have

$$J_i \otimes J_j \cong J_i \cdot J_j.$$

Now consider the short exact sequence

$$(4.6) \quad 0 \longrightarrow J_i \cdot J_j \longrightarrow J_i \longrightarrow R/J_j \longrightarrow 0.$$

From this sequence, we deduce that

$$[J_i] \cdot [J_j] = [J_i] + [J_j] - [R(0)]$$

in $K_1(R - \mathbb{C}^*)$. Equivalently, $(U_i - 1)(U_j - 1) \in \mathcal{J}$. We shall now consider relations in the Grothendieck ring involving $[J_i] \cdot [J_i]$ for $1 \leq i < \rho$. Consider the sequence

$$(4.7) \quad 0 \longrightarrow (t^{\alpha_2-1}, v - c_i)(c^*) \xrightarrow{f} R \oplus R(c^*) \xrightarrow{g} J_i \longrightarrow 0,$$

where $f(v - c_i) = -t \oplus (v - c_i)$ and $f(t^{\alpha_2-1}) = (-xq(v)) \oplus (v - c_i)t^{\alpha_2-1}$, and $g(1 \oplus 0) = v - c_i$ and $g(0 \oplus 1) = t$. When we localize at $v - c_i$ this sequence just becomes

$$0 \longrightarrow R_{v-c_i}(c^*) \longrightarrow R_{v-c_i} \oplus R_{v-c_i}(c^*) \longrightarrow R_{v-c_i} \longrightarrow 0.$$

Similarly, if we localize at $xq(v)/(v - c_i)$, then the ideal $(v - c_i, t^{\alpha_2-1})$ becomes the principal ideal (t^{α_2-1}) , since $t^{\alpha_2} = (v - c_i)(xq(v)/(v - c_i))$. Similarly, $(J_i)_{xq(v)/(v-c_i)}$ is generated by t . Hence our sequence becomes the exact sequence

$$0 \longrightarrow R_{xq(v)/(v-c_i)}(\alpha_2 c^*) \longrightarrow R_{xq(v)/(v-c_i)}(0, c^*) \longrightarrow R_{xq(v)/(v-c_i)}(c^*) \longrightarrow 0.$$

(Note that $R_{xq(v)/(v-c_i)}(\alpha_2 c^*) = R_{xq(v)/(v-c_i)}(0)$, since $xq(v)/(v - c_i)$ has weight $\alpha_2 c^*$.) Hence the sequence (4.7) is exact (see for example [4] Corollary 2.9, page 68). Tensoring this short exact sequence with the R -flat module J_i over R yields the fact that

$$[J_i] \cdot [J_i] = [J_i(0)] + [J_i(c^*)] - [(v - c_i, t^{\alpha_2-1})(c^*) \cdot J_i].$$

Notice that

$$\begin{aligned} (v - c_i, t^{\alpha_2-1}) \cdot J_i &= ((v - c_i)^2, (v - c_i)t, t^{\alpha_2}) \\ &= ((v - c_i)^2, (v - c_i)t, xq(v)) \\ &= (v - c_i) \quad (\text{as } q \text{ has simple roots}) \\ &\cong R(0). \end{aligned}$$

Thus

$$[J_i]^2 = [J_i(0)] + [J_i(c^*)] - [R(c^*)]$$

in $K_1(R - \mathbb{C}^*)$. Equivalently, $(U_i - 1)(U_i - W^{c^*}) \in \mathcal{J}$. ■

We are ready to prove the main theorem of this section.

Theorem 4.1 $K_1(R - \mathbb{C}^*) \cong \mathbb{Z}[W, W^{-1}, U_1, \dots, U_{\rho-1}]/\mathcal{J}$, where \mathcal{J} is the ideal generated by

$$\left\{ (U_i - 1)(U_j - 1), (U_i - 1)(U_i - W^{c^*}), \frac{W^{\alpha_2\alpha_3} - 1}{W^{\alpha_3} - 1}(U_i - 1) : 1 \leq i < j < \rho \right\}.$$

The isomorphism is given by the map which sends $[J_1], \dots, [J_{\rho-1}]$ and $[R(1)]$ to $U_1, \dots, U_{\rho-1}$ and W respectively.

Proof Lemma 4.3 shows that any additional relations in the equivariant Grothendieck ring can be assumed to correspond to linear polynomials in $[J_1], \dots, [J_{\rho-1}]$ over $\mathbb{Z}[[R(1)], [R(-1)]]$. In light of this remark and Corollary 4.2, it suffices to analyze under which conditions

$$(4.8) \quad Q := \left(\bigoplus_{j=1}^{m_0} I(\beta_{0,j}) \right) \oplus \left(\bigoplus_{i=1}^{\rho-1} \bigoplus_{j=1}^{m_i} J_i(\beta_{i,j}) \right)$$

is equivariantly free. By equations (4.4) and (4.5) we may assume that $0 \leq \beta_{i,j} < \alpha_2\alpha_3$ for $0 \leq i < \rho$. Thus if we let $B_{i,j} = \text{card}\{\beta_{i,k} : \beta_{i,k} = j\}$, then we may write Q as

$$(4.9) \quad Q = \bigoplus_{j=0}^{\alpha_2\alpha_3-1} \left(I(j)^{B_{0,j}} \oplus \left(\bigoplus_{i=1}^{\rho-1} J_i(j)^{B_{i,j}} \right) \right).$$

Now by (4.1) we know that $I(-c^*\alpha_2) \oplus (\bigoplus_{j=1}^{\rho-1} J_j)$ is a free $R\text{-}\mathbb{C}^*$ -module. Using this fact along with equation (4.5) and Corollary 4.2, we can in fact say that $I \oplus (\bigoplus_j J_j)$ is equivariantly free. Thus using Corollary 4.2 again if necessary, we may assume that for each j there exists an i with $0 \leq i < \rho$, such that $B_{i,j} = 0$.

Also by assumption Q is equivariantly free, and hence there exist l_1, \dots, l_n such that $Q \cong R(l_1, l_2, \dots, l_n)$. Now let $\mathcal{M} = (x, y, t)$. Notice $I/\mathcal{M}I = \mathbb{C}(c^*)$ and $J_i/\mathcal{M}J_i \cong \mathbb{C}(0)$. Hence using equation (4.8) we see

$$\begin{aligned} \left(\bigoplus_{j=1}^{m_0} \mathbb{C}(\beta_{0,j} + c^*) \right) \oplus \left(\bigoplus_{i=1}^{\rho-1} \bigoplus_{j=1}^{m_i} \mathbb{C}(\beta_{i,j}) \right) &= Q/\mathcal{M}Q \\ &= \mathbb{C}(l_1, \dots, l_n). \end{aligned}$$

Hence the sequence $\{l_1, \dots, l_n\}$ must be a permutation of the sequence $\{\beta_{i,j} + c^*\delta_{0,i} : 0 \leq i < \rho\}$. It follows that

$$\begin{aligned} Q &\cong \bigoplus_{j=1}^{m_0} (R(\beta_{0,j} + c^*)) \oplus \left(\bigoplus_{i=1}^{\rho-1} \bigoplus_{j=1}^{m_i} R(\beta_{i,j}) \right) \\ &\cong \bigoplus_{j=0}^{\alpha_2\alpha_3-1} (R(j + c^*)^{B_{0,j}} \oplus R(j)^{\sum_{i \geq 1} B_{i,j}}). \end{aligned}$$

