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ON TWISTED ORBITAL INTEGRAL IDENTITIES FOR $PGL(3)$ OVER A p-ADIC FIELD

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ABSTRACT. The object of this paper is to prove certain p-adic orbital integral identities needed in order to accomplish the symmetric square transfer via the twisted Arthur trace formula. Only §5 of this article contains original material, the rest of it is due to R. Langlands. Very briefly, we reduce the problem of proving certain orbital integral identities for "matching" functions in the respective Hecke algebras to two counting problems on the buildings. We give Langlands' solution of one of these problems in the case of the unit elements of the respective Hecke algebras and §5 provides the solution to the other one, again, in the unit element case. The main results assume $p \neq 2$.

0. Contents. Only §5 of this article contains original material, the rest of it can be essentially be found in the unpublished notes [**12**]. Very briefly, §1 provides an introduction including the definition (due to H. Jacquet and J. Shalika) of the "norm map", §2 applies the Satake transform to proving some simple orbital integral identities in the "split case", §3 reduces the problem of proving these identities in the "non-split case" to two counting problems on the buildings, §4 recalls R. Langlands' solution of one of these problems for the unit element of the Hecke algebra, and, finally, §5 provides the solution to the other one (for the unit element of the Hecke algebra). In sections 4 and 5 we assume that $p \neq 2$.

This paper was written independently of a recent paper [**4**] which also relied on Langlands' unpublished manuscript [**12**].

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1. Introduction.

1.1. History. Let F denote a p-adic field and $\Pi(G(F))$ denote the set of equivalence classes of admissible irreducible representations of a reductive al-

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gebraic group G. Typically, F will be a completion of a number field K at a place v; let $\mathbb{A} = \mathbb{A}_K$ denote the adele ring of K.

One example of the local functoriality conjecture ([**2**], [**13**]) says that the L-homomorphism

$$(1.1) \qquad r : GL(2, \mathbb{C}) \longrightarrow GL(2n+1, \mathbb{C}),$$

given by $r := \mathrm{Sym}^{\otimes 2n} \otimes (\det)^{-n}$ should yield a transfer (the "symmetric nth power lifting")

$$(1.2) \qquad r_* : \Pi(GL(2, F)) \longrightarrow \Pi(GL(2n+1, F)).$$

Using L-parameters one can directly "lift" or transfer unramified principal series from $GL(2, F)$ to $GL(2n+1, F)$, so there is some evidence for the validity of this conjecture. There is of course a global analog of this conjectural transfer, predicted by the global functionality principle.

If $n = 1$ then S. Gelbart and H. Jacquet [**5**] proved the existence of a global representation $\Pi \in \Pi(GL(3, \mathbb{A}))$ associated to a cuspidal $\pi \in \Pi(GL(2, \mathbb{A}))$ using the theory of L-functions, converse Hecke theory on $GL(3, \mathbb{A})$ [**6**], and an idea of G. Shimura [**17**]. Let me reformulate this in terms of functoriality: when $n = 1$, the L-map (1.1) becomes

$$(1.3) \qquad r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} a^2 & -ab & -b^2 \\ -2ac & ad+bc & 2bd \\ -c^2 & cd & d^2 \end{pmatrix}$$

the adjoint representation of $sl(2, \mathbb{C})$ with respect to the basis

$$\underline{e}_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \underline{e}_2 := \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \ \underline{e}_3 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and this factors through $PGL(2)$ giving us

$$(1.4) \qquad \lambda_1 : {}^L SL(2, \mathbb{C}) = PGL(2, \mathbb{C}) \hookrightarrow SO(3, \mathbb{C}) \hookrightarrow SL(3, \mathbb{C}) = {}^L PGL(3, \mathbb{C}).$$

The functoriality conjecture then predicts a local and global transfer from $SL(2)$ to $PGL(3)$. On the other hand, the natural embedding gives us an L-map

$$(1.5) \qquad \lambda_2 : {}^L PGL(2, \mathbb{C}) \hookrightarrow {}^L PGL(3, \mathbb{C}),$$

In this case the functoriality conjecture predicts a local and global transfer from $PGL(2)$ to $PGL(3)$. The global transfer associated to (1.5) is in fact the Gelbart–Jacquet lift. The local transfer associated to (1.4) will be elaborated upon in subsection 1.2 below.

Around 1976, H. Jacquet suggested that there should be a twisted trace formula for $PGL(3)$, corresponding to the outer automorphism

$$\sigma : g \longmapsto J^t g^{-1} J, \ \sigma^2 = 1,$$

where J is the idempotent matrix given by

$$J := \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

and that this trace formula could be applied to proving the symmetric square transfer in a fashion analogous to the base change lift of Saito–Shintani–Langlands. Moreover, $PGL(2)$ and $SL(2)$ should be σ-endoscopic groups for $PGL(3)$ and the symmetric square transfer should be a consequence of stabilizing this twisted $PGL(3)$-trace formula. The notes [**12**], completed a year or so later, were motivated by Jacquet's suggestion.

1.2. L-parameters and the transfer from $SL(2)$ to $PGL(3)$. Let F be a p-adic field with ring of integers O_F, let G denote one of the groups $SL(n)$ or $PGL(n)$, let K denote a maximal compact subgroup of $G(F)$, and let $\mathcal{H}(G,K)$ denote the **Hecke algebra** of compactly supported locally constant functions on the double coset space $K \setminus G(F)/K$. Let $K_0 := G(O_F)$ denote the hyperspecial maximal compact subgroup; when $K = K_0$, we write $\mathcal{H}(G) := \mathcal{H}(G, K_0)$. Let $C_c^\infty(G)$ denote the algebra of **Schwartz–Bruhat functions** of locally constant functions of compact support on G.

Corresponding to the L-map (1.4), the Satake transform (defined in §2 below) gives us a homomorphism of Hecke algebras

$$(1.6) \qquad \lambda_1^* : \mathcal{H}(PGL(3)) \longrightarrow \mathcal{H}(SL(2)), \qquad \phi \longmapsto f := \lambda_1^*(\phi),$$

determined by $\hat{f}(t) = \hat{\phi}(\lambda_1(t)), t \in {}^L SL(2, \mathbb{C})$. When this identity holds, one says that the spherical function ϕ on $PGL(3,F)$ **corresponds** with the spherical function f on $SL(2,F)$. Similarly, corresponding to the L-map (1.5) there is the L-homomorphism of Hecke algebras

$$(1.7) \qquad \lambda_2^* : \mathcal{H}(PGL(3)) \longrightarrow \mathcal{H}(PGL(2)), \qquad \phi \longmapsto f := \lambda_2^*(\phi),$$

determined by $\hat{f}(t) = \hat{\phi}(\lambda_2(t)), t \in {}^L PGL(2, \mathbb{C})$. Using this one can define a "correspondence", as above, between spherical functions on $PGL(3,F)$ and spherical functions on $PGL(2,F)$.

According to the local Langlands correspondence for $SL(2)$, to each unramified admissible homomorphism $\psi : W_F \longrightarrow {}^L SL(2, \mathbb{C})$ there is an L-packet of admissible irreducible unramified representations $\Pi(\theta) \in \Pi(SL(2,F))_L$ of $SL(2,F)$. Here W_F denotes the absolute local Weil group of F and the subscript L on the Π signifies that we are identifying L-indistinguishable representations. Each homomorphism $\theta : W_F \longrightarrow {}^L SL(2,\mathbb{C})$ yields via (1.4) a homomorphism $\theta^* : W_F \longrightarrow {}^L PGL(3,\mathbb{C})$, and if θ is admissible then so is θ^*. The local functoriality conjecture predicts that the L-map (1.4) should yield a map of tempered L-packets

$$(1.8) \qquad \Lambda_1 : \Pi^{\text{temp}}(SL(2,\ F))_L \longrightarrow \Pi^{\text{temp}}(PGL(3,\ F)), \qquad \Pi(\theta) \longmapsto \pi_{\theta^*}.$$

We say that $\pi \in \Pi(SL(2,F))$ **transfers** to $\Pi \in \Pi(PGL(3,F))$ if $\pi \in \Pi(\theta)$ and $\Pi = \pi_{\theta^*}$, for some admissible θ as above. The L-packets of $SL(2)$ have been described by J.-P. Labesse and R. Langlands [**10**].

1.3. The Jacquet–Shalika norm maps. The basic ideas of this subsection are, I believe, due to H. Jacquet and J. Shalika and may be found in [**12**].

Let $G := PGL(3), H_1 := SL(2), H_2 := PGL(2)$. Associated to the two σ-endoscopic groups H_i are "norm" maps

$$(1.9) \qquad \begin{aligned} N_i :& \{\text{stable } \sigma\text{-conjugacy classes in } G\} \\ &\longrightarrow \{\text{stable conjugacy classes in } H_i\}, \end{aligned}$$

which (hopefully) allow one to relate stable class functions on G to stable class functions on H_i. The explicit construction of these norm maps is the object of this subsection.

LEMMA 1.10. *If* $[A] \in PGL(2n+1,F)$ *is represented by* $A \in GL(2n+1,F)$ *then* $[A]\sigma([A])$ *is represented by* $A\cdot {}^tA^{-1}$ *and* $A\cdot {}^tA^{-1}$ *has at least one eigenvalue equal to one.*

This is an immediate consequence of the fact that $A^{-1} - {}^tA^{-1}$ is singular (since $2n+1$ is odd), hence has 0 as an eigenvalue.

Let $V := F^{2n+1}$ and identify, by fixing a basis of V, $GL(2n+1,F)$ with $Aut_F(V)$. For the $[A]$ and A in the lemma, let $V_A \subset V$ be a 1-dimensional subspace on which $A\cdot {}^tA^{-1}$ acts as the identity. Let W_A be the orthogonal complement of ${}^tA^{-1}V_A$ with respect to the inner product $(v_1,v_2) := {}^tv_1\cdot v_2$. Observe that $A\cdot {}^tA^{-1}$ acts on V_A trivially, therefore on V/V_A, and $A{}^tA^{-1}$ acts on W_A.

Let me now restrict to the case $n = 1$.

LEMMA 1.11. *Suppose that* $A\cdot {}^tA^{-1}$ *induces the linear transformation* $B_1 \in Aut_F(W_A)$ *and* $B_2 \in Aut_F(V/V_A)$. *Then*

(a) $\det B_1 = \det B_2 = 1$;

(b) there is an isomorphism $\theta : W_A \xrightarrow{\sim} V/V_A$ *such that*

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$$\begin{array}{ccc} W_A & \xrightarrow{B_1} & W_A \\ \theta \downarrow & & \uparrow \theta \\ V/V_A & \xrightarrow{B_2} & V/V_A \end{array}$$

commutes;

(c) as elements of $SL(2, F), B_1$ *and* B_2 *are stably conjugate;*

(d) the stable conjugacy class of B_1 *is independent of the choice of* V_A.

For the proof, I refer to [**12**].

Definition 1.12. Choose a basis of V/V_A or of W_A and represent the transformation induced by $A \cdot {}^tA^{-1}$ by a 2×2 matrix. The stable $SL(2)$-conjugacy class of this matrix is well-defined and depends only on the stable σ-conjugacy class of $[A]$. This stable conjugacy class in $SL(2, F)$ is called the H_1-**norm** of $[A]$, written as $N_1([A]) \subseteq H_1(F)$. For the H_2-norm, first identify H_2 with $SO(3)$ via the adjoint representation (1.3), where $SO(3)$ denotes the connected component of the orthogonal group for the skew-diagonal matrix J above. If the eigenvalues of $A \cdot {}^tA^{-1}$ are α, 1, α^{-1} and $\alpha \neq \pm 1$, then define the H_2-**norm** of $[A]$, written $N_2([A])$, to be the stable H_2-conjugacy class which contains a matrix with α, 1, α^{-1} as eigenvalues.

As a matter of notation, let

$$Cl_\sigma^{st}(G) := \{\text{stable } \sigma\text{-conjugacy classes in } G\},$$

$$Cl^{st}(H_i) := \{\text{stable conjugacy classes in } H_i\}.$$

Proposition 1.13 (Jacquet–Shalika). *The norm map*

$$N_1 \,:\, Cl_\sigma^{st}(G) \longrightarrow Cl^{st}(H_1)$$

is a bijection.

Remarks. (1) This is proven in [**12**] by explicit matrix manipulations.

(2) For H_2 stable conjugacy and ordinary conjugacy are essentially the same, so the statement analogous to the proposition is false for N_2.

