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THE DEGREE OF HOLOMORPHIC APPROXIMATION ON A TOTALLY REAL SET

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Abstract

Let E be a totally real set on a Stein open set Ω on a complete noncompact Kähler manifold (M, g) with nonnegative holomorphic bisectional curvature such that (Ω, g) has bounded geometry at E. Then every function f in a C^p class with compact support on Ω and $\overline{\partial}$-flat on E up to order $p-1$, $p \geq 2$ (respectively, in a Gevrey class of order $s > 1$, with compact support on Ω and $\overline{\partial}$-flat on E up to infinite order) can be approximated on compacts subsets of E by holomorphic functions f_k on Ω with degree of approximation equal $k^{-p/2}$ (respectively, $\exp(-c(s)k^{1/2(s-1)})$).

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1. Introduction

Let Ω be a Stein open set on a complete noncompact Kähler manifold (M, g) with nonnegative holomorphic bisectional curvature. Let $\phi \in C^2(\Omega)$ be a nonnegative strictly plurisubharmonic function on Ω such that $i\partial\overline{\partial}\phi \geq \delta g$ where $\delta > 0$. Then

$$E = \{z \in \Omega \mid \phi(z) = 0\}$$

is a totally real set. Let $k \geq 1$ be a integer and P_k the orthogonal projection of

$$L^2(\Omega, e^{-k\phi}\, dV_g)$$

to

$$A^2(\Omega, e^{-k\phi}\, dV_g),$$

the latter space being the Bergman space, that is the subspace of $L^2(\Omega, e^{-k\phi}\, dV_g)$ consisting of holomorphic functions in $L^2(\Omega, e^{-k\phi}\, dV_g)$ which is nontrivial since ϕ is strictly plurisubharmonic. If $D > 0$ is large enough, set

$$\Omega_k = \left\{ z \in \Omega \;\middle|\; d(z, M \setminus \Omega) \geq \frac{D}{\sqrt{k}} \right\}$$

where d is the geodesic distance associated to g.

DEFINITION 1.1. The manifold (Ω, g) has bounded geometry at E in the sense of Chang–Yau [3] if there is a positive real number R such that, for every point $a \in E$, there is an open neighborhood U_a of a in Ω and a biholomorphic mapping $\Psi_a : U_a \to B_e(0, R)$ of U_a onto $B_e(0, R)$, the ball of radius R centered at $0 \in \mathbb{C}^n$, such that if g_e is the Euclidean metric in $\mathbb{C}^n$, then:

(i) $\Psi_a(a) = 0$;
(ii) $A\Psi_a^* g_e \leq g \leq B\Psi_* g_e$ on U_a where the constants A and B are independent of a.

In other words, there exist a covering of E by coordinate Euclidean balls of a fixed radius in which the corresponding Euclidean metrics are uniformly comparable to the metric g. We refer to the number R and the (nonunique) choice of constants in (ii) as the constants associated with the bounded geometry of (Ω, g) at E.

We suppose that (Ω, g) has bounded geometry at E. Our main results are as follows.

THEOREM 1.2. *Let $f \in C^{p\geq 2}(\Omega)$ with compact support and $\overline{\partial}$-flat at E up to order $p-1$. Then for every compact K of E there exist $C > 0$ and $k_0 \in \mathbb{N}$ such that for $k \geq k_0$*

$$\sup_{K\cap\Omega_k} |f - P_k(f)| \leq Ck^{-p/2}.$$

If $M = \mathbb{C}^n$ and $i\partial\overline{\partial}\phi \geq \delta i\partial\overline{\partial}\|z\|^2$, Theorem 1.2 was established by Berndtsson [1] without $\overline{\partial}$-flatness of $f \in C_0^1(\Omega)$ where the maximum is taken over $E \cap \Omega_k$. However, the constant C depends on the maximum of the second derivative of ϕ on $E \cap \Omega_k$.

THEOREM 1.3. *Let $f \in G^s(\Omega)$, the Gevrey class of order $s > 1$, with compact support and $\overline{\partial}$-flat at E up to infinite order. Then for every compact $K \subset E$ there exist $C > 0$ and $k_0 \in \mathbb{N}$ such that for $k \geq k_0$*

$$\sup_{K\cap\Omega_k} |f - P_k(f)| \leq C\exp(-c(s)k^{1/2(s-1)}).$$

If $\Omega = M$ and ϕ has Logarithmic growth at infinity, then $P_k(f) \in \mathcal{O}_k(M)$. The latter space is the complex linear space of all holomorphic functions on M of polynomial growth of degree at most k. By [2, Theorem 1.2, p. 2] we have $\dim_{\mathbb{C}} \mathcal{O}_k(M) \leq \dim_{\mathbb{C}} \mathcal{P}_k(\mathbb{C}^n)$ and if the equality holds for some positive integer k then M is holomorphically isometric to the complex Euclidean space $\mathbb{C}^n$ with the standard flat metric.