Using (4.9) we can say

$$(4.10) \quad \bigoplus_{j=0}^{\alpha_2\alpha_3-1} \left(I(j)^{B_{0,j}} \oplus \left(\bigoplus_{i=1}^{\rho-1} J_i(j)^{B_{i,j}} \right) \right) \cong \bigoplus_{j=0}^{\alpha_2\alpha_3-1} (R(j + c^*)^{B_{0,j}} \oplus R(j)^{\sum_{i \geq 1} B_{i,j}}).$$

For $1 \leq i < \rho$ we define R_i to be the quotient ring, $R/(v - c_i, t) \cong \mathbb{C}[x, y]/(v - c_i)$, of R . Notice that

$$\begin{aligned} (v - c_i, t)^2 &= ((v - c_i)^2, (v - c_i)t, t^2) \\ &= ((v - c_i)^2, (v - c_i)t, t^2, t^{\alpha_2}) \quad (\text{since } \alpha_2 \neq 1) \\ &= ((v - c_i)^2, xq(v), (v - c_i)t, t^2) \\ &= (v - c_i, t^2) \quad (\text{since } q \text{ has simple roots}). \end{aligned}$$

Thus $J_i/J_i^2 \cong R(c^*)/J_i(c^*)$. Using this fact along with equation (4.6), we see

$$R_i \otimes_R J_j \cong \begin{cases} R_i & \text{if } i \neq j, \\ R_i(c^*) & \text{if } i = j. \end{cases}$$

Also $R_i \otimes_R I \cong R_i$ for all i (see Lang [9] Proposition 2.7, page 612).

From these facts, we see that if we tensor both sides of (4.10) with R_{i_0} for some i_0 , then

$$(4.11) \quad \bigoplus_{j=0}^{\alpha_2\alpha_3-1} R_{i_0}(j)^{B_{i_0,j-c^*} + \sum_{i \neq i_0} B_{i,j}} \cong \bigoplus_{j=0}^{\alpha_2\alpha_3-1} R_{i_0}(j)^{B_{0,j-c^*} + \sum_{i>0} B_{i,j}}.$$

Here, $B_{i,j-c^*}$ is understood to mean B_{i,j_0} , where j_0 is the unique integer satisfying $0 \leq j_0 < \alpha_2\alpha_3$ and j_0 congruent to $j - c^* \pmod{\alpha_2\alpha_3}$. From Proposition 3.1 we have

$$K_1(R_{i_0} - \mathbb{C}^*) \cong \mathbb{Z}[T, T^{-1}]/(T^{\alpha_2\alpha_3} - 1)$$

with the isomorphism given by the map sending $[R_{i_0}(1)]$ and $[R_{i_0}(-1)]$ to T and T^{-1} respectively. Hence equation (4.11) says

$$\sum_{j=0}^{\alpha_2\alpha_3-1} T^j \left(B_{i_0,j-c^*} - B_{0,j-c^*} + \sum_{i \neq i_0} B_{i,j} - \sum_{i>0} B_{i,j} \right) \in (T^{\alpha_2\alpha_3} - 1).$$

That is to say $B_{i_0,j-c^*} - B_{i_0,j} = B_{0,j-c^*} - B_{0,j}$ for all j . From this fact it follows that whenever j and j' are congruent mod $\gcd(c^*, \alpha_2\alpha_3) = \alpha_3$, that $B_{0,j} - B_{0,j'} = B_{i,j} - B_{i,j'}$, for all i . Now suppose that $B_{0,j} \neq B_{0,j'}$ for some j and j' that are congruent modulo α_3 . Then without loss of generality we may assume that $B_{0,j} > B_{0,j'}$. Hence $B_{i,j} = B_{i,j'} + B_{0,j} - B_{0,j'} > 0$ for all i . This contradicts our assumption that $B_{i,j} = 0$ for some i . It follows that $B_{i,j} = B_{i,j'}$ whenever j and j' are congruent modulo α_3 . Thus any additional existing relations in the equivariant Grothendieck ring must arise from isomorphisms of the form

$$\begin{aligned} & \bigoplus_{j=0}^{\alpha_3-1} \bigoplus_{k=0}^{\alpha_2-1} \left(I(j + k\alpha_3)^{B_{0,j}} \oplus \left(\bigoplus_{i=1}^{\rho-1} J_i(j + k\alpha_3)^{B_{i,j}} \right) \right) \\ & \cdot \bigoplus_{j=0}^{\alpha_3-1} \bigoplus_{k=0}^{\alpha_2-1} R(j + k\alpha_3)^{B_{0,j-c^*} + \sum_{i>0} B_{i,j}}. \end{aligned}$$

That is to say, that any additional relations in $K_1(R - \mathbb{C}^*)$ can be assumed to be of the form

$$\left(\sum_{k=0}^{\alpha_2-1} R(\alpha_3 k) \right) \left(\sum_{j=0}^{\alpha_3-1} \left(B_{0,j} ([I(j)] - [R(j+c^*)]) + \sum_{i=1}^{\rho-1} B_{i,j} ([J_i(j)] - [R(j)]) \right) \right) = 0.$$

Equivalently, by virtue of (4.1) any additional generators of \mathcal{J} may be assumed to be of the form

$$\sum_{i=1}^{\rho-1} p_i(W) \frac{W^{\alpha_2 \alpha_3} - 1}{W^{\alpha_3} - 1} (U_i - 1)$$

for some polynomials $p_1, \dots, p_{\rho-1}$. But we showed in Lemma 4.2 that $\frac{W^{\alpha_2 \alpha_3} - 1}{W^{\alpha_3} - 1} (U_i - 1) \in \mathcal{J}$ for $1 \leq i < \rho$; thus we have all relations in $K_1(R - \mathbb{C}^*)$. ■

5 Third Computation

In this section $X = \text{Spec}(A)$ will always denote a Russell-Koras contractible threefold, where A is of the form

$$\mathbb{C}[x, y, z, t] / (t^{\alpha_2} - G(x, y^{\alpha_1}, z^{\alpha_3})) = \mathbb{C}[x', y, z, t] / (z^{\alpha_3} - F(y^{\alpha_1}, x, t^{\alpha_2})).$$

We shall compute the equivariant Grothendieck ring of A . To do this, we shall first examine the homogeneous prime ideals in A .

Proposition 5.1 *A homogeneous prime ideal in A that is not one of the following ideals:*

- i. (x, z, t) ;
- ii. $\mathcal{J}_i := (v - c_i, z, t)$ for $1 \leq i < \rho$, has a finite resolution by free $A - \mathbb{C}^*$ -modules.

