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PARAREDUCTIVE OPERATORS ON BANACH SPACES

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ABSTRACT. This note gives a Banach space extension of the Hilbert space result due to P. A. Fillmore (see [3]). In particular, it is shown that the adjoint $T^* = A - iB$ of an operator $T = A + iB$ (with A and B hermitian) is a polynomial in T if and only if T^* leaves invariant every linear subspace invariant under T, and this is equivalent to the assertion that T^* leaves invariant every paraclosed subspace invariant under T.

Let X be a complex Banach space. The Banach algebra of all bounded linear operators on X is denoted by $\mathcal{B}(X)$. A linear subspace of the space X is called *paraclosed* if it is the range of some bounded linear mapping from some Banach space into X. For $T \in \mathcal{B}(X)$, let $\mathrm{Lat}_0\, T$ denote the lattice of all (not necessarily closed) subspaces invariant under T. By $\mathrm{Lat}_{1/2}\, T$ and $\mathrm{Lat}\, T$ we denote the sublattices of $\mathrm{Lat}_0\, T$ consisting of paraclosed and closed subspaces respectively.

We now recall the notion of hermitian operators on a Banach space. An operator $H \in \mathcal{B}(X)$ is called *hermitian* if

$$\lim_{t \to 0} \frac{\|I + itH\| - 1}{t} = 0,$$

where t approaches zero through real values. Let $\mathcal{H}(X) \subseteq \mathcal{B}(X)$ be the real Banach space of all hermitian operators on X. It is well-known that an operator $H \in \mathcal{B}(X)$ is hermitian if and only if $\|\exp(itH)\| = 1$ for all $t \in \mathbb{R}$. For other equivalent definitions and basic properties of hermitian operators see [1] and [2].

Let $\mathcal{J}(X)$ denote the subspace of all operators $T \in \mathcal{B}(X)$ of the form $T = A + iB$ with A and B hermitian, or shortly $\mathcal{J}(X) = \mathcal{H}(X) + i\mathcal{H}(X)$. The space $\mathcal{J}(X)$ with the norm of $\mathcal{B}(X)$ is a complex Banach space, but it need not be a subalgebra of $\mathcal{B}(X)$. Since each element of $\mathcal{J}(X)$ has a unique representation of the form $A + iB$ with A and B hermitian, we may define a continuous linear involution on $\mathcal{J}(X)$ by

$$(A + iB)^* = A - iB.$$

An operator $T = A + iB$ with A and B hermitian is said to be *normal* if $T^*T = TT^*$, or equivalently $AB = BA$.

In the proof of the main theorem of this note we need the following result concerning a normal operator, the spectrum of which is a finite set.

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LEMMA. *Suppose that an operator $T \in \mathcal{J}(X)$ is normal and that its spectrum is a finite set, i.e. $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for some complex numbers $\{\lambda_k\}$. Then*

$$T = \sum_{k=1}^{n} \lambda_k P_k \quad \text{and} \quad T^* = \sum_{k=1}^{n} \bar{\lambda}_k P_k,$$

where $\{P_k\}$ are the spectral projections corresponding to $\{\lambda_k\}$. Furthermore, the operator T is algebraic.

PROOF. Let $T = A + iB$, where $A, B \in \mathcal{H}(X)$ and $AB = BA$. It is well-known that

$$T = \sum_{k=1}^{n} (\lambda_k P_k + Q_k),$$

where $\{Q_k\}$ are quasinilpotents, and $\{P_k\}$ are the spectral projections corresponding to $\{\lambda_k\}$ and satisfying the following $P_k^2 = P_k$, $P_i P_j = 0$ for $i \neq j$ and $P_1 + P_2 + \cdots + P_n = I$. The facts that $AT = TA$ and $BT = TB$ imply $AP_k = P_k A$ and $BP_k = P_k B$ respectively; therefore (for any $k = 1, 2, \ldots, n$) $A_k = A|_{\operatorname{Im} P_k}$ and $B_k = B|_{\operatorname{Im} P_k}$ are hermitian operators on $\operatorname{Im} P_k$. We then have

$$A_k + iB_k = \lambda_k I_k + Q_k,$$

where I_k is the identity operator on $\operatorname{Im} P_k$. From this it follows that

$$Q_k = (A_k - \operatorname{Re} \lambda_k I_k) + i(B_k - \operatorname{Im} \lambda_k I_k).$$