(3) In general, one expects to be able to define norms from $PGL(r)$ to any of its σ-endoscopic groups. One expects that the norm map associated to the "largest" σ-endoscopic group should also yield a bijection.

The following corollary of Langlands' proof of (1.13) is sometimes useful.

Corollary 1.14. *If* $N_1([A])$ *does not contain a unipotent, then there is a* $[A']$ *in the same stable* σ*-conjugacy class as* $[A]$ *and represented by* $A' \in GL(3, F)$, *for which* *(a)* $V = V_{A'} \oplus W_{A'}$;

(b) $W_{A'}^{\perp} = V_{A'}$ *with respect to the inner product* $(v_1, v_2) := {}^t v_2 \cdot v_1$;

(c) $V_{A'}$ *and* $W_{A'}$ *are both invariant under* A *and* ${}^t A^{-1}$ *(i.e., A can be put in (2,1)-block form).*

1.4. The fundamental lemmas. Recall $G := PGL(3), H_1 := SL(2), H_2 := PGL(2)$. Let us denote the **twisted centralizer** by

$$(1.15) \quad G(g\sigma,\ F) := \{x \in G(F) | x^{-1} g\sigma(x) = g\},$$

and the (ordinary) **centralizer** by

$$(1.16) \quad H_i(h,\ F) := \{x \in H_i(F) | x^{-1} hx = h\}, \quad i = 1.2.$$

Let $\omega_{G(g\sigma)}$, ω_G be fixed nonzero forms of maximal degree on $G(g\sigma, F), G(F)$, respectively, and let

$$d\bar{x} := \frac{d\omega_G(x)}{d\omega_{G(g\sigma)}(x)}$$

denote the quotient measure. For $\phi \in C_c^{\infty}(G)$, define the **twisted orbital integral** by

$$(1.17) \quad \Phi(g,\ \phi,\ \omega_G,\ \omega_{G(g\sigma)}) = \Phi(g,\ \phi) := \int_{G(g\sigma,F)\backslash G(F)} \phi(g^{-1} g\sigma(g)) d\bar{g},$$

provided it converges. Similarly, let $\omega_{H_i(h)}, \omega_{H_i}$ be fixed nonzero forms of maximal degree on $H_i(h\sigma, F), H_i(F)$, respectively, and let

$$d\bar{x} := \frac{d\omega_{H_i}(x)}{d\omega_{H_i(h)}(x)}$$

denote the quotient measure. In fact, we don't take any $\omega_{H_i(h)}$, but only one obtained by pulling back $\omega_{G(g\sigma)}$ via an étale surjective homomorphism over $\bar{F}$

$$\psi_i \ :\ H_i(h)^0 \longrightarrow G(g\sigma),$$

namely, it must satisfy

$$\omega_{H_i(h)} = |\ker \psi_i|^{-1} \psi_i^* \omega_{G(g\sigma)},$$

for all $h \in N_i(g)$. Here N_i denotes the norm map defined above. Note that if $h \in H_i(F)$ is regular semi-simple then we may assume that ψ_i is an isomorphism. For $f \in C_c^{\infty}(H_i)$ and $h \in H_i(F)$ regular, define the **orbital integral** by

$$(1.18) \quad \Phi(h,\ f,\ \omega_{H_i}, \omega_{H_i(h)}) = \Phi(h,\ f) := \int_{H_i(h,F)\backslash H_i(F)} f(x^{-1} hx) d\bar{x}.$$

This defines the orbital integrals which will occur in the fundamental lemmas below. Note that, whereas $\Phi(h,f)$ is a class function on $H_i(F)^{reg}$, $\Phi(g,\phi)$ may be regarded as a class function on the nonconnected reductive group

$$\bar{G} := G \times \{1,\ \sigma\},\ \text{(semi-direct product)},$$

because of the identity

$$(x \times \sigma)^{-1}(g \times \sigma)(x \times \sigma) = \sigma(x^{-1} g\sigma(x)) \times \sigma.$$

To analyse these integrals further we of course should know something about the conjugacy classes on H_i and on $\bar{G}$. Actually, from now on I shall pretty much ignore $\bar{G}$ since it seems easier to regard conjugacy classes on $\bar{G}$ as σ-conjugacy classes on G.

Let me leave aside the notion of σ-conjugacy for the moment and first discuss the (untwisted) notions of conjugacy and stable conjugacy. Two elements $h_1, h_2 \in SL(2,F) = H_1(F)$ can be **stably conjugate** (meaning that there is an $x \in H_1(\bar{F})$ such that $h_1 = x^{-1}h_2x$) but not conjugate (over F). This is not true for $PGL(n)$. In other words, if two elements of $PGL(n,F)$ are $PGL(n,\bar{F})$-conjugate then they are necessarily $PGL(n,F)$-conjugate (see, for example, [**11**, ch. 15, §3, p. 543]). Equivalently, for $PGL(n)$, the notions of stable conjugacy and (ordinary) conjugacy coincide. For groups other than $PGL(n)$, however, these notions do not generally coincide. As a measure of the difference between these two notions, one may consider the set of conjugacy classes of an element within a given stable conjugacy class of that element. For $h \in H_i(F)$, the set of $H_i(F)$-conjugacy classes of h_i within the stable conjugacy class of h is parameterized by

$$(1.19) \qquad \mathcal{D}(h,\ H_i) := H_{i,h}(\bar{F}) \setminus St(h, H_i)/H_i(F),$$

where

$$St(h,\ H_i) := \{x \in H_i(\bar{F}) \,|\, x^{-1}hx \in H_i(F)\}$$

denotes those elements over $\bar{F}$ which leave invariant the stable conjugacy class of $h \in H_i(F)$ under conjugation. If $i = 2$ then $\mathcal{D}(h, H_2)$ is a singleton since in that case the notions of stable conjugacy and ordinary conjugacy coincide.

Now let me turn to the twisted analogs of these notions. Although $PGL(n)$ has "no L-indistinguishability" it *does* have "twisted L-indistinguishability". In other works, it is possible for $g_1, g_2 \in G(F)$ to be stably σ-conjugate (meaning that there is some $g \in G(\bar{F})$ such that $g_1 = g^{-1}g_2\sigma(g)$) but not σ-conjugate over F. For $g \in G(F)$, the set of σ-conjugacy classes of g within the stable σ-conjugacy class of g is parameterized by

$$(1.20) \quad \mathcal{D}_\sigma(g, G) := G_{g\sigma}(F) \setminus St_\sigma(g)/H_i(F),$$

where

$$St_\sigma(g, G) := \{x \in G(\bar{F}) \mid x^{-1}gx \in G(F)\}$$

denotes those elements over $\bar{F}$ which leave invariant the stable σ-conjugacy class of g.

LEMMA 1.21. *There are isomorphisms (as pointed sets)*

$$\mathcal{D}(h, H_i) \cong H^1(F, H_i(h)),$$
$$\mathcal{D}_\sigma(g, G) \cong H^1(F, G(g\sigma))$$

For the proof of this see, for example, [**14**]. We will use this to compute the cardinality of these sets later.

For fixed $g \in G(F)$ and arbitrary $g' \in \mathcal{D}_\sigma(g, G)$, all the twisted centralizers $G(g'\sigma)$ are (well-defined and) isomorphic over $\bar{F}$. When considering the twisted orbital integrals $\Phi(g', \omega_{G(g'\sigma)}, \omega_G)$, $g' \in \mathcal{D}_\sigma(g, G)$, we will always assume that the measures $\omega_{G(g'\sigma)}$ are determined from the given $\omega_{G(g\sigma)}$ by pulling it back via some isomorphism $G(g'\sigma) \xrightarrow{\sim} G(g\sigma)$. A similar remark pertains to the orbital integrals attached to the $h' \in \mathcal{D}(h, H_i)$.

If $N_1(g)$ is regular, $g \in G(F)$ then define the **stable twisted orbital integral** of $\phi \in C_c^\infty(G)$ by

$$(1.22) \quad \Phi^{T,1}(G, \phi) := \sum_{g' \in \mathcal{D}_\sigma(g,G)} \Phi(g', \phi),$$

where T denotes $G(g\sigma)$. Similarly, if $h \in H_i(F)$ is regular then define the **stable orbital integral** of $f \in C_c^\infty(H_i)$ by

$$(1.23) \quad \Phi^{T_i,1}(h, f) := \sum_{h' \in \mathcal{D}(h,H_i)} \Phi(h', f),$$

where T_i denotes the centralizer $H_i(h)$.

FUNDAMENTAL LEMMA FOR H_1 (LANGLANDS). *If $\phi \in \mathcal{H}(G)$ and $f_1 = \lambda_1^*(\phi) \in \mathcal{H}(H_1)$ then, for all non-trivial $h \in N_1(g)$, we have*

$$(1.24) \quad \Phi^{T,1}(g, \phi) = \Phi^{T_1,1}(h, f_1).$$

We will prove this in §4 in the case where ϕ is the unit element of $\mathcal{H}(G)$ and $p \neq 2$.

The analogous statement for H_2 requires the introduction of "unstable" orbital integrals. These are the same as stable orbital integrals except that the sum is "twisted" by certain roots of unity (± 1's in our case). To define these, we use the property that the elements $g \in G(F)$ with $N_1(g) = 1$ have the property that $|\mathcal{D}_\sigma(g, G)| = 2$, and we can define a bijection of sets

$$\kappa : \mathcal{D}_\sigma(g, G) \longrightarrow \{\pm 1\}, \tag{1.25}$$

by

$$\kappa(g) := \begin{cases} +1, & \text{if } G(g\sigma) \text{ is split over } F, \\ -1, & \text{if } G(g\sigma) \text{ is non-split over } F \end{cases}.$$

If $N_1(g) \neq 1$ is regular then replace g by $g_s := (g + {}^t g)/2$ to get a regular element with $N_1(g_s) = 1$. In this case, define $\kappa(g) = \kappa(g_s)$. This depends only on the σ-conjugacy class of g. This sign is closely related to the Hilbert norm residue symbol of E/F, where E is the quadratic extension obtained from adjoining the eigenvalues of $h \in N_1(g)$ to F, see §4 below. The **unstable** or **κ-orbital integral** of $\phi \in C_c^\infty(G)$ is

$$\Phi^{T,\kappa}(g,\ \phi) := \sum_{g' \in \mathcal{D}_\sigma(g,G)} \kappa(g')\Phi(g',\ \phi), \tag{1.26}$$

where $T := G(g\sigma)$.

FUNDAMENTAL LEMMA FOR H_2. *(conjecture) Assume $N_2(g) \neq \pm 1$. If $\phi \in \mathcal{H}(G)$ and $f_2 := \lambda_2^*(\phi) \in \mathcal{H}(H_2)$ then, for any $h \in N_2(g)$, we have*

$$\Phi^{T,\kappa}(g,\ \phi) = \tau(g)\Phi(h,\ f_2), \tag{1.27}$$

where τ is the transfer factor

$$\tau(h) := \pm|(1+\beta_1)(1+\beta_2)|^{1/2},$$

denoting the eigenvalues of h by β_1, β_2. The $\pm$ sign will be explained in (3.18) *below.*

This main result of this paper is, as mentioned earlier, the proof that this holds when the residual characteristic of F is greater than 2 and $\phi \in \mathcal{H}(G)$ is the unit element.

2. The identities in the split case.

2.1. Summary. The object is to briefly sketch, following [**12**], a proof of the stable and unstable fundamental lemmas

$$(2.1) \qquad \Phi^{T,1}(g,\ \phi) = \Phi^{T_1,1}(h,\ f),$$

for $h \in N_1(g) \cap A_1(F)$ regular semi-simple, and

$$(2.2) \qquad \Phi^{T,\kappa}(g,\ \phi) = |(1+\beta_1)(1+\beta_2)|^{-1/2}\Phi(h',\ f),$$

for $h' \in N_2(g) \cap A_2(F)$ regular, semi-simple. Here $A_i(F)$ denotes the maximal split torus in $H_i(F)$; let $A(F)$ denote the maximal split torus of $G(F)$. The norms have been defined in such a way that, if $g \in A(F)$ then $N_i(g) \cap A_i(F) \neq \emptyset$. We use Satake transforms, a method which has the advantage of yielding, for the split groups we consider, an explicit expression for the absolute value of the transfer factor (the precise transfer factor is very difficult to predict in general, see the recent work of R. Langlands and D. Shelstad [**15**]).