Proof Notice that $A' = \mathbb{C}[x, y, z] \subseteq A$ is a polynomial ring and that

$$A = \left\{ \sum_{i=0}^{\alpha_2-1} b_i t^i : b_0, \dots, b_{\alpha_2-1} \in A' \right\} \cong \bigoplus_{i=0}^{\alpha_2-1} A'(c^* i)$$

is a free A' -module. Now any element γ of A can be expressed as $\gamma = \sum_{j=0}^{\alpha_2-1} t^j h_j(x, y, z)$ for some polynomials $h_0, \dots, h_{\alpha_2-1}$. If γ is homogeneous, then looking at its weight mod α_2 shows that it is necessarily of the form $t^i h(x, y, z)$ for some i , $0 \leq i < \alpha_2$, and some homogeneous polynomial h . The reason for this is that α_2 and c^* are relatively prime, and the weights of x, y and z are multiples of α_2 . Suppose that $\wp \in A$ is a homogeneous prime ideal such that $t \notin \wp$. Then \wp is generated by polynomials in x, y and z . Let us define \mathcal{P} to be $\wp \cap A'$. Notice that $\mathcal{P} \otimes_{A'} A \cong \wp$, with the isomorphism given by the map $p \otimes a \mapsto a \cdot p$. As A' is a polynomial ring with a linear \mathbb{C}^* -action, Theorem 1.1 of [2] shows that any $A' - \mathbb{C}^*$ -module has a finite resolution by free $A' - \mathbb{C}^*$ -modules. Hence if

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow \mathcal{P} \longrightarrow 0$$

is an equivariantly free resolution of \mathcal{P} , then tensoring with A over A' yields an equivariantly free resolution of \wp . Thus we have that any homogeneous prime ideal in A that does not contain t will have a finite resolution by free $A - \mathbb{C}^*$ -modules. Recall that there exists a homogeneous variable x' such that $A =$

$\mathbb{C}[x', y, z, t]/(z^{\alpha_3} - F(y^{\alpha_1}, x', t^{\alpha_2}))$ for some polynomial F . Moreover, the weights of x', y and t are all multiples of α_3 , and $\gcd(b^*, \alpha_3) = 1$. Hence we have that any homogeneous element of A can be expressed as $z^i h(x', y, t)$ for some homogeneous polynomial h and some $i, 0 \leq i < \alpha_3$. It follows that any homogeneous prime ideal that does not contain z is generated by polynomials in x', y and t . Using the same reasoning employed when we were considering homogeneous prime ideals that do not contain t , we see that any homogeneous prime ideal in A that does not contain z must have a finite resolution by free $A\text{-}\mathbb{C}^*$ -modules. Therefore any homogeneous prime ideal in A that does not contain (z, t) will necessarily have a finite resolution by equivariantly free A -modules. Recall that if \wp is a homogeneous prime that contains (z, t) , then \wp must contain one of $(x, z, t), (v - c_1, z, t), \dots, (v - c_{\rho-1}, z, t)$. Each of these ρ ideals is a height two homogeneous prime ideal. It follows that if \wp properly contains one of these ideals, then it must be maximal. The only homogeneous maximal ideal is $(x, y, z, t) = (y, z, t)$, which has a free resolution, as $\{y, z, t\}$ is a regular sequence (see [11] Corollary to Theorem 43, page 136). ■

Let P_j be the kernel of the canonical surjection of the free module $A(0, b^*, c^*)$ onto the ideal $(v - c_j, z, t)$ for $1 \leq j < \rho$. From equation (1.4), we know that

$$G(y^{\alpha_1}, x, z^{\alpha_3}) = xq(v) + z^{\alpha_3} G_1(y^{\alpha_1}, x, z^{\alpha_3})$$

for some polynomial G_1 . By localizing at $v - c_j$ at and $xq(v)/(v - c_j)$, we see that P_j is the $A\text{-}\mathbb{C}^*$ -submodule of $A(0, b^*, c^*)$ generated by

$$\begin{aligned} e_{j,1} &= (xq(v)/(v - c_j), z^{\alpha_3-1} G_1(y^{\alpha_1}, x, z^{\alpha_3}), -t^{\alpha_2-1}) \quad \text{having weight } \alpha_2 c^*, \\ e_{j,2} &= (t, 0, c_j - v) \quad \text{having weight } c^*, \\ e_{j,3} &= (z, c_j - v, 0) \quad \text{having weight } b^*, \text{ and} \\ (5.1) \quad e_{j,4} &= (0, t, -z) \quad \text{having weight } b^* + c^*. \end{aligned}$$

Moreover, $(P_j)_{v-c_j}$ and $(P_j)_{xq(v)/(v-c_j)}$ are both free $A\text{-}\mathbb{C}^*$ -modules, and hence P_j is a projective $A\text{-}\mathbb{C}^*$ -module. These projective modules will serve as generators for $K_1(A - \mathbb{C}^*)$ as a \mathbb{Z} -algebra.

Lemma 5.1 $K_1(A - \mathbb{C}^*)$ is generated as \mathbb{Z} -algebra by $[A(1)], [A(-1)]$ and $[P_1], \dots, [P_{\rho-1}]$, where P_i is as in (5.1).

Proof By Proposition 2.2, $K(A - \mathbb{C}^*)$ is generated as a \mathbb{Z} -module by elements of the form $[A(s)]$ and $[\wp \otimes A(r)]$, where \wp is a homogeneous prime and $r, s \in \mathbb{Z}$. Notice any homogeneous prime \wp that is not either (x, z, t) or one of the J_i 's has a finite resolution by free $A\text{-}\mathbb{C}^*$ -modules and hence $[\wp \otimes A(m)]$ belongs to the \mathbb{Z} -algebra $\mathbb{Z}[[A(1)], [A(-1)]]$. From the definition of P_j , we obtain the relation $[(v - c_j, z, t)] = [A(0, b^*, c^*)] - [P_j]$ in $K(A - \mathbb{C}^*)$. Finally, let $I_j = (x \prod_{k=1}^j (v - c_k), z, t)$ for $0 \leq j < \rho$. We have that $I_0 = (x, z, t)$ and $I_{\rho-1} = (z, t)$. Consider the filtration

$$(x, z, t) = I_0 \supseteq I_1 \supseteq \dots \supseteq I_{\rho-1} = (z, t) \supseteq (0).$$

This leads to the relation

$$[(x, z, t)] = [(z, t)] + \sum_{i=0}^{\rho-2} [I_i/I_{i+1}]$$

in $K(A - \mathbb{C}^*)$. Moreover, $I_i/I_{i+1} \cong A(\alpha_2 c^*)/J_{i+1}(\alpha_2 c^*)$, and hence we can write our relation as

$$[(x, z, t)] = [(z, t)] + \sum_{i=1}^{\rho-1} ([A(\alpha_2 c^*)] - [J_i(\alpha_2 c^*)]).$$

And since (z, t) has the free resolution

$$0 \longrightarrow A(c^* + b^*) \longrightarrow A(c^*, b^*) \longrightarrow (z, t) \longrightarrow 0,$$

we see that $[(x, z, t)]$ and each of the classes $[(v - c_1, z, t), \dots, (v - c_{\rho-1}, z, t)]$ are in the \mathbb{Z} -algebra generated by $[A(1)]$, $[A(-1)]$ and $[P_1], \dots, [P_{\rho-1}]$. By the regularity of A , it follows that $K_1(A - \mathbb{C}^*)$ is generated as a \mathbb{Z} -algebra by $[A(1)]$, $[A(-1)]$ and $[P_1], \dots, [P_{\rho-1}]$. ■