2. Proofs of Theorems 1.2 and 1.3

As in [1], the proofs are based on Hörmander's L^2 estimates with weights for the $\overline{\partial}$ operator [5].

THEOREM 2.1. *Let X be a weakly 1-complete manifold equipped with a Kähler metric h possibly noncomplete. Let Ψ be a C^2 function on X such that*

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$\mathrm{Ric}(h) + i\partial\overline{\partial}\Psi \geq \lambda\omega_h$, *where* λ *is a positive continuous function on* X. *Then if* $v \in L^2_{(0,1)}(X, e^{-\Psi}\, dV_h)$ *is* $\overline{\partial}$*-closed there exists a solution* u *of* $\overline{\partial}u = v$ *such that*

$$\int_X |u|^2 e^{-\Psi}\, dV_h \leq \int_X |\overline{\partial}v|_h^2 e^{-(\Psi+\log\lambda)}\, dV_h$$

provided that the right-hand side is finite.

Theorem 2.1 implies an Agmon-type estimate for the minimal solution of $\overline{\partial}u = f$.

PROPOSITION 2.2. *Let* Ω *be a Stein open set on a complete noncompact Kähler manifold* (M, g) *with nonnegative holomorphic bisectional curvature. Let* ϕ *be a* C^2 *strictly plurisubharmonic function on* Ω *such that* $i\partial\overline{\partial}\phi \geq \delta g$ *where* $\delta > 0$. *If* $u \in L^2(\Omega, e^{-k\phi}\, dV_g)$ *is the minimal solution of* $\overline{\partial}u = v$, *then for all* $a \in M$

$$\int_\Omega |u|^2 e^{-(k\phi+\sqrt{k}d(\cdot,a))}\, dV_g \leq \frac{C_\delta}{k}\int_\Omega |v|_g^2 e^{-(k\phi+\sqrt{k}d(\cdot,a))}\, dV_g.$$

PROOF. For the proof, we need the following lemma [2, Lemma 4.1, p. 17]. □

LEMMA 2.3. *Let* (M, g) *be a complete noncompact Kähler manifold of complex dimension* n *with nonnegative holomorphic bisectional curvature. Then there exists a positive constant* $C(n)$ *depending only on the dimension* n *such that for every* $a \in M$ *and* $k \geq 1$, *there is a smooth function* d_k *on* M *satisfying:*

(1) $C(n)^{-1}(1 + \sqrt{k}d(z, w)) \leq d_k(z) \leq C(n)(1 + \sqrt{k}d(z, w))$, $z \in M$;
(2) $|\overline{\partial}d_k|_g \leq C(n)\sqrt{k}$, *on* M;
(3) $|\partial\overline{\partial}d_k|_g \leq C(n)k$, *on* M.

First, suppose that Ω is bounded on M. Let $u \in L^2(\Omega, e^{-k\phi}\, dV_g)$ be the minimal solution of $\overline{\partial}u = v$. Put

$$u_k = ue^{-d_k}$$

where d_k as in Lemma 2.3. Since u is orthogonal to all holomorphic functions on $L^2(\Omega, e^{-k\phi}\, dV_g)$ and Ω is bounded, then u_k is orthogonal to all holomorphic functions on $L^2(\Omega, e^{-k\phi+d_k})$. By Theorem 2.1 u_k is the minimal solution for some $\overline{\partial}$-equation. Since

$$ki\partial\overline{\partial}\phi - i\partial\overline{\partial}d_k \geq k(\delta - C(n))g \geq C_\delta g$$

if δ is large enough, since a positive multiple of ϕ does not change the set E. So it follows from Theorem 1.2 that

$$\int_\Omega |u_k|^2 e^{-k\phi+d_k}\, dV_g \leq \frac{1}{C_\delta k}\int_\Omega |\overline{\partial}u_k|_g e^{-k\phi+d_k}\, dV_g. \qquad (*)$$

Since

$$\overline{\partial}u_k = (v - u_k\overline{\partial}d_k)e^{-d_k}$$

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and $|\overline{\partial} d_k|_g \le C(n)\sqrt{k}$, taking C_δ large enough, we can absorb the contribution to $(*)$ coming from the second term $u_k \overline{\partial} d_k$ in the left-hand side of $(*)$. By Lemma 2.3(1) d_k is comparable to $\sqrt{k} d(z, a)$, then Proposition 2.2 follows if Ω is bounded.