2.2. Background on twisted integration formulas. All Haar measures in this subsection are normalized as in [**3**].

In this section, let $G' = SL(n)$ or $PGL(n)$ and consider the outer automorphism $\sigma(g) = J' \cdot {}^t g^{-1} \cdot J'$ where J' denotes the $n \times n$ skew-diagonal matrix with alternating ± 1's on the skew-diagonal:

$$J' := \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdot & & \\ & \cdot & & & \\ (-1)^{n-1} & & & & \end{pmatrix}$$

Let Ad denote the action of $G'(F)$ on Lie G' : $Ad(g)X := gXg^{-1}$. The Jacobian of the change-of-variables $n \longmapsto nmn^{-1}m^{-1} = \exp\,[(1 - Ad(m))\,\log n]$, where $\log n \in$ Lie N' is given by $\exp\log n := n$, is of course $\Delta(m) := |\det(Ad(m) - 1)_{\mathrm{Lie}\,N'}|_F$. Similarly, the Jacobian of $n \longmapsto nm\sigma(n)^{-1}m^{-1} = \exp\,[(1 - Ad(m)\sigma)\,\log n]$ is $\Delta_\sigma(m) := |\det\,(Ad(m)\sigma - 1)_{\mathrm{Lie}\,N'}|_F$.

To be concrete, if $\alpha_1, \ldots, \alpha_n$ are the simple roots of G' then σ acts on their associated root groups via $\sigma(\exp\,(tX_{\alpha_i})) = J'(\exp\,(tX_{\alpha_i}))^{-1}J' = (-1)^{n-1}\exp\,(-tJ'{}^tX_{\alpha_i}J)$, or equivalently, via the action of $\sigma \in Out(G') \cong Aut(\mathcal{D})$ on the Dynkin diagram $\mathcal{D}$ of G'. For $PGL(3)$, σ exchanges the two simple roots; on the level of the Lie algebra, it exchanges the root vectors $X_{\alpha_1}, X_{\alpha_2}$ and changes the sign of $X_{\alpha_1+\alpha_2}$ in Lie N' since $X_{\alpha_1+\alpha_2} = [X_{\alpha_1}, X_{\alpha_2}]$. Therefore the analog to

$$(2.3) \qquad \int_{N'} \phi(nmn^{-1}m^{-1})dn = \Delta(m)^{-1}\int_{N'} \phi(n)dn$$

becomes

$$\text{(2.4)} \qquad \int_{N'} \phi(nm\sigma(n)^{-1}m^{-1})dn = \Delta_\sigma(m)^{-1} \int_{N'} \phi(n)dn$$

Let $\alpha \in A'(F)$ belong to the maximal split torus of $G'(F)$ and let ϕ be a spherical function on $G'(F)$. As a straightforward application of (2.3-4), we have

$$\text{(2.5)} \qquad \int_{G'(g\sigma,F)\backslash G'(F)} \phi(x^{-1}g\sigma(x))d\bar{x} = \int_{G'(g\sigma,F)\backslash A'(F)} \Delta_\sigma(m)^{-1}\Delta(m) \times \int_{A'(F)\backslash G'(F)} \phi(x^{-1}mx)d\bar{x}d\bar{t},$$

where $m := t^{-1}a\sigma(t)$. Notice that the outer integral on the right is a finite sum. This fact will be used in a later subsection.

The remainder of this subsection is devoted to the calculation of the twisted Jacobian $\Delta_\sigma(m)$, when $G' = G = PGL(3)$. Write Lie $N = F \cdot X_{\alpha_1} \oplus F \cdot X_{\alpha_2} \oplus F \cdot X_{\alpha_1+\alpha_2}$ and identify its elements as column vectors. In terms of this basis, σ is represented by the matrix

$$\begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix}$$

and $Ad(a)$ is represented by the matrix

$$\begin{pmatrix} \alpha_1(a) & & \\ & \sigma_2(a) & \\ & & (\alpha_1+\alpha_2)(a) \end{pmatrix},$$

where of course $\alpha_1(a) = a_1/a_2, \alpha_2(a) = a_2/a_3, (\alpha_1+\alpha_2)(a) = a_1/a_3$. Combining this together, we find that det $(Ad(a)\sigma - 1)_{Lie\ N} = (a_1/a_3)^2 - 1$. In particular, with $a = m = t^{-1}g\sigma(t)\begin{pmatrix} \gamma_1 t_1^{-1}t_3^{-1} & & \\ & \gamma_2 t_2^{-2} & \\ & & \gamma_3 t_1^{-1}t_3^{-1} \end{pmatrix}$, where γ_i denote the eigenvalues of a representative of $g \in A(F)$, we find that

$$\text{(2.6)} \qquad \Delta_\sigma(m) = |1 - (\gamma_1/\gamma_3)^2|_F.$$

2.3. The Satake transform on $PGL(3)$, $SL(2)$, $PGL(2)$. There are, as we mentioned already in §1, L-maps

$$\lambda_1 : {}^LH_1(\mathbb{C}) \longrightarrow {}^LG(\mathbb{C}), \qquad {}^LH_1 = SO(3) \cong PGL(2)$$

$$\lambda_2 : {}^LH_2(\mathbb{C}) \longrightarrow {}^LG(\mathbb{C}), \qquad {}^LH_2 = SL(2),$$

given on the maximal split torus by

$$\lambda_1 : \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \longmapsto \begin{pmatrix} t_1 t_2^{-1} & & \\ & 1 & \\ & & t_2 t_1^{-1} \end{pmatrix},$$

$$\lambda_2 : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix}.$$

Define the **Satake transform** by

$$\hat{\phi}(t) := \sum_{\Lambda \in X^*({}^L A)} \alpha(\Lambda)\Lambda(t), \; t \in {}^L A(\mathbb{C}),$$

$$(2.7) \qquad \hat{f}_1(t) := \sum_{\mu \in X^*({}^L A_1)} \eta_1(\mu)\mu(t), \; t \in {}^L A_1(\mathbb{C}),$$

$$\hat{f}_2(t) := \sum_{\nu \in X^*({}^L A_2)} \eta_2(\nu)\nu(t), \; t \in {}^L A_2(\mathbb{C}).$$

Here

$$(2.8) \qquad \alpha(\Lambda) := \Delta_G(g) \int_{A(F)\backslash G(F)} \phi(x^{-1} g x) d\bar{x},$$

where Λ is that element $\Lambda = \Lambda(g) = \Lambda(k_1, k_2, k_3) \in X^*({}^L A) \cong \mathbb{Z}^3$ is that element corresponding to $(k_1, k_2, k_3) \in \mathbb{Z}^3$ for k_i given by

$$g = \begin{pmatrix} u_1 \pi^{k_1} & & \\ & u_2 \pi^{k_2} & \\ & & u_3 \pi^{k_3} \end{pmatrix}, \; u_i \in O_F^\times.$$

We also define $\Delta_G(g) := \prod_{\text{all roots } \alpha} |1 - \alpha(g)|^{1/2} = |1 - \gamma_1/\gamma_3||1 - \gamma_2/\gamma_3||1 - \gamma_1/\gamma_3||\gamma_3/\gamma_1|$. Similarly, for $H_1(F)$ we have

$$(2.9) \qquad \eta_1(\mu) := \Delta_{H_1}(h) \int_{A_1(F)\backslash H_1(F)} f_1(x^{-1} h x) d\bar{x},$$

where $\mu = \mu(h) = \mu(m) \in X^*({}^L H_1) \cong \mathbb{Z}$ corresponds to $h = \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} = \begin{pmatrix} u_1' \pi^m & \\ & u_2' \pi^{-m} \end{pmatrix}$, $u_i' \in O_F^\times$. Here we define $\Delta_{H_1}(h) := \prod_{\text{all roots } \alpha} |1 - \alpha(h)|^{1/2}$ $= |1 - \beta_1/\beta_2/\beta_2|\beta_1|^{1/2}$. For $H_2(F)$ we define $\Delta_{H_2}(h') := \prod_{\text{all roots } \alpha} |1 - \alpha(h')|^{1/2}$ $= |1 - \beta_1'||\beta_1'|^{-1/2}$, and

$$(2.10) \qquad \eta_2(\nu) := \Delta_{H_2}(h') \int_{A_2(F)\backslash H_2(F)} f_2(x^{-1} h' x) d\bar{x},$$

where $\nu = \nu(n_1, n_2) \in X^*({}^LH_2)$ corresponds to $h' = \begin{pmatrix} \lambda\beta_1' & \\ & \lambda \end{pmatrix} = \begin{pmatrix} u_1''\pi^{n_1} & \\ & u_2''\pi^{n_2} \end{pmatrix}$, where $u_i'' \in O_F^\times$. Here the pair $(n_1, n_2) \in \mathbb{Z}^2$ is not well-defined but $n_1 - n_2$ is.

Note that, via the induced map $\lambda_1^* : X^*({}^LA) \longrightarrow X^*({}^LA_1)$, two $\Lambda(k_1, k_2, k_3)$, $\Lambda(k_1', k_2', k_3')$ belonging to $X^*({}^LA)$ have the same image if and only if $k_1 - k_3 = k_1' - k_3'$. Similarly, via the induced map $\lambda_2^* : X^*({}^LA) \longrightarrow X^*({}^LA_2)$, two $\Lambda(k_1, k_2, k_3)$, $\Lambda(k_1', k_2', k_3') \in X^*({}^LA)$ have the same image if and only if $k_1 + k_2 = k_1' + k_2' = 2k_3 = 2k_3'$.

LEMMA 2.11. *(a) For f_1 and ϕ spherical, the following are equivalent:*

(i) $f_1 = \lambda_1^*\phi$,
(ii) $\hat{f}_1(t) = \hat{\phi}(\lambda_1(t)),\ t \in {}^LH_1$,
(iii)

$$\sum_{\substack{\Lambda=\Lambda(k_1,k_2,k_3)\\ k_1-k_3=m}} \alpha(\Lambda) = \eta_1(\mu),\ \forall\mu = \mu(m).$$

(b) For f_2 and ϕ spherical, the following are equivalent:

(i) $f_2 = \lambda_2^*\phi$,
(ii) $\hat{f}_2(t) = \hat{\phi}(\lambda_2(t)),\ t \in {}^LH_2$,
(iii)

$$\sum_{\substack{\Lambda=\Lambda(k_1,k_2,k_3)\\ k_1=n_1,k_2=n_2}} \alpha(\Lambda) = \eta_2(\nu),\ \forall\nu = \nu(n_1,\ n_2).$$

The proofs are omitted.

2.4. The fundamental identities in the split case. We may always represent $t \in A(F)$ by a diagonal matrix in $GL(3, F)$ whose middle entry t_2 is 1. This gives

$$t^{-1}\sigma(t) = \begin{pmatrix} t_1^{-1}t_3^{-1} & & \\ & 1 & \\ & & t_1^{-1}t_3^{-1} \end{pmatrix}.$$

Because of this, the right side of the expression in (2.5) is

$$\Delta_\sigma(m)^{-1} \sum_{t(\mathrm{mod}\, A\cap K)} \delta(gt^{-1}\sigma(t))\Phi(gt^{-1}\sigma(t))$$
$$= |\gamma_3/\gamma_1|^{-1}\Delta_\sigma(m)^{-1} \sum_{\substack{\Lambda'=\Lambda'(k,0,k)\in X^*({}^LA)\\ k\in\mathbf{Z}}} \alpha(\Lambda + \Lambda'),\ \Lambda = \Lambda(g),$$

since $\Delta_G(gt^{-1}\sigma(t))^{-1}\Delta(gt^{-1}\sigma(t)) = |\gamma_3/\gamma_1|^{-1}$. By the previous lemma, the left side of (2.5) is

$$\begin{aligned}&\Delta_\sigma(m)^{-1}|\gamma_3/\gamma_1|^{-1}\eta_1(\mu)\\ &\quad= \Delta_\sigma(m)^{-1}|\gamma_3/\gamma_1|^{-1}|1-\beta_1/\beta_2||\beta_2/\beta_1|^{1/2}\int_{A_1(F)\backslash H_1(F)} f_1(x^{-1}hx)d\bar{x},\end{aligned}$$

where $h = \begin{pmatrix}\gamma_1/\gamma_3 & \\ & \gamma_3/\gamma_1\end{pmatrix} \in N_1(g)$. Since $\beta_1/\beta_2 = \gamma_1^2/\gamma_3^2$, these equations combine to give, for $g \in A(F)$ and $h \in N_1(g) \cap H_1(F)$,

$$\int_{G(g\sigma,F)\backslash G(F)} \phi(x^{-1}g\sigma(x))d\bar{x} = \int_{H_1(h,F)\backslash H_1(F)} (\lambda_1^*\phi)(x^{-1}hx)d\bar{x}. \tag{2.12}$$

This proves the $SL(2)$-fundamental lemma in the split case since the stable conjugacy class of h contains only one element.

