R_1-TOPOLOGICAL SPACES[1]

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In his paper "Indexed systems of neighborhoods for general topological spaces" (Amer. Math. Monthly 68, (1961), 886-893), A.S. Davis defined a hierarchy of what he called regularity axioms. The R_1-axiom is independent of both T_0 and T_1, but is strictly weaker than T_2. In this note, we propose to study the properties of the spaces satisfying the R_1-axiom. In particular, we will show that in many well-known results, the hypothesis can be weakened from T_2 to R_1, which is part of our motivation in studying R_1-spaces.

1. Definition. A topological space (X, τ) is said to be $\underline{R_1}$ iff for every pair of points x, y of X, $\bar{x} \neq \bar{y}$ implies x and y have disjoint neighborhoods.

It is easy to see that $T_2 = R_1 + T_1$. That R_1 is independent of T_0 and of T_1 is shown by the following examples. An infinite set with the finite complement topology is T_1 but not R_1. On the other hand, the set $\{a, b, c\}$ equipped with the topology consisting of $\emptyset$, $\{a, b\}$, $\{c\}$ is an R_1-space but not T_0.

2. LEMMA. In an R_1-space, if G is an open neighborhood of a point x, then $\bar{x} \subset G$. (In the terminology of Davis: every R_1-space is R_0).

Proof. If $y \in G^c$, then $\bar{y} \subset G^c$. This means $\bar{x} \neq \bar{y}$ and

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so x and y have disjoint neighborhoods. Therefore $y \notin \bar{x}$ and $\bar{x} \subset G$.

3. COROLLARY. A topological space is R_1 iff whenever $\bar{x} \neq \bar{y}$, $\bar{x}$ and $\bar{y}$ have disjoint neighborhoods.

4. THEOREM. The R_1 property is hereditary, productive, projective and a topological invariant.

The routine verification of this theorem is omitted.

Since every finite set is compact, the following theorem follows from the fact that $T_2 = T_1 + R_1$.

5. THEOREM. An R_1-space is T_2 iff every compact subset is closed.

6. THEOREM. An R_1-space is T_2 iff every sequence has at most one limit.

Proof. Such a space is necessarily T_1.

7. LEMMA. If, in an R_1-space, A is a compact subset $\bar{x} \cap A = \emptyset$, then x and A have disjoint neighborhoods.

Proof. For each $y \in A$, $\bar{x} \neq \bar{y}$ and hence x and y have disjoint neighborhoods, U_y and V_y respectively. Since A is compact, there exist points $y_1, y_2, \ldots, y_n$ of A such that $A \subset \bigcup_{i=1}^{n} V_{y_i} = V$. If $U = \bigcap_{i=1}^{n} U_{y_i}$, then U and V are disjoint neighborhoods of x and A respectively.

8. THEOREM. A compact R_1-space is normal.

This can be proved by using Lemma 7 and by repeating the technique used in its proof.

If X is a topological space and $X^* = X \cup \{\infty\}$, $\infty \notin X$, we topologize X^* as follows: a set G is open in X^* iff (i) G is open in X or (ii) $X^* - G$ is a closed, compact subset of X.

The space X^* is called <u>the one-point compactification</u> of X.

9. THEOREM. The one-point compactification of X is R_1 iff X is R_1 and locally compact. (A space is said to be locally compact iff every point has a closed, compact neighborhood).

<u>Proof.</u> The "only if" part is trivial. For the converse, if $x, y \in X^*$ and $\bar{x} \neq \bar{y}$, then we must show that x and y have disjoint neighborhoods. If $x, y \in X$, there is nothing to prove. Suppose then that $x \in X$ and $y = \infty$. Since $\{\infty\}$ is a closed subset of X^*, $\bar{\infty} = \infty \neq \bar{x}$. Let U be a closed compact neighborhood (in the topology of X) of x. Then U and $X^* - U$ are disjoint neighborhoods of x and ∞ respectively.

10. COROLLARY. A locally compact R_1-space is completely regular.

<u>Proof.</u> The one-point compactification of such a space is normal and hence completely regular. Since complete regularity is hereditary, the given space is completely regular.

11. THEOREM. An R_1 paracompact space is normal.

<u>Proof.</u> Since a regular paracompact space is normal, it suffices to prove that an R_1 paracompact space is regular. Let A be a closed set of an R_1 paracompact space X and $x \in A^c$. For each $y \in A$, $\bar{y} \subset A$ and $\bar{x} \subset A^c$ (by Lemma 2). Since the space is R_1, x and y have disjoint neighborhoods U_y and V_y respectively. Then $A^c \cup \{V_y \mid y \in A\}$ is an open cover of X and hence must have an open locally finite refinement $\{V_\alpha\}$. $V = \{V_\alpha \mid V_\alpha \cap A \neq \emptyset\}$ is an open neighborhood of A. Now, there exists a neighborhood W of x which meets only finitely many sets $V_1, V_2, \ldots, V_n$ of $\{V_\alpha\}$. Each such V_i that meets A must lie in some V_{y_i}, $y_i \in A$. Therefore the intersection of W and $\bigcap_{i=1}^{n} U_{y_i}$ is an open neighborhood of x not intersecting V. This proves regularity of X.

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