Let $R = A/(z)$. Consider the ring homomorphism

$$\phi: K_1(A - \mathbb{C}^*) \rightarrow K_1(R - \mathbb{C}^*)$$

given by $[Q] \mapsto [Q \otimes_A R]$. ϕ is well-defined, as a projective module is necessarily flat. From Theorem 4.1, we know that $K_1(R - \mathbb{C}^*) \cong \mathbb{Z}[W, W^{-1}, U_1, \dots, U_{\rho-1}]/\mathcal{J} = \mathbb{Z}[w, w^{-1}, u_1, \dots, u_{\rho-1}]$, where W corresponds to $[R(1)]$, U_j corresponds to $[(v - c_j, t)]$ for $1 \leq j < \rho$, and \mathcal{J} is the ideal generated by

$$\left\{ (U_i - 1)(U_j - 1), (U_i - 1)(U_i - W^{c^*}), \frac{1 - W^{\alpha_2 \alpha_3}}{1 - W^{\alpha_3}}(U_1 - 1) : 1 \leq i < j < \rho \right\}.$$

Now $\phi([A(1)]) = [R(1)]$ and we have a map f from $P_j \subseteq A(0, b^*, c^*)$ into $R(b^*) \oplus R(c^*)$ given by $f(\alpha, \beta, \gamma) = (\bar{\beta}, \bar{\gamma})$, where $\bar{\beta}$ is the restriction of β to the surface $z = 0$ and $\bar{\gamma}$ is the restriction of γ to $z = 0$. From the generating set for P_j given in the equations of (5.1), we have that the image of f is just $(t, v - c_j)(b^*) \oplus (t^{\alpha_2 - 1}, v - c_j)(c^*) \subseteq R(b^*) \oplus R(c^*)$. Now suppose $(\alpha, \beta, \gamma) \in \ker(f)$. If this is the case, then $z|\beta$ and $z|\gamma$. Moreover, since $(v - c_j)\alpha + z\beta + t\gamma = 0$, we have that $(v - c_j)\alpha \in (z)$. (z) is a prime ideal, and hence $\alpha \in (z)$. It follows that there exist $\alpha', \beta', \gamma' \in A$ such that $(\alpha, \beta, \gamma) = z(\alpha', \beta', \gamma')$. Now clearly $(v - c_j)\alpha' + z\beta' + t\gamma' = 0$, and therefore $(\alpha, \beta, \gamma) \in zP_j$. Conversely, if $p \in zP_j$, then $p \in \ker(f)$. Thus we see that

$$P_j \otimes_R A \cong P_j/zP_j \cong (v - c_j, t)(b^*) \oplus (v - c_j, t^{\alpha_2 - 1})(c^*).$$

This shows that $\phi([P_j]) = [(v - c_j, t)(b^*)] + [(v - c_j, t^{\alpha_2 - 1})(c^*)]$. Now we have the exact sequence

$$0 \longrightarrow (v - c_j)(c^*) \longrightarrow (v - c_j, t^{\alpha_2 - 1})(c^*) \longrightarrow R(\alpha_2 c^*)/(v - c_j, t)(\alpha_2 c^*) \longrightarrow 0$$

holds. Thus $\phi([P_j]) = [(v - c_j, t)(b^*)] + [R(c^*, \alpha_2 c^*)] - [(v - c_j, t)(\alpha_2 c^*)]$. Hence the image of ϕ is the ring

$$\mathbb{Z}[w, w^{-1}, u_i(w^{b^*} - w^{\alpha_2 c^*}) + w^{\alpha_2 c^*} + w^{c^*} : 1 \leq i < \rho] \subseteq \mathbb{Z}[w, w^{-1}, u_1, \dots, u_{\rho-1}].$$

Henceforth, h will denote the surjective homomorphism from $\mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}]$ onto

$$\mathbb{Z}[w, w^{-1}, u_1(w^{b^*} - w^{\alpha_2 c^*}) + w^{\alpha_2 c^*} + w^{c^*}, \dots, u_{\rho-1}(w^{b^*} - w^{\alpha_2 c^*}) + w^{\alpha_2 c^*} + w^{c^*}]$$

which maps T, T^{-1} and S_i to w, w^{-1} and $u_i(w^{b^*} - w^{\alpha_2 c^*}) + w^{\alpha_2 c^*} + w^{c^*}$ respectively, for $1 \leq i < \rho$. \mathcal{J} will denote the kernel of the surjective map from $\mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}]$ onto $K_1(A - \mathbb{C}^*)$, sending $S_1, \dots, S_{\rho-1}$ and T to $[P_1], \dots, [P_{\rho-1}]$ and $[A(1)]$ respectively. We prove the following lemma.

Lemma 5.2 $\mathcal{J} \subseteq \ker(h)$.

Proof Notice that for $i \neq j$,

$$h((S_i - T^{c^*} - T^{b^*})(S_j - T^{c^*} - T^{b^*})) = (w^{b^*} - w^{\alpha_2 c^*})^2(u_i - 1)(u_j - 1) = 0.$$

Furthermore,

$$\begin{aligned} & h((S_i - T^{c^*} - T^{b^*})(S_i - T^{\alpha_2 c^*} - T^{c^* - \alpha_2 c^* + b^*})) \\ &= (w^{b^*} - w^{\alpha_2 c^*})^2(u_i - 1)(u_i - w^{c^* - \alpha_2 c^*}) \\ &= (w^{b^*} - w^{\alpha_2 c^*})^2(u_i - 1)(u_i - w^{c^*}) \\ &= 0. \end{aligned}$$

The penultimate step follows from the fact that

$$(w^{\alpha_2 c^*} - 1)(u_i - 1) = 0.$$

We shall now find the remaining elements in the kernel of h . Using what we have just shown, any additional generators of the kernel of h can be assumed to have the form

$$q_0(T) + \sum_{j=1}^{\rho-1} q_j(T)(S_j - T^{c^*} - T^{b^*}).$$

Notice that

$$h\left(q_0(T) + \sum_{j=1}^{\rho-1} q_j(T)(S_j - T^{c^*} - T^{b^*})\right) = 0$$

if and only if

$$q_0(w) + \sum_j q_j(w)(w^{b^*} - w^{\alpha_2 c^*})(u_j - 1) = 0.$$

From Theorem 4.1 we know that this happens if and only if $q_0(w) = 0$ and $\frac{w^{\alpha_2\alpha_3} - 1}{w^{\alpha_3} - 1} \mid q_j(w)(w^{b^*} - w^{\alpha_2c^*})$ for $1 \leq j < \rho$. Observe that

$$\begin{aligned} \gcd\left(\frac{w^{\alpha_2\alpha_3} - 1}{w^{\alpha_3} - 1}, w^{b^*} - w^{\alpha_2c^*}\right) &= \gcd\left(\frac{w^{\alpha_2\alpha_3} - 1}{w^{\alpha_3} - 1}, w^{\alpha_1\alpha_2(a'_2\alpha_3 - a'_2)} - 1\right) \\ &= \frac{w^{\alpha_2} - 1}{w - 1}, \end{aligned}$$