For the $SO(3)$-fundamental lemma, we have $h' = \begin{pmatrix}\gamma_1/\gamma_3 & & \\ & 1 & \\ & & \gamma_3/\gamma_1\end{pmatrix} \in N_1(g)$. (We identify $SO(3)$ with $PGL(2)$ via (1.3).) Due to the symmetry under the Weyl group, we can replace the sum

$$\sum_{\substack{M=M(k,0,k)\\ k\in Z}} \alpha(\Lambda+M) = \sum_{\substack{M=M(0,k,0)\\ k\in Z}} \alpha(\Lambda+M),$$

by a sum over the end coordinate: $(0,0,k)$. By lemma 2.11, this sum satisfies

$$\eta_2(\nu) = \sum_{\substack{M=M(0,0,k)\\ k\in Z}} \alpha(\Lambda+M).$$

A simple computation verifies that $\Delta_\sigma(m)^{-1}|\gamma_3/\gamma_1|^{-1}|1-\beta_1||\beta_1|^{-1/2} = |(1+\beta_1)(1+\beta_2)|^{-1/2}$. From this, (2.9), and the above expression for h', we have

$$\Phi(g,\ \phi) = |(1+\beta_1)(1+\beta_2)|^{-1/2}\Phi(h',\ \lambda_2^*\phi), \tag{2.13}$$

for $h' \in N_2(g) \cap H_2(F)$. This proves the $SO(3)$-fundamental lemma in the split case.

3. The reduction in the non-split case to buildings.

3.1. Summary. R. Langlands [**12**], [**14**] has shown how to exploit buildings to prove fundamental lemmas (see also R. Kottwitz [**8**], [**9**]). Following [**12**], this section is concerned with reducing the individual terms of the identities

$$\Phi^{T_1,1}(h,\ \lambda_1^*(\phi),\ \omega_h) = \Phi^{T,1}(g,\ \phi,\ \omega_g), \tag{3.1}$$

and

$$\Phi(h',\ \lambda_2^*(\phi),\ \omega_{h'}) = \pm|(1+\beta_1)(1+\beta_2)|^{1/2}\Phi^{T,\kappa}(g,\ \phi,\ \omega_g), \tag{3.2}$$

down to finite sums, in case ϕ is the characteristic function of a double coset KtK, where $t \in A(F)$ and $K := G(O_F)$. We will only by interested in the case $t = 1$ but the simple reduction below is just as easy to carry out for general t. For background, some references are [**14**] and [**8**].

Notation. Recall $H_1 := SL(2)$ and $H_2 := PGL(2)$. Let

$$Q(t) = Q(\Lambda) = Q(k_1,\ k_2,\ k_3) := KtK,$$

where $T \in A(F)$ is represented by a diagonal matrix with entries $t_i \in F^\times |t_i| = q^{k_i}$, and $\Lambda \in X^*({}^LA)$ is the character associated to the triple $(k_1,\ k_2,\ k_3)$ by duality. We assume that the entries have been ordered in such a way that $k_1 \geqq k_2 \geqq k_3$. Of course, the triple $(k_1, k_2, k_3) \in \mathbb{Z}^3$ is only well-defined up to translation by $(n, n, n), n \in \mathbb{Z}$. Let

$$\begin{aligned}
\phi_\Lambda &:= (meas\ Q(\Lambda))^{-1} char\ Q(\Lambda),\\
f_{1,\Lambda} &:= \lambda_1^*(\phi_\Lambda),\\
f_{2,\Lambda} &:= \lambda_2^*(\phi_\Lambda),\\
a(\Lambda) &:= \Phi^{T_1,1}(h,\ f_{1,\Lambda}),\\
b(\Lambda) &:= \Phi(h',\ f_{2,\Lambda}),\\
A(\Lambda) &:= \Phi^{T,1}(g,\ \phi_\Lambda),\\
B(\Lambda) &:= \pm|(1+\beta_1)(1+\beta_2)|^{1/2}\Phi^{T,\kappa}(g,\ \phi_\Lambda),\\
r(\Lambda) &:= meas\,(H_1(h,\ F) \cap H_1(O_F))Qq^{\lambda_1-\lambda_3}a(\Lambda),\\
s(\Lambda) &:= meas\,(H_2(h',\ F)^0 \cap H_2(O_F))Qq^{\lambda_1-\lambda_3}b(\Lambda),\\
R(\Lambda) &:= meas\,(G(g\sigma,\ F) \cap G(O_F))Qq^{\lambda_1-\lambda_3}A(\Lambda),\\
S(\Lambda) &:= meas\,(G(g\sigma,\ F) \cap G(O_F))Qq^{\lambda_1-\lambda_3}B(\Lambda),
\end{aligned}$$

where $Q := (1+q^{-1})(1+q^{-1}+q^{-2})$ and where the $\pm$ sign will be explained in (3.18) below. In terms of the above notation, the object of section 5 of this paper is to prove that $S(0) = s(0)$.

3.2. The reduction. To reduce $a(\Lambda)$ and $b(\Lambda)$ down to finite sums, one may use Macdonald's formula [**16**] and Plancherel's formula for the Satake transform, following [**12**]. The only special case of this we shall need is the following

LEMMA 3.3. *Suppose that both $h \in H_1(F)$ and $h' \in H_2(F)$ are elliptic regular. In the notation above, we have*

$$r(0) = \begin{cases} Q|Fix\,(h)|, & \text{if } E = F(\beta_1) \text{ is unramified,} \\ 2^{-1}Q|Fix\,(h)|, & \text{if } E \text{ is ramified,} \end{cases}$$

and

$$s(0) = \begin{cases} r(0), & \text{if } \Delta(h) = \Delta(h'), \\ 0, & \text{if } h' \text{ has no fixed points,} \\ Q, & \text{otherwise.} \end{cases}$$

I shall indicate two different proofs of the first part of (3.3) (the first is Lemma 3.4b and the second is (3.12-14) below).

The proof of the following lemma, due to R. Langlands, provides an interesting application of the Macdonald and Weyl character formulas to the computation of the number of fixed points in the unramified case.

LEMMA 3.4. *First, assume that E/F is unramified, where $E := F[h] = F[h']$ denotes the splitting field of the torus $T_1 := H_{1,h}$ determined by h.*

(a) If $f_1 \in \mathcal{H}(SL(2)), h \in GL(2, O_F)$ then

$$\Phi^{T_1,1}(h, f_1) = \frac{1}{meas\ T(O_F)} \int_{{}^LS(\mathbb{C})} \hat{f}_1(s)\, \frac{q+1}{q-1} \left[\frac{\Delta(h)^{-1}}{q} - \frac{1}{q}\,|c(s)|^{-2}\right] ds,$$

where the notation is explained below.

(b) If $f_1 = f_{1,(0,0,0)}$ is the characteristic function of $SL(2, O_F)$ divided by the measure of $SL(2, O_F)$ then

$$\Phi^{T_1,1}(h, f_1) = Q|Fix\,(h)|.$$

Notation. Here T_1 denotes the centralizer of h in H_1 and ${}^LS(\mathbb{C})$ is the maximal compact subgroup of ${}^LH_1(\mathbb{C})$ consisting of all $s \in {}^LH_1(\mathbb{C})$ with eigenvalues a, b having the same absolute value. The Haar measure of ${}^LS(\mathbb{C})$ is normalized so that its total volume is 1. Also, here

$$c(s) := \frac{1 - q^{-1}ab^{-1}}{1 - ab^{-1}}$$

is the p-adic analog of the Harish–Chandra c-function for $SL(2)$ ([**16**, p. 51]), so that the Plancherel measure is given by

$$\frac{1+q^{-1}}{2}\,|c(s)|^{-2} ds$$

(see [**16**, p. 65], [**14**, p. 46]).

Proof of 3.4. (a) Substitute Lemmas 5.3 and 5.5 into Lemma 5.6 of [**14**], observing that $\Phi^{T_1,1}(h,f_1) = \Delta(h)^{-1}F(h,f_1)$ (in the notation of [**14**, p. 48]).
(b) We want to calculate the Satake transform of $f_{1,\Lambda}$, $\Lambda = (k_1, k_2, k_3) \in \mathbb{Z}^3, k_1 \geqq k_2 \geqq k_3$, in the case where $\Lambda = (0,0,0)$. By Macdonald's formula [**16**, p. 52],

$$\hat{f}_{1,\Lambda}(s) = \mathcal{S}\left(s_1^{k_1}s_2^{k_2}s_3^{k_3}\prod_{\alpha>0}\frac{1-q^{-1}\alpha(s)^{-1}}{1-\alpha(s)^{-1}}\right)\cdot Q^{-1}q^{k_3-k_1}, \tag{3.5}$$

where the product runs over all positive roots of $sl(3)$ and $\mathcal{S}$ is the **symmetry operator** on polynomials in the s_i defined by

$$\mathcal{S}(P(s_1,\ s_2,\ s_3)) := \sum_{\sigma\in S_3} P(s_{\sigma(1)},\ s_{\sigma(2)},\ s_{\sigma(3)}), \tag{3.6}$$

where S_3 denotes the symmetric group on three letters. Let $\mathcal{A}$ denote the **anti-symmetry operator** defined by

$$\mathcal{A}(P(s_1,\ s_2,\ s_3)) := \sum_{\sigma\in S_3}(sgn\ \sigma)P(s_{\sigma(1)},\ s_{\sigma(2)},\ s_{\sigma(3)}), \tag{3.7}$$

so that

$$\mathcal{A}(\sigma P) = (sgn\ \sigma)\mathcal{A}(P), \quad \sigma \in S_3. \tag{3.8}$$

Let ρ denote half the sum of the positive roots. Observe that the **Weyl function**,

$$\begin{aligned} q(s) &:= \rho(s)\prod_{\alpha>0}(1-\alpha(s)^{-1}) \\ &= s_1s_3^{-1}(s_1^{-1}s_2)(1-s_2^{-1}s_3)(1-s_1^{-1}s_3), \end{aligned} \tag{3.9}$$

is anti-symmetric in the sense of (3.8) and, by the Weyl character formula,

$$\begin{aligned} \mathcal{S}\left(\prod_{\alpha>0}\frac{1-q^{-1}\alpha(s)^{-1}}{1-\alpha(s)^{-1}}\right) &= \frac{\mathcal{A}(\rho(s)\prod_{\alpha>0}(1-q^{-1}\alpha(s)^{-1}))}{\rho(s)\prod_{\alpha>0}(1-\alpha(s)^{-1})} \\ &= Q\cdot \mathrm{tr}\,(r_{trivial}) \\ &= Q, \end{aligned} \tag{3.10}$$

where $r_{trivial}$ is the trivial representation on ${}^LH_1$. (In fact, for arbitrary Λ, $\hat{f}_\Lambda$ can be expressed as a linear combination of characters of finite dimensional representations of ${}^LH$.) The result now follows from (3.12) (or Lemma 4.5) below and Plancherel's formula [**14**, Lemma 5.4]. Q.E.D.

For the moment, E/F need not be unramified. Let $h \in SL(2,F)$ be non-split, so E/F is quadratic and $T := H_1(h) = H_1(h)^0$ is a non-split Cartan. We have

$$(3.11) \quad \int_{T(F)\backslash H_1(F)} f_{1,(0,0,0)}(x^{-1}hx)d\bar{x} = \frac{1}{meas\ T(F)} \int_{H_1(F)} f_{1,(0,0,0)}(x^{-1}hx)d\bar{x}$$
$$= \frac{1}{meas\ T(F)} |\mathrm{Fix}_0(h)|,$$

where, in the notation of [**9**],

$$\mathrm{Fix}_0(h) := \{x \in X_{H_1}(0) | hx = x\}.$$

If $h^\delta \in \mathcal{D}(h)$, T^δ the associated Cartan, for some $\delta \in GL(2,F)$, then

$$\int_{T^\delta(F)\backslash H_1(F)} f_{1,(0,0,0)}(x^{-1}h^\delta x)d\bar{x} = \frac{1}{meas\ T(F)} |\mathrm{Fix}_0(h^\delta)|,$$

since the Haar measures on T and T^δ have been chosen so that $meas\ T(F) = meas\ T^\delta(F)$ (we have $T^\delta \cong T$ over $\bar{F}$ and the Haar measure on T^δ is defined by the pull-back of that on T).