Since $\{A_k - \operatorname{Re} \lambda_k I_k, B_k - \operatorname{Im} \lambda_k I_k, Q_k\}$ is also a commutative triple of two hermitian operators and a quasinilpotent, they are all equal to zero by [2, Proposition 4.20]. Therefore, $T = \sum_{k=1}^{n} \lambda_k P_k$ and $T^* = \sum_{k=1}^{n} \bar{\lambda}_k P_k$. Clearly, the operator T is algebraic, and the proof is completed. ■

Since the spectrum of an algebraic operator is a finite set, the following assertion clearly holds.

COROLLARY. *A normal operator $T \in \mathcal{J}(X)$ is algebraic if and only if T^* is algebraic.*

The following theorem is a generalization of the theorem in [3].

THEOREM. *If $T \in \mathcal{J}(X)$, then each of the following conditions implies all the others:*

1. $\operatorname{Lat}_0 T \subseteq \operatorname{Lat}_0 T^*$;
2. $T^* = p(T)$ *for some polynomial* p;
3. $\operatorname{Lat}_{1/2} T \subseteq \operatorname{Lat}_{1/2} T^*$;
4. $T^* = u(T)$ *for some entire function* u;
5. *Either T is normal and algebraic, or else $T = aH + bI$ for some hermitian operator H and complex numbers a and b.*

Moreover, each of these conditions is equivalent to the symmetric condition obtained by interchanging T and T^*.

By the analogy with the notion of reductive operators on a Hilbert space, an operator $T \in \mathcal{J}(X)$ satisfying any (and therefore all) of the conditions of this theorem is called *parareductive*. (A bounded operator A on a Hilbert space is called *reductive* if $\operatorname{Lat} A = \operatorname{Lat} A^*$.)

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PROOF OF THE THEOREM. We will first show the equivalence of 1 and 2. Since 2 obviously implies 1, we suppose that 1 holds. By [4, Theorem 2] it is enough to show that T is normal. For any vector x, let us define the (not necessarily closed) cyclic subspace $C_x = \mathrm{Lin}\{x, Tx, T^2x, T^3x, \ldots\}$ and assume first that C_x is finite-dimensional. Since $T = A + iB$ for some hermitian operators A and B, it follows that

$$A = \frac{1}{2}(T + T^*) \quad \text{and} \quad B = \frac{1}{2i}(T - T^*),$$

so that (using 1) $\mathrm{Lat}_0\, T \subseteq \mathrm{Lat}_0\, A$ and $\mathrm{Lat}_0\, T \subseteq \mathrm{Lat}_0\, B$. Thus C_x is an invariant subspace under A and B, and hence the restrictions $A|_{C_x}$ and $B|_{C_x}$ are hermitian operators on C_x. By the Jordan canonical form there exists a basis for C_x so that $C_x = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and the matrix of $T|_{C_x}$ is of the form $T_1 \oplus T_2 \oplus \cdots \oplus T_n$, where T_k are matrices of the restrictions of T to V_k. Moreover, for some $\alpha_k, \beta_k \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$ we have

$$T_k = (\alpha_k + i\beta_k)I_k + Q_k,$$

where I_k is the identity and Q_k a strictly upper triangular nilpotent matrix. Since $\mathrm{Lat}_0(T|_{C_x}) \subseteq \mathrm{Lat}_0(A|_{C_x})$ and $\mathrm{Lat}_0(T|_{C_x}) \subseteq \mathrm{Lat}_0(B|_{C_x})$, the matrices of $A|_{C_x}$ and $B|_{C_x}$ are also of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ respectively, where A_k and B_k are (because of the same reason) upper triangular matrices of some hermitian operators. Then