where the last step follows from Lemma 2.1 *a*, taking X to be w and l, m and n to be $\alpha_1(a'_2\alpha_3 - a'_2), \alpha_3$ and α_2 respectively. Thus $h(q_0(T) + \sum_{i>1} q_i(T)(S - T^{c^*} - T^{b^*})) = 0$ if and only if $q_0(T) = 0$ and

$$\frac{T^{\alpha_2\alpha_3} - 1}{T^{\alpha_3} - 1} \mid q_j(T) \frac{T^{\alpha_2} - 1}{T - 1}$$

for $1 \leq j < \rho$. From this it follows that the kernel of h is generated by

$$\left\{ (S_i - T^{b^*} - T^{c^*})(S_i - T^{\alpha_2c^*} - T^{c^*+b^*}), \frac{(T^{\alpha_2\alpha_3} - 1)(T - 1)}{(T^{\alpha_2} - 1)(T^{\alpha_3} - 1)} \cdot (S_i - T^{b^*} - T^{c^*}) \right\}$$

and

$$\{(S_i - T^{b^*} - T^{c^*})(S_j - T^{b^*} - T^{c^*})\}$$

for $1 \leq i < j < \rho$. Thus

$$\begin{aligned} \mathbb{Z}[w, w^{-1}, u_i(w^{b^*} - w^{\alpha_2c^*}) + w^{\alpha_2c^*} + w^{c^*} : 1 \leq i < \rho] &\cong \\ &\mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}] / (\ker(h)), \end{aligned}$$

where T and S_i correspond to $[A(1)]$ and $[P_i]$ respectively. We have just shown that the map from $\mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}] / \mathcal{J}$ to $\mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}] / (\ker(h))$, which sends T to itself and S_i to itself for each i , is a well-defined surjective ring homomorphism. Hence $\mathcal{J} \subseteq \ker(h)$. ■

We shall now show that $\ker(h) \subseteq \mathcal{J}$. We accomplish this via the following lemmas.

Lemma 5.3 $(S_i - T^{b^*} - T^{c^*}) \frac{(T-1)(T^{\alpha_2\alpha_3}-1)}{(T^{\alpha_2}-1)(T^{\alpha_3}-1)} \in \mathcal{J}$ for $1 \leq i < \rho$.

Proof We define $J_{i,k} = (v - c_i, z, t^k)$ for $0 \leq k \leq \alpha_2$. Notice that $J_{i,0} = (1), J_{i,1} = J_i$. Consider the filtration

$$J_i = J_{i,1} \supseteq J_{i,2} \supseteq \dots \supseteq J_{i,\alpha_2} = (v - c_i, z) \supseteq (0).$$

From this we see that we have the relation

$$[J_i] = [(v - c_i, z)] + \sum_{k=1}^{\alpha_2-1} [J_{i,k} / J_{i,k+1}]$$

in $K(A - \mathbb{C}^*)$. We have an isomorphism

$$J_{i,k}/J_{i,k+1} \cong A(c^*k)/J_i(c^*k).$$

Moreover $[(v - c_i, z)]$ has a free resolution by A - \mathbb{C}^* -modules given by

$$0 \longrightarrow A(b^*) \longrightarrow A(0, b^*) \longrightarrow (v - c_i, z) \longrightarrow 0.$$

Hence our relation becomes

$$[J_i] = [A(0)] + \sum_{k=1}^{\alpha_2-1} ([A(c^*k)] - [J_i(c^*k)]).$$

Using the fact that $[J_i] = [A(0, b^*, c^*)] - [P_i]$, we deduce that

$$[A(b^*, c^*)] - [P_i] = \sum_{k=1}^{\alpha_2-1} ([P_i(c^*k)] - [A(c^*k + b^*, c^*k + c^*)])$$

in $K(A - \mathbb{C}^*)$. By the regularity of A , we have that this relation also holds in $K_1(A - \mathbb{C}^*)$. Using the same style of argument as was used in obtaining equation (4.3), we find that this corresponds to the fact that

$$\left(\frac{1 - T^{\alpha_2 c^*}}{1 - T^{c^*}} \right) (S_i - T^{c^*} - T^{b^*}) \in \mathcal{J}.$$

Next, from equation (1.5) we have $(v - c_i, z, t) = (v' - d_i, z, t)$; moreover $z^{\alpha_3} \in (v' - d_i, t)$. Proceeding in the same manner, using the filtration

$$(v' - d_i, z, t) \supseteq (v' - d_i, z^2, t) \supseteq \dots \supseteq (v' - d_i, z^{\alpha_3}, t) = (v' - d_i, t) \supseteq 0,$$

we find that

$$\left(\frac{1 - T^{\alpha_3 b^*}}{1 - T^{b^*}} \right) (S_i - T^{c^*} - T^{b^*}) \in \mathcal{J}.$$

Also, notice that $\alpha_2 \alpha_3$ is the gcd of the weight of x and the weight of y ; hence there exist positive integers γ and δ such that $x^\gamma y^\delta$ has weight $\alpha_2 \alpha_3$. We have the exact sequence

$$0 \longrightarrow A(\alpha_2 \alpha_3) \longrightarrow J_i(\alpha_2 \alpha_3) \oplus A \longrightarrow J_i \longrightarrow 0,$$

where the map from $J_i(\alpha_2 \alpha_3) \oplus A$ onto J_i is given by $(j, a) \mapsto jx^\gamma y^\delta + (v - c_i)a$. Notice that this map is surjective as $x^\gamma y^\delta$ and $v - c_i$ generate the unit ideal. The map from $A(\alpha_2 \alpha_3)$ into $J_i(\alpha_2 \alpha_3) \oplus A$ is given by $a \mapsto (a(v - c_i), -ax^\gamma y^\delta)$. The fact that A is a UFD shows that this sequence is exact. From this exact sequence we shall deduce that $(T^{\alpha_2 \alpha_3} - 1)(S_i - T^{c^*} - T^{b^*}) \in \mathcal{J}$. Consider the ideal

$$\left(T^{\alpha_2 \alpha_3} - 1, \frac{T^{\alpha_2 c^*} - 1}{T^{c^*} - 1}, \frac{T^{\alpha_3 b^*} - 1}{T^{b^*} - 1} \right) \subseteq \mathbb{Z}[T, T^{-1}].$$

Notice by Lemma 2.1, taking X to be T^{α_3} and l, m and n to be $1, a_2^{\alpha_1}$ and α_2 respectively, we have

$$\left(T^{\alpha_2\alpha_3} - 1, \frac{T^{\alpha_2c^*} - 1}{T^{c^*} - 1} \right) = \left(\frac{T^{\alpha_2\alpha_3} - 1}{T^{\alpha_3} - 1} \right).$$

Similarly,

$$\left(T^{\alpha_2\alpha_3} - 1, \frac{T^{\alpha_3b^*} - 1}{T^{b^*} - 1} \right) = \left(\frac{T^{\alpha_2\alpha_3} - 1}{T^{\alpha_2} - 1} \right).$$

Therefore our ideal is just the ideal

$$\left(\frac{T^{\alpha_2\alpha_3} - 1}{T^{\alpha_2} - 1}, \frac{T^{\alpha_2\alpha_3} - 1}{T^{\alpha_3} - 1} \right).$$