The following facts (which can be found in [**1**, ch. 1]), play a role here: if h splits over a ramified extension, then the fixed points of h (or of h^δ) in the tree of $H_1(F)$ are those less than a certain distance from a certain point in the first barycentric subdivision of $X_{H_1(F)}$; if h splits over an unramified extension then the fixed points are those less than a certain distance from a certain vertex in $X_{H_1(F)}$. In the ramified case the fixed points occur in pairs, one in $X(0)$ and the other in $X(1)$. In particular, if T is ramified then $|\mathrm{Fix}_0(h)| = |\mathrm{Fix}_0(h^\delta)|$ and $|\mathrm{Fix}_1(h)| = |\mathrm{Fix}_1(h^\delta)|$. Writing

$$\mathrm{Fix}\ (h) = \mathrm{Fix}_0(h) \cup \mathrm{Fix}_1(h), \quad \text{(disjoint union)},$$

where $\mathrm{Fix}_i(h) := X_{H_1(F)}(i) \cap \mathrm{Fix}\ (h)$, we have in the unramified case

$$|\mathrm{Fix}_0(h)| = |\mathrm{Fix}_1(h^\delta)|, \qquad |\mathrm{Fix}_0(h^\delta)| = |\mathrm{Fix}_1(h)|,$$

and therefore, by (1.23),

$$(3.12) \quad \Phi^{T,1}(h, f_{1,(0,0,0)}) = \frac{1}{meas\ T(F)} |\mathrm{Fix}\ (h)|.$$

A similar result holds for $\Phi_{H_2}(h', f_{2,(0,0,0)})$, where $h' \in N_2(g) \cap H_2(F)$.

Suppose $|\det h| = |\pi|^{2n}$, for some $n \in \mathbb{Z}$. This is necessary for $\mathrm{Fix}\ (h) \neq \emptyset$. If E/F is ramified then the orbits on X_{H_1} under $T(F)$ are twice as large as the orbits under $T(F) \cap H_1(O_F)$, so that

$$(3.13) \quad meas\ T(F) = 2 \cdot meas\,(T(F) \cap H_1(O_F))$$

(see [**14**, p. 52]); if E/F is unramified then

$$(3.14)\qquad meas\ T(F) = meas\,(T(F) \cap H_1(O_F)).$$

These follow from the classification of the Cartan subgroups of $H_1(F)$ in terms of the extensions of F. Recall that Haar measures have already been choosen so that

$$meas\,(G(g\sigma,\ F) \cap G(O_F)) = meas\,(H_1(h,\ F) \cap H_1(O_F)).$$

This proves the first part of (3.3). The second part of (3.3) follows along similar lines.

Langlands' proof that $R(\Lambda) = r(\Lambda)$ uses the identity (actually a facsimile thereof, given below)

$$(3.15)\qquad R(\Lambda) = Q(\Lambda)q^{k_3-k_1} \sum_{g' \in \mathcal{D}(g)} \ \sum_{\substack{P \\ Inv(g'\sigma(P),P)=\Lambda}} 1,$$

where

$$Q(\Lambda) = \begin{cases} 1, & k_1 > k_2 > k_3, \\ 1 + q^{-1}, & k_1 = k_2, \text{ or } k_2 = k_3, \text{ but } k_1 \neq k_3, \\ Q, & \Lambda = (0,0,0), \end{cases}$$

and the inner sum runs over the vertices in the building for $G(F)$ (the inner sum is independent of the representative g' choosen for the σ-conjugacy class). In case $N_1(g)$ has distinct eigenvalues not in F, by explicitly analyzing the proof of Lemma 1.21 one can show that, as sets,

$$(3.16)\qquad \mathcal{D}_\sigma(g) \cong H^1(F,\ G(g\sigma)) \cong F^\times/N_{E/F}(E^\times),$$

where E denotes the quadratic extension determined by the eigenvalues of $N_1(g)$. Very briefly, one uses (3.16) to parameterize $\mathcal{D}_\sigma(g)$ in terms of certain representatives $g_a, a \in F^\times$. One can show that this parameterization satisfies the following property: two such g_a, g_b belong to the same class in $\mathcal{D}_\sigma(g)$ if and only if $a \cdot N_{E/F}(E^\times) = b \cdot N_{E/F}(E^\times)$. Rather than (3.15), it is actually necessary to use the more general identity

$$(3.17)\qquad R(\Lambda) = \frac{2}{[F^\times : U]}\, Q(\Lambda)q^{k_3-k_1} \sum_{a \in F^\times/U} \ \sum_{\substack{P \\ Inv(g_a\sigma(P),P)=\Lambda}} 1,$$

where $U \subseteq N_{E/F}(E^\times) \subseteq F^\times$ is a compact open subgroup of finite index. Similarly, we have

$$(3.18)\quad S(\Lambda) = \frac{\pm 2}{[F^\times : U]} |(1+\beta_1)(1+\beta_2)|^{1/2} Q(\Lambda) q^{k_3-k_1} \times \sum_{a\in F^\times/U} \sum_{\substack{P \\ Inv(g_a\sigma(P),P)=\Lambda}} \kappa(g_a),$$

where $\kappa : \mathcal{D}_\sigma(g) \longrightarrow \{\pm 1\}$ is defined to be +1 on the element of $H^1(F, G(g\sigma))$ corresponding to the split inner form of $G(g\sigma))$ and to be -1 on the other element. The $\pm$ sign is choosen so that $\pm\kappa(g_a) = (a, \epsilon)_2$, where $E = F(\sqrt{\epsilon})$ and $(,)_2$ denotes the quadratic Hilbert symbol attached to F. The sign κ satisfies the following property:

Fact 3.19. *For any $g', g'' \in G(F)$, we have $\kappa(g') = \kappa(g'')$ if and only if g' and g'' belong to the same class in $\mathcal{D}_\sigma(g)$.*

The expressions (3.17), (3.18), and those in Lemma 3.3 constitute the desired reductions.

4. The Buildings for *PGL*(3) and *SL*(2) For background, see for example [**8**] and [**18**]. The results (and pictures) in this section are, as mentioned earlier, due to R. Langlands [**12**]. We assume $p \neq 2$ throughout this section.

Fix a vector space V over F of dimension either 2 or 3. We may consider the buildings $\mathcal{B}(G)$, $\mathcal{B}(H_1)$ for G, H_1 as graphs of lattice classes; if L is a rank two or three O_F-lattice in V then its class is denoted by $[L]$. Recall that two vertices $[L_1]$, $[L_2]$ are to be joined by an **edge** if and only if there exist $\lambda_1, \lambda_2 \in F^\times$ such that

$$(4.1)\quad \pi\lambda_1 L_1 \subseteq \lambda_2 L_2 \subseteq \lambda_1 L_1.$$

The edge from $[L_2]$ to $[L_1]$ is **positively directed** if (4.1) holds and if $\lambda_1 L_1/\lambda_2 L_2$ is a module of rank one over the residue field. Three vertices L_1, L_2, L_3 are the vertices of a 2-simplex or **chamber** of $\mathcal{B}(G)$ if and only if there exists $\lambda_1, \lambda_2, \lambda_3 \in F^\times$ such that

$$(4.2)\quad \pi\lambda_1 L_1 \subseteq \lambda_3 L_3 \subseteq \lambda_2 L_2 \subseteq \lambda_1 L_1.$$

To describe the action of G on $\mathcal{B}(G)$ one must first represent $g \in G(F)$ by a matrix $A \in GL(3, F)$. The point is that the definition of the lattice class implies that the center acts trivially, so the action of $g = A \cdot Z(F)$ is well-defined, where $Z(F)$ denotes the center of $G(F)$. Explicitly, the action of $G(F)$ on $\mathcal{B}(G)$ is given by $g : [L] \longmapsto [AL]$. The action of H_1 on $\mathcal{B}(H_1)$ is simpler: $h \in H_1(F)$ sends $[\Lambda]$ to $[h\Lambda]$. These actions send vertices to vertices, edges to edges, chambers to chambers (in the case of $\mathcal{B}(G)$), preserve orientation and are automorphisms of the buildings.

The action of σ on $\mathcal{B}(G)$, however, does not preserve orientation. This "anti-automorphism" sends a lattice L to its dual lattice

(4.3) $$\check{L} := \{v \in V \,|\, {}^t v \cdot v' \in O_F,\ \forall v' \in L\}.$$

If $L = g\Lambda_0$, where Λ_0 denotes the lattice class of O_F^3, then this means

$$\sigma : g\Lambda_0 \longmapsto {}^t g^{-1}\Lambda_0.$$

The composition $g\sigma$ acts by sending $[L]$ to $[A\check{L}]$. It is the fixed point set of $g\sigma$ in $\mathcal{B}(G)$ that we want to describe.

LEMMA 4.4. *(a) Suppose that the number of fixed points of $h \in H_1(F)$ in $\mathcal{B}(H_1)$ is finite: $|Fix\,(h)| < \infty$. (This is the case if h is elliptic regular, i.e., the eigenvalues of h generate a quadratic extension of F.) Then, for any $x \in H_1(F)$, we have $|Fix\,(h)| = |Fix\,(x^{-1}hx)|$, i.e., the number of fixed points depends only on the conjugacy class of h.*
(b) Suppose that the number of fixed points of $g\sigma$ in $\mathcal{B}(G)$ is finite: $|Fix\,(g\sigma)| < \infty$. (This is the case if $N_1(g)$ is elliptic regular.) Then, for any $x \in PGL(3, F)$, we have $|Fix\,(g\sigma)| = |Fix\,((x^{-1}g\sigma(x))\sigma)|$, i.e., the number of fixed points of $g\sigma$ depends only on the σ-conjugacy class of g.

Note. The proof of this lemma is easy, hence is omitted. If h, h' are stably conjugate but not conjugate over F one may have $|\text{Fix}\,(h)| \neq |\text{Fix}\,(h')|$, and a similar statement holds for σ-conjugacy.

The following result is well-known.

LEMMA 4.5. *Let $h \in GL(2, F)$ be an elliptic regular element with $\Delta_{H_2}(h) = q^{-k}$, for some $k \geqq 0$. Let E/F denote the quadratic extension generated by h.*

(1) If E/F is unramified then

$$|Fix\,(h)| = \frac{q^{k+1} + q^k - 2}{q - 1}.$$

(2) If E/F is ramified then

$$|Fix\,(h)| = \frac{2(q^k - 1)}{q - 1}.$$

(We have used the fact that the conductor of E/F, as defined in **[1]***, is* 1 *in the ramified case.)*

We now describe an embedding

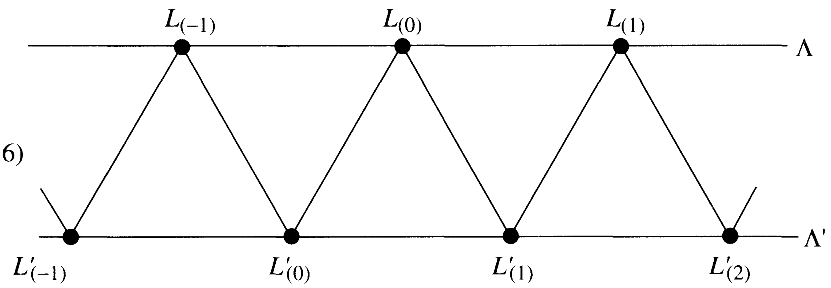
$$\mathcal{B}(H_1) \hookrightarrow \mathcal{B}(G).$$

First, replace the tree $\mathcal{B}(H_1)$ by the product of itself with the affine line: replace all vertices by lines (copies of V_A running perpendicular to the edges, as visualized in three dimensions) and all edges by strips. This figure may be regarded as $\mathcal{B}(GL(2))$. In order to embed this into $\mathcal{B}(G)$, one must impose a simplicial structure on it compatible with that of $\mathcal{B}(G)$. Each line $\bar{\Lambda}$ of $\mathcal{B}(GL(2))$ associated to a vertex Λ of $\mathcal{B}(H_1)$ must first be provided with its own vertices. We define the vertices on $\bar{\Lambda}$ to be $\Lambda_{(n)} := \pi^{-n}O_F + \Lambda$, for fixed embeddings $\pi^{-n}O_F \hookrightarrow V_A, \Lambda \hookrightarrow W_A$. Here V_A, W_A are as in subsection 1.3. Given two neighboring vertices Λ, Λ' of $\mathcal{B}(H_1)$ we obtain neighboring lines $\bar{\Lambda}, \bar{\Lambda}'$ and the vertices of these lines are joined by an edge if they are neighbors. This simplicial structure on $\mathcal{B}(GL(2))$ defines an embedding of $\mathcal{B}(H_1)$ into $\mathcal{B}(G)$ whose image we denote by $\mathcal{Z} = \mathcal{Z}(V_A, W_A)$. Let

$$\pi\lambda\Lambda \subseteq \lambda'\Lambda' \subseteq \lambda\Lambda,$$

and let $L'_{(n)} := \pi^{-n}O_F + \Lambda', L_{(n)} := \pi^{-n}O_F + \Lambda$. The band in $\mathcal{Z}$ associated to the edge joining Λ and Λ' in $\mathcal{B}(H_1)$ is represented by a picture of the following sort:

(4.6)



The lines in $\mathcal{Z}$ have been drawn in such a way that the positive direction on them is from left to right. As mentioned earlier, the action of $\mathrm{Aut}_F(W_A) \hookrightarrow \mathrm{Aut}_F(W_A + V_A)$ on $\mathcal{Z}$ preserves orientation. The action of σ on $\mathcal{B}(G)$ does not even restrict to an action on $\mathcal{Z} = \mathcal{Z}(V_A, W_A)$. However, the action of $g\sigma$, with $g = [A] \in G(F)$ represented by $A \in GL(3, F)$, does send $\mathcal{Z}(V_A, W_A)$ to $\mathcal{Z}(AW_A^{\perp}, AV_A^{\perp}) = \mathcal{Z}(V_A, W_A)$, reversing orientation. Here we have used the fact the $AW_A^{\perp} = V_A$ and $AV_A^{\perp} = ({}^tA^{-1}V_A)^{\perp}$, by definition of W_A. As we shall see, thanks to Corollary (1.14) it turns out to be sufficient for our purpose of relating fixed points of $g\sigma$ in $\mathcal{B}(G)$ with fixed points of $h \in N_1(g)$, $h \in H_1(F)$, in $\mathcal{B}(H_1)$ to assume that the matrix A in $GL(3, F) \cong \mathrm{Aut}_F(V_A + W_A)$ is in (1,2)-block form.

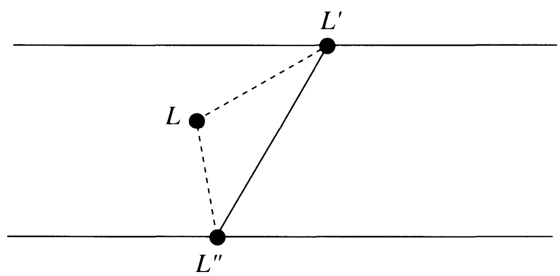
For such matrices, it is possible to explicitly realize the action of $g\sigma$ on $\mathcal{Z}$ in terms of the geometry described above. In fact, this is what we do next.

In [**12**], R. Langlands introduced the notion of a "characteristic leaf" to understand explicitly vertices in $\mathcal{B}(G)$ not in $\mathcal{Z}$ in terms of vertices in $\mathcal{Z}$. This notion is very useful for relating fixed points of $g\sigma$ in $\mathcal{B}(G)$ to fixed points of $h \in N_1(g)$ in $\mathcal{B}(H_1)$. Given $L := O_F + \pi^b O_F + \pi^n O_F$, let $M_1 := \pi^{-b} O_F, M_2 := \pi^b O_F, M_3 := \pi^n O_F$. Consider the vertices of the **characteristic leaf** for L:

$$\begin{aligned} L' &:= M_1 + M_2 + M_3 \in \mathcal{Z}, \\ L &:= \pi^b M_1 + M_2 + M_3 \notin \mathcal{Z}, \qquad (4.7) \\ L'' &:= \pi^b M_1 + \pi^b M_2 + M_3 \in \mathcal{Z}, \end{aligned}$$

depicted

(4.8)



The vertices of the simplices forming the equilateral triangle are given by $\pi^{n_1} M_1 + \pi^{n_2} M_2 + M_3, 0 \leqq n_2 \leqq n_1 \leqq b$. The segment of length $|b|$ joining L' to L'' is called the **characteristic base** or **segment**. This base degenerates to a vertex if $L \in \mathcal{Z}$ and otherwise it may be regarded as an equilateral triangle, under a suitable metric structure on $\mathcal{B}(G)$.

LEMMA 4.9 ([**12**]). *Suppose $g \in G(F)$ is such that $N_1(g)$ is elliptic regular. Then the following hold:*

(a) $g\sigma$ sends the characteristic leaf of L to the characteristic leaf of $g\check{L}$;

(b) For each vertex Λ in $\mathcal{B}(H_1)$, $g\sigma$ sends the line $\{\pi^{-n} O_F + \Lambda | n \in \mathbb{Z}\}$ of $\mathcal{Z}$ with positive orientation to another line $\{\pi^m O_F + \Lambda' | m \in \mathbb{Z}\}$ in $\mathcal{Z}$ with reversed orientation;

(c) $g\sigma$ sends the characteristic base of the leaf for L to the characteristic base (with orientation reversed) for $g\check{L}$;

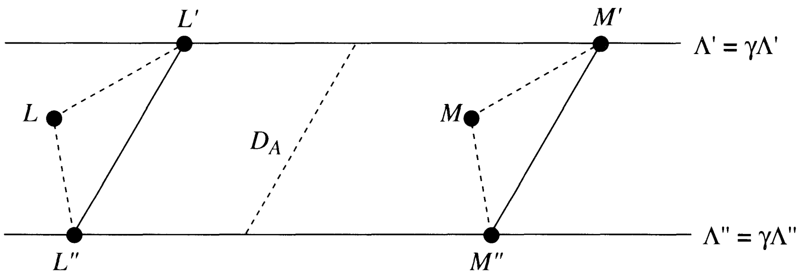
(d) for any $h \in SL(2,F)$ representing $N_1(g)$, the action of h on $\mathcal{Z}$ is the same as the action of $(g\sigma)^2$ restricted to $\mathcal{Z}$.

Remark. The $h \in H_1(F)$ in (d) is fixed for the rest of this section.

Proof. We may assume by Corollary 1.14 that g is represented by a (2,1)-block matrix $A \in GL(3, F)$ in the basis giving the embedding $\mathcal{Z}$. From this the four statements (a)–(d) follow without difficulty from simple facts about buildings. Q.E.D.

In particular, (d) implies that if $\overline{\Lambda'}, \overline{\Lambda''}$ are lines associated to two fixed points Λ', Λ'' of $h \in N_1(g)$ and if the endpoints L', L'' of the vertices of the characteristic base lie on $\overline{\Lambda'}, \overline{\Lambda''}$, respectively, then $g\sigma$ "flips" L and its leaf about a certain "dotted" line D_A (which depends only on A, not on L, L', L'', and lies in the first barycentric subdivision of $\mathcal{B}(G)$ but does not necessarily lie in $\mathcal{B}(G)$):

(4.10)



If $N_1(g)$ does not contain a unipotent then by Corollary (1.14) we may assume that the matrix $A \in GL(3, F) \cong \mathrm{Aut}_F(V_{A'} + W_{A'})$ representing $g = [A]$ is in block form:

(4.11) $\quad A = \begin{pmatrix} u\pi^c & \\ & C \end{pmatrix}, \; C \in GL(2,\ F),\ u \in O_F^\times,\ c \in \mathbb{Z}.$

(Here $[A']$ is the matrix in (1.14) which belongs to the same stable σ-conjugacy class as $[A]$.) In this case, $A\sigma$ sends the lattice $\pi^{-n}O_F + \Lambda$ in $\mathcal{Z}$ to $\pi^{c+n}O_F + \Lambda'$, for some Λ'. If we parameterize the vertices in the lines (4.8) by the integers then, in a sense, D_A may be thought of as the "$c/2$-line". It belongs to $\mathcal{B}(G)$ if and only if c is even. Suppose that c is even and that

$$(4.12) \qquad \begin{aligned} A\sigma &: [\pi^{-n}O_F + \Lambda] \longmapsto [\pi^{c+n}O_F + \Lambda'], \\ A\sigma &: [\pi^{-n}O_F + \Lambda'] \longmapsto [\pi^{c+n}O_F + \Lambda]. \end{aligned}$$

It can be shown without much effort that, in this case, $[\Lambda], [\Lambda']$ in $\mathcal{Z}$ correspond to two fixed points of $h \in N_1(g)$ in $\mathcal{B}(H_1)$. Also, it is not hard to see that,

FACT 4.13. *Suppose c is even and L is a fixed point of $[A]\sigma$ whose characteristic base has endpoints lying on the lines $\overline{\Lambda'} \neq \overline{\Lambda''}$. Then*

(a) D_A intersects the lines $\overline{\Lambda'}, \overline{\Lambda''}$ at the two vertices which determine the endpoints of the characteristic base of L and

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(b) the lines $\overline{\Lambda'}, \overline{\Lambda''}$ *in* $\mathcal{Z}$ *correspond to two fixed points of* h *in* $\mathcal{B}(H_1)$.

Remark. There is a simpler version of this dealing with the case $\overline{\Lambda'} = \overline{\Lambda''}$. In this way, we will obtain a "2-1" correspondence, when c is even, between *pairs* of fixed of H in $\mathcal{B}(H_1)$ and fixed points of $[A]\sigma$ in $\mathcal{B}(G)$. One important point to bear in mind is that when c is *odd* then the line D_A cannot intersect $\mathcal{B}(G)$ and therefore there can be *no* fixed points of $[A]\sigma$ in this case. So far we have only shown that (when c is even) given a fixed point of $[A]$ in $\mathcal{B}(G)$ there are associated two fixed points of h in $\mathcal{B}(H_1)$.

From the reduction of §3 one sees that it is also necessary to know what happens when A in (4.11) is replaced by a matrix A' stably σ-conjugate to A but not σ-conjugate to A over F. Of course, $N_1([A]) = N_1([A'])$ so A' may be put in (1,2)-block form as in (4.11), for some $c' \in \mathbb{Z}, C' \in GL(2,F), u' \in O_F^\times$. If moreover the eigenvalues of h determine an unramified extension of F then c' and c have different parity. On the other hand, if A' and A are σ-conjugate over F then c and c' do have the same parity, so

FACT 4.14. *The parity of* c *depends only on the stable* σ*-conjugacy class of* $[A]$. *If the eigenvalues of* h *determine an unramified extension of* F *(the "unramified case") and* A, A' *are stably* σ*-conjugate but not* σ*-conjugate over* F *then either* c *or* c' *is even (but not both).*

By the previous paragraph, in this "unramified case", there is either a fixed point of $A\sigma$ (and $A'\sigma$ has no fixed point) or a fixed point of $A'\sigma$ (and $A\sigma$ has no fixed point).

The problem now is to deal with the converse of (4.13): given a pair of fixed points of $h \in N_1(g)$, associate to them fixed point of $A\sigma$. Instead of proving this, we shall prove that this holds "on average", which is sufficient for our purposes. For this the following lemma is crucial, but first we need some notation: Let d be the maximum distance between two fixed points of h, let the eigenvalues of h determine a quadratic extension E of F, and suppose that

$$(4.15)\qquad U(d) := U_d\pi^{2\mathbb{Z}} := \{x \in O_F^\times \mid x \equiv 1 \pmod{\pi^d}\}\pi^{2\mathbb{Z}} \subseteq N_{E/F}(E^\times)$$

is finite index in $F^\times$. Let $\mathcal{S}$ denote the set of $a \in F^\times$ for which $P' \neq P''$ exist (for a fixed p', p'') and satisfying (ai)–(aiv) in the lemma below. Here A_a is defined to be a representative of g_a, where g_a is defined in the paragraph following (3.16) and A is our representative of g choosen at the beginning of this section. We may, by (1.14), choose all the A_a to be in block form as in (4.11). $A_a\sigma$ may have no fixed points; however, if it does, let P_0 denote one of them. Let U_n denote the stabilizer of the action of $T_a := (A_a\sigma)(A\sigma)^{-1}$ on P_0, for some $n \geqq 0$, with U_n as in (4.15).