If Ω is unbounded, then $\Omega = \bigcup \Omega_j$ where (Ω_j) is an exhaustion of Ω by bounded Stein domains on M. We apply the above consideration on each Ω_j and passing to a weak limit, we obtain the conclusion of Proposition 2.2 (see [4, p. 982] for $\Omega \subset \mathbb{C}^n$). □

Since (Ω, g) has bounded geometry at E there is a positive real number R such that, for every point $a \in E$, there is an open neighborhood U_a of a in Ω and a biholomorphic mapping $\Psi_a : U_a \to B_e(0, R)$ of U_a onto $B_e(0, R)$, the ball of radius R centered at $0 \in \mathbb{C}^n$, such that if g_e is the Euclidean metric in $\mathbb{C}^n$, then:

(i) $\Psi_a(a) = 0$;

(ii) $A\Psi_a^* g_e \le g \le B\Psi_* g_e$ on U_a where the constants A and B are independent of a, hence

$$A\|\Psi_a(z)\| \le d(z, a) \le B\|\Psi_a(z)\| \quad \forall z \in U_a.$$

If $k \ge \max(A^{-2}, B^{-2})$, then

$$\Psi_a^{-1}\left(B_e\left(0, \frac{R}{2B\sqrt{k}}\right)\right) \subset B\left(a, \frac{R}{2\sqrt{k}}\right) \subset \Psi_a^{-1}\left(B_e\left(0, \frac{R}{2A\sqrt{k}}\right)\right) \subset\subset U_a.$$

2.1. Proof of Theorem 1.2

PROOF. Let $f \in C_0^p(\Omega)$ and $\overline{\partial}$-flat at E up to order $p-1$. The function $f_a = f \circ \Psi_a^{-1} : B_e(0, R/2A\sqrt{k}) \to \mathbb{C}$ is C^p on a neighborhood of $B_e(0, R/2A\sqrt{k})$ and $\overline{\partial}$-flat at a up to order $p-1$. By Taylor's formula, if $w \in B_e(0, R/2A\sqrt{k})$

$$\left|\frac{\overline{\partial} f_a}{\partial \overline{w}_j}(w)\right| \le C(p)\|f\|_{C_0^p(\Omega)}\|w\|^{p-1}$$

whence by (ii),

$$|\overline{\partial} f(z)|_g \le C(p, f) d(z, a)^{p-1} \quad \forall z \in B\left(a, \frac{R}{2\sqrt{k}}\right).$$

Hence,

$$|\overline{\partial} f(z)|_g \le C(f, A, B, R) k^{(1-p)/2} \quad \forall z \in V := \bigcup_{a \in E} B\left(a, \frac{R}{2\sqrt{k}}\right).$$

By Proposition 2.2 the minimal solution u_k of $\overline{\partial} u = \overline{\partial} f$ verifies

$$\int_\Omega |u_k|^2 e^{-k\phi - \sqrt{k} d(\cdot, a)}\, dV_\omega \le \frac{C}{k} \int_\Omega |\overline{\partial} f|_\omega^2 e^{-k\phi - \sqrt{k} d(\cdot, a)}\, dV_\omega.$$

Let K be a compact subset of E. If $a \in K \cap \Omega_k$, then $B(a, R/2A\sqrt{k}) \subset \Omega_k$ if $D > R/2A$. Also $B(a, R/2A\sqrt{k}) \subset K_k = \{z \in \Omega \mid d(z, K) \leq R/2A\sqrt{k}\} \subset\subset \Omega$ if $k \geq k_0(K)$. Since $\phi(a) = D\phi(a) = 0$, by Taylor's formula, if $z \in B(a, R/2A\sqrt{k})$,

$$\phi(z) \leq C \sup_{z \in K_{k_0}} |D^2\phi| d^2(z, a) \leq C(K) d^2(z, a).$$

Hence,

$$\int_{B(a, R/2\sqrt{k})} |u_k|^2 \, dV_g \leq \frac{C}{k} \int_\Omega |\overline{\partial} f|_g^2 e^{-k\phi - \sqrt{k} d(\cdot, a)} \, dV_g. \qquad (**)$$