By assumption, α_2 and α_3 are relatively prime. Hence there exist positive integers γ and δ such that $\alpha_2\gamma - \alpha_3\delta = 1$. Thus

$$\frac{(T^{\alpha_2\alpha_3} - 1)(T - 1)}{(T^{\alpha_2} - 1)(T^{\alpha_3} - 1)} = \frac{(T^{\gamma\alpha_2} - 1)}{(T^{\alpha_2} - 1)} \cdot \frac{(T^{\alpha_2\alpha_3} - 1)}{(T^{\alpha_3} - 1)} - T^{\delta\alpha_3} \frac{T^{\alpha_3} - 1}{T^{\alpha_3} - 1} \cdot \frac{(T^{\alpha_2\alpha_3} - 1)}{(T^{\alpha_2} - 1)}$$

Hence

$$(S - T^{b^*} - T^{c^*}) \frac{(T^{\alpha_2\alpha_3} - 1)(T - 1)}{(T^{\alpha_2} - 1)(T^{\alpha_3} - 1)} \in \mathcal{J}. \quad \blacksquare$$

Lemma 5.4 *If $i \neq j$, then $(S_j - T^{b^*} - T^{c^*})(S_i - T^{b^*} - T^{c^*}) \in \mathcal{J}$.*

Proof Let $e_{j,1}, e_{j,2}, e_{j,3}, e_{j,4}$ be generators for P_j as given in equation (5.1). Observe that $e_{j,1}, e_{j,4} \in J_iP_j$. To see this, first notice that $(v - c_i)e_{j,1}$ and $-z^{\alpha_3-1}G_1(y^{\alpha_1}, x, z^{\alpha_3})e_{j,3} + t^{\alpha_2-1}e_{j,2} = (v - c_j)e_{j,1}$ are both in J_iP_j , and hence $e_{j,1} \in J_iP_j$. Next, note that $(v - c_i)e_{j,4}$ and $ze_{j,2} - te_{j,3} = (v - c_j)e_{j,4} \in J_iP_j$, and thus we see that $e_{j,4} \in J_iP_j$. It follows that P_j/J_iP_j is generated by the images of $e_{j,2}$ and $e_{j,3}$. Thus

$$(5.2) \quad P_j/J_iP_j \cong A(c^*)/J_i(c^*) \oplus A(b^*)/J_i(b^*).$$

Now $P_j/J_iP_j \cong P_j \otimes_A A/J_i$. Since P_j is A -flat, we have that $[P_j \otimes_A A/J_i] = [P_j] - [P_j \otimes J_i]$. Using the isomorphism (5.2), we find that

$$(5.3) \quad [A(c^*, b^*)] - [J_i(c^*, b^*)] = [P_j] - [P_j \otimes J_i]$$

in $K(A - \mathbb{C}^*)$. Recall that we have the exact sequence

$$0 \longrightarrow P_i \longrightarrow A(0, b^*, c^*) \longrightarrow J_i \longrightarrow 0.$$

Using this fact with the fact that P_j is A -flat, we can rewrite equation (5.3) and obtain the relation

$$[A(c^*, b^*)] - [A(c^*, b^*, 2b^*, 2c^*, c^*+b^*, c^*+b^*)] + [P_i(c^*, b^*)][P_j] - [P_j \otimes A(0, b^*, c^*)] + [P_j \otimes P_i]$$

in $K(A - \mathbb{C}^*)$. Simplifying this expression, we find that

$$[P_j \otimes P_i] - [P_j(b^*, c^*)] - [P_i(b^*, c^*)] + [A(2b^*, 2c^*, c^* + b^*, c^* + b^*)] = 0$$

in $K(A - \mathbb{C}^*)$. Since A is regular, this relation holds in $K_1(A - \mathbb{C}^*)$, and we see that

$$\begin{aligned} 0 &= [P_j] \cdot [P_i] - [P_j] \cdot [A(b^*, c^*)] - [P_i] \cdot [A(b^*, c^*)] + [A(b^*, c^*)] \cdot [A(b^*, c^*)] \\ &= ([P_j] - [A(b^*, c^*)]) ([P_i] - [A(b^*, c^*)]) \end{aligned}$$

in $K_1(A - \mathbb{C}^*)$. Equivalently, $(S_j - T^{b^*} - T^{c^*})(S_i - T^{b^*} - T^{c^*}) \in \mathcal{J}$ for $i \neq j$. ■

To show that $\mathcal{J} \subseteq \ker(h)$, it now suffices to prove the following Lemma.

Lemma 5.5

$$(S_i - T^{b^*} - T^{c^*})(S_i - T^{\alpha_2 c^*} - T^{c^* + b^* - \alpha_2 c^*}) \in \mathcal{J}$$

for $1 \leq i < \rho$.

Proof Let $J'_i = (xq(v)/(v - c_i), z, t)$ and let $e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4}$ be as in (5.1). We claim that $e_{i,1}, e_{i,4} \in J'_i P_i$. To show this, notice that $xq(v)/(v - c_i)e_{i,1}$ and $-z^{\alpha_3 - 1}G_1(y^{\alpha_1}, x, z^{\alpha_3})e_{i,3} + t^{\alpha_2 - 1}e_{i,2} = (v - c_i)e_{i,1}$ are both elements of $J'_i P_i$. Since $v - c_i$ and $xq(v)/(v - c_i)$ generate the unit ideal, it follows that $e_{i,1} \in J'_i P_i$. Similarly, $xq(v)/(v - c_i)e_{i,4}$ and $-te_{i,3} + ze_{i,2} = (v - c_i)e_{i,4}$ are both elements of $J'_i P_i$, and hence $e_{i,4} \in J'_i P_i$. Thus $P_i/J'_i P_i$ is generated by the images of $e_{i,2}, e_{i,3}$, giving us the isomorphism

$$P_i/J'_i P_i \cong A(c^*)/J'_i(c^*) \oplus A(b^*)/J'_i(b^*).$$

Therefore in $K(A - \mathbb{C}^*)$ we have the relation

$$(5.4) \quad [P_i] - [J'_i P_i] = [A(b^*, c^*)] - [J'_i(c^*, b^*)].$$

Notice we have the exact sequence

$$(5.5) \quad 0 \longrightarrow (z, t) \longrightarrow (v - c_i, z, t) \longrightarrow A/J'_i \longrightarrow 0.$$

Tensoring this sequence with the A -flat module P_i , we see that

$$[P_i/J'_i P_i] = [J_i \otimes_A P_i] - [(z, t) \otimes_A P_i]$$

in $K(A - \mathbb{C}^*)$. Combining this relation with relation (5.4) we see that

$$[A(c^*, b^*)] - [J'_i(b^*, c^*)] = [J_i \otimes_A P_i] - [(z, t) \otimes_A P_i]$$

in $K(A - \mathbb{C}^*)$. Now $[(z, t)] = [A(b^*, c^*)] - [A(b^* + c^*)]$ and $[J_i] = [A(0, b^*, c^*)] - [P_i]$, hence

$$\begin{aligned} (5.6) \quad [A(c^*, b^*)] - [J'_i(b^*, c^*)] &= [P_i(0, b^*, c^*)] - [P_i \otimes P_i] \\ &\quad - [P_i(b^*, c^*)] + [P_i(b^* + c^*)] \\ &= [P_i] + [P_i(b^* + c^*)] - [P_i \otimes P_i] \end{aligned}$$