LEMMA 4.16 ([**12**, LEMMA 5.1]). *Suppose that* p', p'' *are vertices in* $\mathcal{B}(H_1)$

(possibly $p' = p''$) and that $[A]\sigma$ "flips" the lines in $\mathcal{B}(G)$ associated to p', p'' onto each other. Then:

(a) For each $a \in F^\times$ there exists at most one pair P', P'' of vertices in $\mathcal{B}(G)$ with

(i) P' lying on the line associated to p',

(ii) P'' lying on the line associated to p'',

(iii) in the notation of §3, $A_a\sigma(P') = P'', A_a\sigma(P'') = P'$, and

(iv) P', P'' form the extreme vertices of a characteristic segment.

(b) Suppose $a \in F^\times$ is such that a pair p', p'' exists for A_a. Then a pair exists for A_b (for the same p', p'') if and only if $aN_{E/F}(E^\times) = bN_{E/F}(E^\times)$.

(c) Let S, P_0, and U_n be as above, and assume that $n \geqq 1$. Then $n \leqq d$ and there are $[O_F^\times : U_n]$ distinct P's with segment (P', P'') such that $(A_a\sigma)P = P$.

(d) If $|\mathrm{Fix}\,(A_a\sigma)| \neq 0$ then $|\mathrm{Fix}\,(A_{a\pi}\sigma)| = 0$, for E/F unramified.

(e) In the notation of part (c),

$$\frac{1}{[O_F^\times : U_d]} \sum_{a \in F^\times/U(d)} \sum_{\substack{P \\ A_a\sigma P = P \\ P \text{ has segment } (P', P'')}} 1$$

$$= \frac{1}{[O_F^\times : U_n]} \sum_{a \in F^\times/U(n)} \sum_{\substack{P \\ A_a\sigma P = P \\ P \text{ has segment } (P', P'')}} 1.$$

There is an obvious "κ-analog" (see (3.18)) of the above equation whose statement is left to the reader.

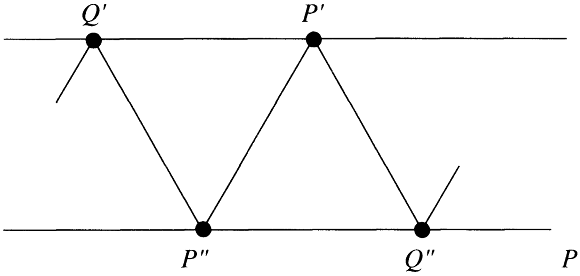
Remark. (a) When applying this lemma to (3.17) or (3.18), one uses the fact that

$$\frac{1}{2}\,[F^\times : U(n)] = [O_F^\times : U_n].$$

(b) For the situation in (4.13c) when $n = 0$, see §5. The "matching" is then $P = P' = P'' \longmapsto p' = p''$.

Let me now sketch the proof of this lemma. Consider the situation described by the following diagram:

(4.17)



$P', P'', Q', Q'' \in Z.$

In this case, as we've noted above, there are three rank one O_F-modules M_1, M_2, M_3 such that

$$P' = [M_1 + M_2 + M_3], \quad P'' = [\pi^b M_1 + \pi^b M_2 + M_3],$$
$$Q' = [\pi^b M_1 + M_2 + M_3], \quad Q'' = [M_1 + \pi^b M_2 + M_3].$$

LEMMA 4.18.

(a) The vertices P in $\mathcal{B}(G)$ with characteristic segment $(P', P''), P' \neq P''$, correspond to rank three lattices $L + M_3$ where L is a lattice satisfying
(i) $\pi^b M_1 + \pi^b M_2 \subseteq L \subseteq M_1 + M_2$,
(ii) $\pi^{b-1} M_1 + \pi^{b-1} M_2 \not\subset L$,
(iii) $L \not\subset \pi M_1 + \pi M_2$,
(iv) $L \not\subset \pi M_1 + M_2 =: M'$,
(v) $L \not\subset M_1 + \pi M_2 =: M''$.
(b) If $u \in O_F^\times$ then the action of $T_u := (A_u\sigma)(A\sigma)^{-1}$ on $\mathcal{Z}$ *induces an action of $O_F^\times$ on the lattices L in (a). This action agrees with that defined by the matrix*

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

and, moreover, it is transitive on the set of L's.

Proof. The stabilizer of any point P with segment (P', P'') under the action of $O_F^\times$ via T_u is of the form U_b, for some $b \geqq 1$. Thus

$$P' = A\sigma T_u(P) = T_u A\sigma(P) = A_u\sigma(P) = P, \tag{4.19}$$

for some $u \in O_F$ uniquely determined modulo U_b. In other words, if P has segment (P', P'') then there is a $u \in O_F^\times$ such that $A\sigma T_u(P) = P$ and

$$\begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix}, \qquad u' \in U_b,$$

not only stabilizes $A\sigma(P)$ but also commutes with $T_u, u \in O_F^\times$. Q.E.D. (Lemma 4.18)

From these facts, lemma 4.16 follows. As a corollary, we obtain $r(0) = R(0)$, using (3.3) and (3.17).

5. The Buildings for $PGL(3)$, $PGL(2)$. Recall $H_1 := SL(2)$, $H_2 := PGL(2)$, $G := PGL(3)$, and N_1, N_2 denote the Jacquet–Shalika norms (1.9). We assume $p \neq 2$ without further mention.

Let $h \in N_1(g) \cap H_1(F)$ have eigenvalues $\beta_1, \beta_2 = \beta_1^{-1}$ and let $h' \in N_2(g) \cap H_2(F)$ be represented by a matrix $B \in GL(2, F)$ having eigenvalues $\lambda\beta_1, \lambda$, for some $\lambda \in \bar{F}^\times$. Recall $\mathcal{B}(H_1) = \mathcal{B}(H_2)$ and that the action of $H_2(F)$ on $\mathcal{B}(H_2)$ is defined analogously to the action of $G(F)$ on $\mathcal{B}(G)$. Let E/F denote the extension of F obtained by adjoining the eigenvalues of h; we may assume E/F is quadratic since the split case was handled in §2. Since the tree $\mathcal{B}(H_2)$ has no closed loops, one can easily show that h has fixed points only if $c\beta_i \in O_E^\times$, for some $c \in F^\times$. This yields $|\beta_1|_E = |\beta_2|_E$, so, because of $\beta_2 = \beta_1^{-1}$, we obtain $|\beta_1|_E = 1$. (Alternatively, if the eigenvalues of h were not units then h could not have fixed points in the building $\mathcal{B}(H_1)_E$ over the tamely ramified extension E and therefore, by Galois descent [**18**, §2.6.1], it cannot have any fixed points in the building over F.)

Our strategy is to prove, in the notation of §3, that $s(0) = S(0)$ in the following manner: (a) if $|1+\beta_1|_E = 1$ then we prove $r(0) = s(0), R(0) = S(0)$, (b) we prove that the case E/F ramified and $|1 + \beta_1| < 1$ cannot occur, (c) if $|1 + \beta_1|_E < 1$ then we prove $s(0) = |\text{Fix}\,(h')| = 1$ and $S(0) = 1$. Thus, in case (c), we may assume that E/F is unramified. (That the case (b) cannot occur was pointed out to me by R. Kottwitz.) The $\pm$ sign in (3.18) has been choosen in such a way that $S(0) \geqq 0$ though we won't prove this until later.

LEMMA 5.1. *(a)* $s(0) \leqq r(0)$.

(b) Assume that $|1+\beta_1| = 1$. *Then the number of fixed points of h is equal to the number of fixed points of h′, i.e.,* $r(0) = s(0)$. *Moreover,* $|S(0)| \leqq R(0) = r(0)$.

Proof. Part (a) follows immediately from Lemma 3.3. For part (b), recall $\Delta H_1(h) := |1-\beta_1^2||\beta_1|^{-1} = |1-\beta_1||1+\beta_1||\beta_1|^{-1}$ and $\Delta_{H_2}(h') := |1-\beta_1||\beta_1|^{-1/2}$, by the definitions in §2. The lemma now follows immediately from the counting formulas (3.17–18) for $R(0)$ and $S(0)$ and the hypothesis. Q.E.D.

LEMMA 5.2. *(a) The set Fix* $(A_a\sigma)$ *is in one-to-one correspondence with the L's such that*

$$[L] = [\{v \in V|{}^t v \cdot {}^t A_a^{-1} \cdot v' \in O_F,\ \forall v' \in L\}],$$

i.e., the class of L is self-dual with respect to the bilinear form of ${}^t A_a^{-1}$.

(b) Assume $|Fix\,(A\sigma)| \neq 0, [L] \in Fix\,(A_u\sigma), u \in O_F^\times$, *and* $[L] \notin \mathcal{Z}$. *Then* $u = v^2$, *for some* $v \in O_F^\times$, *and in particular* $(u, \epsilon)_2 = 1$ *(in the notation of (3.18)).*

Proof.

(a) This is a simple calculation using (4.3): since $A\sigma$ sends $[L]$ to $[AL^\vee], [L]$

is fixed if and only if L belongs to the same class as

$$\begin{aligned} AL^{\vee} &= \{Av \in V | {}^t v \cdot v' \in \mathcal{O}_F,\ \forall v' \in L\} \\ &= \{v_0 \in V | v_0 + Av \text{ and } {}^t v \cdot v' \in \mathcal{O}_F,\ \forall v' \in L\}. \end{aligned}$$

It is easy to see that this is equivalent to the statement in part (a).

(b) We may rephrase the problem in terms of self-dual lattice classes. Let P be an ${}^tA^{-1}$-self-dual lattice class. By (a) above and Lemma 4.14, the transitivity of T_a implies that there is an $a \in \mathcal{O}_F^{\times}$ such that $[L] = T_a P$. It following that $[L]$ is both ${}^tA_u^{-1}$-self-dual and ${}^tA_{a^{-2}}^{-1}$-self-dual, since T_a and $A_b\sigma$ commute. This implies, by (a), that $[L]$ corresponds to a fixed point of T_{ua^2}. However, by Lemma 4.18 and the discussion following (4.15), we know that the stabilizer of T_{ua^2} is U_n, where $n \geqq 1$ is the distance between $[L]$ and its projection $[L']$ in $\mathcal{Z}$ (by hypothesis, $[L] \neq [L']$). Since, by definition, $U_n \subseteq \{\text{squares in } \mathcal{O}_F^{\times}\}$, for $n \geqq 1$, we conclude that u is a square in $\mathcal{O}_F^{\times}$. Q.E.D.

LEMMA 5.3. *Let* $w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, *and let* $\Lambda_0 := \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$ *denote the standard rank 3 lattice written in terms of a fixed basis for* $V_A \oplus W_A$. *The action of* $A_a\sigma$ *on* $\mathcal{B}(G)$ *is given on* $\mathcal{Z}$ *by*

$$\begin{aligned} \left[\begin{pmatrix} r & \\ & C' \end{pmatrix} \Lambda_0\right] &\longmapsto \left[A_a \begin{pmatrix} 1 & \\ & w \end{pmatrix} \begin{pmatrix} r^{-1} & \\ & C' \end{pmatrix} \begin{pmatrix} 1 & \\ & w \end{pmatrix}^{-1} \Lambda_0\right] \\ &= \left[A_a \begin{pmatrix} 1 & \\ & w \end{pmatrix} \begin{pmatrix} r^{-1} & \\ & C' \end{pmatrix} \Lambda_0\right], \end{aligned}$$

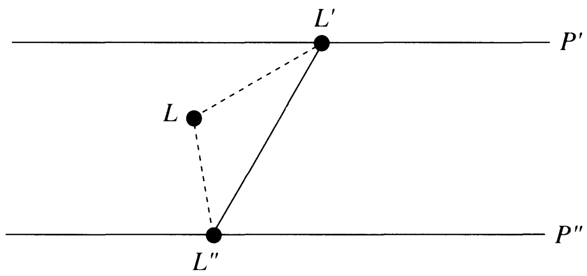
where $r \in F^{\times}, C' \in Aut_F(W_A) \cong GL(2, F)$.

Remark. This follows from (4.3) and the fact that $C^{-1} = (\det\ (C))^{-1t}(w^{-1}Cw)$. This description of the twisted action on the building is especially useful when the action of $A_a\sigma$ is restricted $\mathcal{Z}$.

LEMMA 5.4. *Assume that* $N_1(g)$ *doesn't contain any unipotent element. If* Fix $(A\sigma)$ *contains a vertex not in* $\mathcal{Z}$ *and if* Fix $(A\sigma) \cap \mathcal{Z} = \emptyset$ *then* $|\beta_1'/\beta_2'+1| < 1$, *where* β_i' *are the eigenvalues of* Cw.