Since $\overline{\partial}(u_k \circ \Psi_a^{-1}) = \overline{\partial} f_a$ on $B_e(0, R/2B\sqrt{k})$, we have the following well-known *a priori* estimate

$$|u_k(a)|^2 \leq C\bigg(k^n \int_{B_e(0, R/2B\sqrt{k})} |(\Psi_a^{-1})^* u_k|^2 \, dV_e + \frac{1}{k} \sup_{B_e(0, R/2B\sqrt{k})} |\overline{\partial} f_a|^2\bigg)$$

for some constant $C > 0$ (see Wermer [7, Lemma 16.7, 16.8] or Hörmander and Wermer [6, Lemma 4.4]). By (ii) we deduce

$$|u_k(a)|^2 \leq C\bigg(k^n \int_{B(a, R/2\sqrt{k})} |u_k|^2 \, dV_g + \frac{1}{k} \sup_{B(a, R/2\sqrt{k})} |\overline{\partial} f|^2\bigg).$$

Thus,

$$|u_k(a)|^2 \leq C\bigg(k^n \int_{B(a, R/2\sqrt{k})} |u_k|^2 \, dV_g + \frac{C}{k^p}\bigg).$$

Thanks to (**) we deduce

$$|u_k(a)|^2 \leq k^n \frac{C}{k}\bigg(\sup_{\mathrm{supp}t(f) \cap V} |\overline{\partial} f|^2 e^{-k\phi} + \sup_{\mathrm{supp}(f) \setminus V} e^{-k\phi}\bigg) \int_{\mathrm{supp}(f)} e^{-\sqrt{k} d(\cdot, a)} \, dV_g + \frac{C}{k^p}.$$

Since $\mathrm{Ric}(g) \geq 0$, by the coarea formula and Bishop's comparison theorem

$$\int_M e^{-\sqrt{k} d(z, a)} \, dV_g \leq C k^{-n}.$$

Also since $(\mathrm{supp}(f) \setminus V) \cap E = \emptyset$, we have $e^{-k\phi} \leq e^{-Ck}$ on $\mathrm{supp}(f) \setminus V$. Finally,

$$\sup_{K \cap \Omega_k} |u_k(a)|^2 = \sup_{K \cap \Omega_k} |f - P_k(f)|^2 \leq \frac{C}{k^p} + C e^{-ck} \leq \frac{C}{k^p}.$$

This finishes the proof of Theorem 1.2. □

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2.2. Proof of Theorem 1.3

PROOF. Now let $f \in G^s(\Omega)$ with compact support and $\overline{\partial}$-flat at E up to infinite order. If $p \in \mathbb{N}$ and $w \in B_e(0, R/2A\sqrt{k})$ by Taylor's formula

$$\left|\frac{\partial f_a}{\partial \overline{w}_j}(w)\right| \leq \sum_{|\alpha|=p} \alpha!^{-1} \sup_{w \in B_e(0,R/2A\sqrt{k})} \left| D^\alpha \frac{\partial f_a}{\partial \overline{w}_j}(w) \|w\|^p.\right.$$

Since $C_1^p p! \leq \alpha!$ if $p = |\alpha|$ and $\sum_{|\beta|=p} 1 \leq (p+1)^p \leq C_2 2^p$ where C_1, C_2 are constants, then

$$\left|\frac{\partial f_a}{\partial \overline{w}_j}(w)\right| \leq c\|f\|_s C^{p+1}(p!)^{-1}((p+1)!)^s \|w\|^p$$

where $\|f\|_s$ is the G^s-norm of f. Since $(p+1)! \leq p! 2^p$ and $s > 1$ we have $\inf_{p \in \mathbb{N}}(2^s C\|w\|)^p (p!)^{s-1} \leq A \exp(-B\|w\|^{1/(1-s)})$ where $A, B > 0$ are constants. Hence,

$$|\overline{\partial} f(z)|_g \leq C \exp(-Bd(z,a)^{1/(1-s)}) \quad \forall z \in B\left(a, \frac{R}{2\sqrt{k}}\right).$$

Thus,

$$|\overline{\partial} f(z)|_g \leq C \exp(-Bk^{1/2(1-s)}) \quad \forall z \in V = \bigcup_{a \in E} B\left(a, \frac{R}{2\sqrt{k}}\right).$$

Following the same lines as Section 2.1, we deduce that

$$\sup_{K \cap \Omega_k} |f - P_k(f)| \leq C \exp(-c(s)k^{1/2(1-s)}). \qquad \square$$

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