From the exact sequence (5.5) we know that

$$\begin{aligned} [A(0)] - [J'_i] &= [J_i] - [(z, t)] \\ &= ([A(0, b^*, c^*)] - [P_i]) - ([A(b^*, c^*)] - [A(b^* + c^*)]) \\ &= [A(0)] - [P_i] + [A(b^* + c^*)]. \end{aligned}$$

And so $[J'_i] = [P_i] - [A(b^* + c^*)]$. Using this fact along with equation (5.6) we see that in $K(A - \mathbb{C}^*)$ we have the relation

$$[A(c^*, b^*, 2b^* + c^*, b^* + 2c^*)] - [P_i(b^*, c^*)] = [P_i] + [P(b^* + c^*)] - [P_i \otimes P_i].$$

Therefore in $K_1(A - \mathbb{C}^*)$ we have the relation

$$\begin{aligned} 0 &= [P_i] \cdot [P_i] - [P_i(0, b^*, c^*, b^* + c^*)] + [A(c^*, b^*, 2b^* + c^*, 2c^* + b^*)] \\ &= ([P_i] - [A(b^*, c^*)]) ([P_i] - [A(0, b^* + c^*)]). \end{aligned}$$

Equivalently,

$$(S_i - T^{b^*} - T^{c^*})(S_i - 1 - T^{b^*+c^*}) \in \mathcal{J}$$

for $1 \leq i < \rho$. Since $(T^{\alpha_2 c^*} - 1)(S_i - T^{b^*} - T^{c^*}) \in \mathcal{J}$, we see that

$$(S_i - T^{b^*} - T^{c^*})(S_i - T^{\alpha_2 c^*} - T^{b^*+c^*-\alpha_2 c^*}) \in \mathcal{J}. \quad \blacksquare$$

The three previous lemmas show that $\ker(h) \subseteq \mathcal{J}$. It follows that $\ker(h) = \mathcal{J}$. We record what we have shown in the following theorem.

Theorem 5.1 $K_1(A - \mathbb{C}^*) \cong \mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}]/\mathcal{J}$, where \mathcal{J} is generated by

$$\left\{ (S_i - T^{c^*} - T^{b^*})(S_i - 1 - T^{b^*+c^*}), \frac{(T^{\alpha_2 \alpha_3} - 1)(T - 1)}{(T^{\alpha_2} - 1)(T^{\alpha_3} - 1)}(S_i - T^{b^*} - T^{c^*}) \right\}$$

and

$$\{(S_i - T^{b^*} - T^{c^*})(S_j - T^{b^*} - T^{c^*})\}$$

for $1 \leq i < j < \rho$. The isomorphism is given by the mapping which sends $[P_1], \dots, [P_{\rho-1}]$ and $[A(1)]$ to $S_1, \dots, S_{\rho-1}$ and T respectively.

We shall now try to give a nicer description of $K_1(A - \mathbb{C}^*)$. For the sake of brevity we shall let

$$(5.7) \quad K = \mathbb{Z}[T, T^{-1}, S_1, \dots, S_{\rho-1}]/\mathcal{J}.$$

Let $E_i = S_i - T^{b^*} - T^{c^*}$ for $1 \leq i < \rho$. Notice $E_i E_j = 0$ for $i \neq j$ and that

$$\begin{aligned} E_i^2 &= (S_i - T^{b^*} - T^{c^*})(S_i - T^{b^*} - T^{c^*}) \\ &= (S_i - T^{b^*} - T^{c^*})((S_i - 1 - T^{b^*+c^*}) + 1 + T^{b^*+c^*} - T^{b^*} - T^{c^*}) \\ &= (S_i - T^{b^*} - T^{c^*})(1 - T^{b^*})(1 - T^{c^*}) \\ &= (1 - T^{b^*})(1 - T^{c^*})E_i. \end{aligned}$$

Therefore we have

Theorem 5.2 $K_1(A - \mathbb{C}^*) \cong \mathbb{Z}[T, T^{-1}, E_1, \dots, E_{\rho-1}]/\mathcal{J}'$, where \mathcal{J}' is the ideal generated by

$$\left\{ E_i E_j, E_i^2 - E_i(1 - T^{b^*})(1 - T^{c^*}), \frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} E_i : 1 \leq i < j < \rho \right\}.$$

Notice that there is a surjection from K onto $\mathbb{Z}[T, T^{-1}]$ given by mapping E_i to 0 for $i = 1, \dots, \rho - 1$, and mapping T and T^{-1} to themselves. Hence $\mathbb{Z}[T, T^{-1}]$ is a subalgebra of K . We investigate the structure of K as a $\mathbb{Z}[T, T^{-1}]$ -algebra.

We note that $(1 - T^{b^*})(1 - T^{c^*})$ and $\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})}$ have no common roots and hence there exists a polynomial $\lambda(T) \in \mathbb{Q}[T]$ such that $\lambda(T)(1 - T^{b^*})(1 - T^{c^*})$ is congruent to 1 mod $\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})}$. We will use this fact to give the structure of

$$\tilde{K} := K \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_1(A - \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice from Theorem 5.2 any element σ of K can be written as

$$\sigma = q_0(T) + \sum_{i=1}^{\rho-1} q_i(T) E_i,$$

where $q_j(T)$ is a Laurent polynomial in T for $0 \leq j < \rho$. Now suppose $m\sigma = 0$ for some $m \in \mathbb{Z}$. Then using Theorem 5.2 again, we know that $m q_0(T) = 0$ and that

$$\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \mid m q_i(T)$$

for $1 \leq i < \rho$. Since $\mathbb{Z}[T, T^{-1}]$ is a UFD and m and $(1 - T^{\alpha_2 \alpha_3})(1 - T)(1 - T^{\alpha_2})^{-1} \cdot (1 - T^{\alpha_3})^{-1}$ are relatively prime, we have that $q_0(T) = 0$ and

$$\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \mid q_i(T)$$

for $1 \leq i < \rho$. Hence $\sigma = 0$. Thus K doesn't have any \mathbb{Z} -torsion, and $K_1(A - \mathbb{C}^*)$ injects into $K_1(A - \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Notice that $\lambda(T)E_1, \dots, \lambda(T)E_{\rho-1}$ and $E_{\rho} := 1 - \lambda(T)(E_1 + \dots + E_{\rho-1})$ form a complete system of orthogonal idempotents in \tilde{K} . Now

$$\lambda(T)E_i \tilde{K} \cong \mathbb{Q}[T, T^{-1}] \Big/ \left(\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \right)$$

for $1 \leq i < \rho$. Also

$$E_{\rho} \tilde{K} \cong \mathbb{Q}[T, T^{-1}].$$

Hence we have the isomorphism

$$\begin{aligned} \tilde{K} &\cong \left(\times_{i=1}^{\rho-1} \lambda(T)E_i \tilde{K} \right) \times E_{\rho} \tilde{K} \\ &\cong \mathbb{Q}[T, T^{-1}] \times \left(\mathbb{Q}[T, T^{-1}] \Big/ \left(\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \right) \right)^{\rho-1} \end{aligned}$$

(see [9] page 411). We record this fact in the following theorem.