Proof. By hypothesis, we may assume that A is of the form $\begin{pmatrix} 1 & \\ & C \end{pmatrix}$, thanks to Corollary 1.14. Suppose that there is an $[L] \in \text{Fix}\,(A\sigma)$ not in $\mathcal{Z}$. Lemma 4.16 implies that we can find $[L'], [L''] \in \mathcal{Z}$ forming the characteristic segment of $[L]$ such that $[A\sigma L'] = [L''], [A\sigma L''] = [L']$.

Denote by p', p'' the vertices of $\mathcal{B}(SL(2))$ associated to L', L'':



where $Aw(p') = p'', Aw(p'') = p'$. We may consider the midpoint p_0 of p', p'' in the first barycentric subdivision of $\mathcal{B}(SL(2,F))$ as a vertex of the tree over E, since E/F is ramified. By Lemma 5.3 and a well-known result on trees, p_0 in $\mathcal{B}(SL(2,E))$ is a fixed point of Cw (Cw acts without inversion on the first barycentric subdivision of $\mathcal{B}(SL(2,F))$). By Lemma 5.3, this midpoint is associated to a fixed point $[L_0]$ over E of ($A\sigma$ and) $A \cdot \begin{pmatrix} 1 & \\ & w \end{pmatrix}$. By hypothesis $[L_0] \notin \mathcal{Z}$.

Let p_1, p_2 be two neighbors of p_0, with p_i a vertex in $\mathcal{B}(SL(2,F))$ and with $Cw(p_1) = p_2, Cw(p_2) = p_1$. There is an apartment containing p_1, p_2 in which we can write

$$(5.5)\qquad \begin{aligned} p_1 &= [\pi^{b_1+1} e_1 O_F + \pi^{b_2} e_2 O_F], \\ p_2 &= [\pi^{b_1} e_1 O_F + \pi^{b_2} e_2 O_F], \end{aligned}$$

for some $b_i \in \mathbb{Z}$ and a suitable basis $\{e_1, e_2\}$. It follows that Cw is conjugate over F to $\begin{pmatrix} u_1\pi^{-1} & u_2\pi^{b_1-b_2} \\ u_3\pi^{b_2-b_1-1} & u_4 \end{pmatrix}$, for some $u_i \in O_F^\times$, hence projectively conjugate over F to $\begin{pmatrix} u_1\pi^{b_1-b_2} & u_2\pi^{2(b_1-b_2)+1} \\ u_3 & u_4\pi^{b_1-b_2+1} \end{pmatrix}$. If $b_1 - b_2 \leqq -2$ or $b_1 - b_2 \geqq 1$ then the quotient of the eigenvalues of this matrix are equal to $-1 \pm \epsilon$, for some $|\epsilon| < 1$. In this case, from Lemma 4.5 we find that Cw can have at most one fixed point in the tree over E, and none in the tree over F. In this case, we conclude that there is no fixed point of $A\sigma$ in $\mathcal{Z}$. The cases $b_1 - b_2 = -1$ and $b_1 - b_2 = 0$ are similar: if $b_1 - b_2 = -1$ then $p_1 = [\Lambda_0]$ and $p_2 = [\pi^{-1} e_1 O_F + \Lambda_0]$. It follows that Cw is conjugate over F to $\begin{pmatrix} u_1 & u_2\pi^{-1} \\ u_3 & u_4 \end{pmatrix}$, for some $u_i \in O_F^\times$, and the same conclusion may be drawn. The case $b_1 - b_2 = 0$ is similar and therefore omitted. Q.E.D.

PROPOSITION 5.6. *If* $|1 + \beta_1|_E = 1$ *then* $|S(0)| = s(0)$.

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Proof. Again, by hypothesis, we may assume that A is of the form $\begin{pmatrix} 1 & \\ & C \end{pmatrix}$, thanks to Corollary 1.14.

Note that by the reduction formulas and Lemma 5.2(b), we have

$$(5.7)\qquad R(0) - |S(0)| \leqq 2Q(|\mathrm{Fix}\,(A\sigma) \cap \mathcal{Z}| + |\mathrm{Fix}\,(A_a\sigma) \cap \mathcal{Z}|),$$

where A and A_a represent $\mathcal{D}_\sigma(g)$ (see (1.25)). Thus if $\mathrm{Fix}\,(A\sigma) \cap \mathcal{Z} = \emptyset$ were true, Lemma 5.1 would imply the result. Moreover, since $|S(0)| \leqq R(0) = r(0) = s(0)$, we may immediately dispose of the case $R(0) = 0$. We may therefore assume that $\Delta(h) < 1$, so that $|1 - \beta_1| < 1$ and that $|\mathrm{Fix}\,(A\sigma)| > 0$.

Suppose for the moment that $\mathrm{Fix}\,(A\sigma) \subset \mathcal{Z}$. By Lemma 4.16(b) it follows that $\mathrm{Fix}\,(A_a\sigma) = \emptyset$, so we have $S(0) = R(0)$.

Suppose next that $\mathrm{Fix}\,(A\sigma)$ is not a subset of $\mathcal{Z}$. Since $\mathrm{Fix}\,(A_a\sigma) \subset \mathcal{Z}$ would contradict Lemma 4.16, we must then have that $\mathrm{Fix}\,(A_a\sigma)$ is not a subset of $\mathcal{Z}$. From Lemma 4.16 we have that $\mathrm{Fix}\,(A\sigma) \cap \mathcal{Z} \neq \emptyset$ or $\mathrm{Fix}\,(A_a\sigma) \cap \mathcal{Z} \neq \emptyset$. Suppose without loss of generality that $\mathrm{Fix}\,(A\sigma) \cap \mathcal{Z} \neq \emptyset$. By (1.25) and Lemma 4.16(b), if E/F was ramified then we may take $a \in O_F^\times$ (since $F^\times/N(E^\times)$ is represented by a unit in the case of a ramified extension E/F). In this case, Lemma 5.2(b) implies all the signs in (3.18) are the same, so $|S(0) = R(0)$. On the other hand, if E/F were unramified then $a \notin O_F^\times$ by (4.14) and then Lemma 4.16(d) forces $\mathrm{Fix}\,(A_a\sigma) \cap \mathcal{Z} = \emptyset$. Then Lemma 5.4 implies that Cw has no fixed points, which by Lemma 5.3 would contradict our assumption. Q.E.D.

Next we must consider the case where $|1 + \beta_1| < 1$ and E/F is unramified. In this case, $\Delta(h') = 1$, so by the formula for the number of fixed points we have $|\mathrm{Fix}\,(h')| = 1$. On the other hand, the same formula implies that $|\mathrm{Fix}\,(h)| > 1$, so it is clear that in this case $r(0) > s(0)$. We of course want to show that the sum in the formula for $S(0)$ is equal to 1, so $s(0) = Q$. We want to show that, if $|1 + \beta_1| = q^{-m}$ with $m > 0$ and $|\mathrm{Fix}\,(A\sigma)| \neq 0$ then

$$(5.10)\qquad \begin{aligned} S(0) := &\frac{\pm 2}{[F^\times : U]}\, |(1+\beta_1)(1+\beta_2)|^{1/2} \\ &\times \sum_{a \in F^\times/U} \sum_{\substack{P \\ Inv(A_a\sigma(P),P)=0}} \kappa(A_a) = 1. \end{aligned}$$

In other words, we want to show that

$$(5.11)\qquad \begin{aligned} \frac{2}{[F^\times : U]} \sum_{\substack{a \in F^\times/U \\ (a,\epsilon)_2=+1}} \sum_{\substack{P \\ Inv(A_a\sigma(P),P)=0}} 1 = q^m + &\frac{2}{[F^\times : U]} \\ &\times \sum_{\substack{a \in F^\times/U \\ (a,\epsilon)_2=-1}} \sum_{\substack{P \\ Inv(A_a\sigma(P),P)=0}} 1. \end{aligned}$$

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For this it clearly suffices to prove that

$$(5.12)\qquad \frac{2}{[F^\times : U]} \sum_{\substack{a\in F^\times/U \\ (a,\epsilon)_2=+1}} \sum_{\substack{P \\ Inv(A_a\sigma(P),P)=0}} 1 = \frac{q^{m+1}+q^m-2}{q-1}+R,$$

(the "unramified case"), and

$$(5.13)\qquad \frac{2}{[F^\times : U]} \sum_{\substack{a\in F^\times/U \\ (a,\epsilon)_2=-1}} \sum_{\substack{P \\ Inv(A_a\sigma(P),P)=0}} 1 = \frac{2(q^m-1)}{q-1}+R,$$

(the "ramified case") for some R. These are due to the fact that every fixed point P of A gives rise to a characteristic leaf by Lemma 4.16, hence to a fixed point in the first barycentric subdivision of $\mathcal{B}(SL(2,F))$. The difference of (5.12) and (5.13) merely counts the difference between the sets of fixed points in the first barycentric subdivision of $\mathcal{B}(SL(2,F))$ in the unramified case and in the ramified case, respectively. The number of such fixed points is as stated in (5.12) by Lemma 4.5(a), in the unramified case, and by Lemma 4.5(b) in the ramified case.

It remains to dispose of the case $|1+\beta_1| < 1$ and E/F ramified. Although one must be able to deal with this case by a more direct argument, I will give the following geometric proof. Since $|1-\beta_1| = 1$ and E/F is ramified, it follows from [**14**, p. 50], for example, that h' has no fixed points. This implies that $s(0) = 0$ and (by the proof of Lemma 5.4) that $A\sigma$ has no fixed points in $\mathcal{Z}$. Thus each fixed point of $A\sigma$, which we may assume exists, is associated to a characteristic base which is not a vertex. By the remarks preceeding (4.10), the fact that $A\sigma$ has no fixed points in $\mathcal{Z}$ implies that, in the notation of (5.8), $A_a\sigma$ will have fixed points in $\mathcal{Z}$ (since E/F is ramified, there is an element of valuation one in $N_{E/F}(E^\times)$ and therefore we may take A_a to correspond to this element; the midpoint of the characteristic base isn't a vertex, whereas the midpoint of the dotted line D_{A_a} associated to the fixed point L of $A\sigma$ in (4.10) is a vertex in $\mathcal{Z}$ since the valuation of a is one). By Lemma 5.3, this implies that, in the notation of the proof of Proposition 5.7, Cw has fixed points. By Lemma 5.3, this contradicts the fact that $A\sigma$ has no fixed points in $\mathcal{Z}$.

This completes the proof that $S(0) = s(0)$.

References

1. C. Blondel, *Les représentations supercuspidales des groupes métaplectiques sur GL(2) et leurs caractéres*, Bull. Soc. Math. de France, (Memoire 18) *113* (1985).

2. A. Borel, *Automophic L-functions*, in Proc. Symp. Pure Math., vol 33, part 2, AMS, Providence, RI, 1979.

3. P. Cartier, *Representations of p-adic groups*, in Proc. Symp. Pure Math., vol 33, part 2, AMS, Providence, RI, 1979.

4. Y. Flicker, *On the symmetric square: orbital integrals*, Math. Ann. *279* (1987), 173–191.
5. S. Gelbart and H. Jacquet, *A relation between automorphic forms on GL(2) and GL(3)*, Ann. Sci. École Norm. Sup. *11* (1978), 471–552.
6. H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on GL(3), I, II*, Ann. Math. *109* (1979), 169–258.
7. R. Kottwitz and D. Shelstad, in preparation.
8. R. Kottwitz, *Orbital integrals on GL(3)*, Amer. J. Math. *102* (1980), 327–384.
9. R. Kottwitz, *Unstable orbital integrals on SL(3)*, Duke Math. J. *48* (1981), 649–664
10. J.-P. Labesse and R. Langlands, *L-indistinguishability for SL(2)*, Can. J. Math., *31* (1979), 726–785.
11. S. Lang, *Algebra*, 2nd ed., Addison-Wesley, 1984.
12. R. Langlands, *Some identities for orbital integrals attached to GL(3)*, manuscript.
13. R. Langlands, *Les Debut d'une Formule des Traces Stable*, Publ. Math. Univ. Paris VII, 1980.
14. R. Langlands, *Base Change for GL(2)*, Ann. Math. Studies, Princeton Univ. Press, 1980.
15. R. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann (1987).
16. I. Macdonald, *Spherical Functions on a Group of p-adic Type*, Publ. Ramanujan Inst., no. 2, Madras, 1971.
17. G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. *31* (1975) 79–98.
18. J. Tits, *Reductive groups over local fields*, in Proc. Symp. Pure Math., vol 33, part 1, AMS, Providence, RI, 1979.

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