Theorem 5.3

$$K_1(A - \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[T, T^{-1}] \times \left(\mathbb{Q}[T, T^{-1}] / \left(\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \right) \right)^{\rho-1}.$$

Moreover, $K_1(A - \mathbb{C}^*)$ injects into $K_1(A - \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Corollary 5.1 *If $A \not\cong \mathbb{C}^{[3]}$ (i.e. if $\varepsilon \neq 0$), then the number ρ and the sequence $\{\alpha_2, \alpha_3\}$ are uniquely determined by the equivariant Grothendieck ring of A . In particular, they are equivariant invariants.*

Proof By the previous theorem, the polynomial $\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})}$ and ρ are uniquely determined by the structure of the equivariant Grothendieck ring of A . Moreover, if $\varepsilon \neq 0$, then the polynomial $(1 - T^{\alpha_2 \alpha_3})(1 - T)(1 - T^{\alpha_2})^{-1}(1 - T^{\alpha_3})^{-1}$ determines the sequence $\{\alpha_2, \alpha_3\}$. This completes the proof. ■

We shall now show that if $\varepsilon \neq 0$, then $E_1, \dots, E_{\rho-1}$ form a minimal set of generators for K as a $\mathbb{Z}[T, T^{-1}]$ -algebra.

Theorem 5.4 *If $A \not\cong \mathbb{C}^{[3]}$, then $E_1, \dots, E_{\rho-1}$ form a minimal generating set for K as $\mathbb{Z}[T, T^{-1}]$ -algebra.*

Proof We argue by contradiction. Suppose that $\{e_1, \dots, e_n\}$ is a generating set. Let $\{\bar{e}_1, \dots, \bar{e}_n\}$ be the images of $\{e_1, \dots, e_n\}$ in

$$K / ((1 - T^{b^*})(1 - T^{c^*})) \cong \mathbb{Z}[T, T^{-1}, E_1, \dots, E_{\rho-1}] / (\mathcal{J}', (1 - T^{b^*})(1 - T^{c^*})).$$

In this ring, $\bar{e}_i \bar{e}_j$ is in the $\mathbb{Z}[T, T^{-1}]$ -module generated by $\{1, \bar{e}_1, \dots, \bar{e}_n\}$, for all i and j . To see this, let \bar{E}_i denote the image of E_i in this quotient ring. Notice that if $\bar{e}_i = \lambda_0(T) + \sum_{k=1}^{\rho-1} \lambda_k(T) \bar{E}_k$ and $\bar{e}_j = \gamma_0(T) + \sum_{k=1}^{\rho-1} \gamma_k(T) \bar{E}_k$, then since $\bar{E}_i \bar{E}_j = 0$ for all i and j in this ring, we have that

$$\begin{aligned} \bar{e}_i \bar{e}_j &= \lambda_0(T) \gamma_0(T) + \sum_{k=1}^{\rho-1} (\lambda_0(T) \gamma_k(T) \bar{E}_k + \gamma_0(T) \lambda_k(T) \bar{E}_k) \\ &= \lambda_0(T) \bar{e}_j + \gamma_0(T) \bar{e}_i - \lambda_0(T) \gamma_0(T). \end{aligned}$$

Therefore, $\{1, \bar{e}_1, \dots, \bar{e}_n\}$ must generate $K / ((1 - T^{b^*})(1 - T^{c^*}))$ as a $\mathbb{Z}[T, T^{-1}]$ -module. Notice that $K / ((1 - T^{b^*})(1 - T^{c^*})) \cong \mathbb{Z}[T, T^{-1}] \oplus (\mathbb{Z}[T, T^{-1}] / V)^{\rho-1}$, where V is the ideal generated by $(1 - T^{b^*})(1 - T^{c^*})$ and $\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})}$. Since $\{1, \bar{e}_1, \dots, \bar{e}_n\}$ generate $K / ((1 - T^{b^*})(1 - T^{c^*}))$ as a $\mathbb{Z}[T, T^{-1}]$ -module, it follows that $\{1 \otimes 1, \bar{e}_1 \otimes 1, \dots, \bar{e}_n \otimes 1\}$ must generate

$$K / ((1 - T^{b^*})(1 - T^{c^*})) \bigotimes_{\mathbb{Z}[T, T^{-1}]} \mathbb{Z}[T, T^{-1}] / V \cong (\mathbb{Z}[T, T^{-1}] / V)^{\rho}$$

as a $\mathbb{Z}[T, T^{-1}]/V$ -module. By Lemma 2.1 *b*, we know that when $\varepsilon \neq 0$ that $V \neq (1)$. Since $V \neq (1)$ and $(\mathbb{Z}[T, T^{-1}]/V)^\rho$ is a free module of rank ρ , it follows that $\text{card}\{1 \otimes 1, \bar{e}_1 \otimes 1, \dots, \bar{e}_n \otimes 1\} \geq \rho$. Hence $n \geq \rho - 1$. ■

Finally, let us make the remark that $1, E_1, \dots, E_{\rho-1}$ generate K as a $\mathbb{Z}[T, T^{-1}]$ -module. This module is easily seen to be isomorphic to

$$\mathbb{Z}[T, T^{-1}] \oplus \left(\mathbb{Z}[T, T^{-1}] / \left(\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \right) \right)^{\rho-1}.$$

Now $\frac{(1 - T^{\alpha_2 \alpha_3})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})}$ is a monic polynomial in T of degree $(\alpha_2 - 1)(\alpha_3 - 1)$ and hence

$$\mathbb{Z}[T, T^{-1}] / \left(\frac{(1 - T^{\alpha_2})(1 - T)}{(1 - T^{\alpha_2})(1 - T^{\alpha_3})} \right) \cong \mathbb{Z}^{(\alpha_2 - 1)(\alpha_3 - 1)}$$

as a \mathbb{Z} -module. Therefore K is just the direct sum of $\mathbb{Z}[T, T^{-1}]$ with a free abelian group of rank $\varepsilon/a'_1 = (\rho - 1)(\alpha_2 - 1)(\alpha_3 - 1)$. This can be interpreted as follows. We have a map from $K(A - \mathbb{C}^*)$ into $K(\mathbb{C} - \mathbb{C}^*)$ given by $M \mapsto M/(y, z, t)M$. This map is clearly surjective. The kernel of this map is then a free abelian group of rank ε/a'_1 . Since $\mathbb{Z}[T, T^{-1}]$ is a projective \mathbb{Z} -module, we know that the sequence

$$0 \longrightarrow \mathbb{Z}^{\varepsilon/a'_1} \longrightarrow K(A - \mathbb{C}^*) \longrightarrow K(\mathbb{C} - \mathbb{C}^*) \longrightarrow 0$$

splits. Hence we have an isomorphism between $K(A - \mathbb{C}^*)$ and $\mathbb{Z}^{\varepsilon/a'_1} \oplus K(\mathbb{C} - \mathbb{C}^*)$. Also, we know that the submodule of $K(A - \mathbb{C}^*)$ generated by

$$\{ [M] - [(M/(y, z, t)M) \otimes_{\mathbb{C}} A] : M \text{ is an } A \text{ module} \}$$

is a free abelian group of rank ε/a'_1 .

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