$$T_k = A_k + iB_k = (\alpha_k + i\beta_k)I_k + Q_k,$$

and hence

$$Q_k = (A_k - \alpha_k I_k) + i(B_k - \beta_k I_k).$$

Since A_k and B_k are matrices of some hermitian operators, they have real eigenvalues on the diagonals. It follows that matrices $A_k - \alpha_k I_k$ and $B_k - \beta_k I_k$ are strictly upper triangular, and therefore nilpotents. Since they are matrices of some hermitian operators as well, we have

$$A_k - \alpha_k I_k = B_k - \beta_k I_k = Q_k = 0.$$

Thus, the matrices of operators $T|_{C_x}$ and $T^*|_{C_x}$ are of the form

$$\bigoplus_{k=1}^{n} (\alpha_k + i\beta_k)I_k \quad \text{and} \quad \bigoplus_{k=1}^{n} (\alpha_k - i\beta_k)I_k$$

respectively, and hence $T^*Tx = TT^*x$. If C_x is infinite-dimensional, the proof of the equation $T^*Tx = TT^*x$ is exactly the same as in the Hilbert space case (see [4, Theorem 3]).

Most of the proof of equivalence of 3 and 4 can be obtained as a special case of [5, Theorem 3.2]. The only exception is the proof that 3 implies 4 in the case that T is algebraic. To prove this, choose any linear submanifold $\mathcal{M} \in \mathrm{Lat}_0\, T$ and any vector $x \in \mathcal{M}$. Since the cyclic subspace C_x generated by x is finite-dimensional, it is closed, so that $C_x \in \mathrm{Lat}_{1/2}\, T$. By 3 we then have $C_x \in \mathrm{Lat}_{1/2}\, T^*$, and $T^*x \in C_x \subseteq \mathcal{M}$. Thus $\mathcal{M}$ is invariant under T^*, and hence 1 holds. Since 1 and 2 are equivalent and 2 clearly implies 4, 3 implies 4.

We next show that 4 implies 5. If $T^* = u(T)$, then T is normal, and so $T = A + iB$ for some commutative hermitian operators A and B. Let $\mathcal{A}$ be the maximal commutative Banach sub-algebra of $\mathcal{B}(X)$ containing A and B. By $\mathcal{G}$ we denote the corresponding Gelfand transform, and we observe that $\mathcal{G}$ commutes with any polynomial and therefore with any entire function. Since A is hermitian, $\|\exp(itA)\| = 1$ for all $t \in \mathbb{R}$. It follows that

$$\left\|\exp\big(it\mathcal{G}(A)\big)\right\|_\infty = \left\|\mathcal{G}\big(\exp(itA)\big)\right\|_\infty \leq \|\exp(itA)\| = 1 \quad \text{for all } t \in \mathbb{R},$$

so that the function $\mathcal{G}(A)$ is real. Similarly, $\mathcal{G}(B)$ is a real function. Therefore, we have

$$\mathcal{G}(T) = \mathcal{G}(A) + i\mathcal{G}(B), \qquad \mathcal{G}(T^*) = \mathcal{G}(A) - i\mathcal{G}(B) \quad \text{and} \quad \mathcal{G}(T^*) = \overline{\mathcal{G}(T)}.$$

Define an entire function u^* by $u^*(z) = \overline{u(\bar{z})}$. Then,

$$\mathcal{G}(T) = \overline{\mathcal{G}(T^*)} = \overline{\mathcal{G}\big(u(T)\big)} = \overline{u\big(\mathcal{G}(T)\big)} = u^*\big(\overline{\mathcal{G}(T)}\big) = u^*\big(\mathcal{G}(T^*)\big) = \mathcal{G}\Big(u^*\big(u(T)\big)\Big),$$

and so

$$\mathcal{G}\Big(u^*\big(u(T)\big) - T\Big) = 0.$$

Since the entire function $v(z) = u^*\big(u(z)\big) - z$ satisfies the equation $\mathcal{G}\big(v(T)\big) = 0$, the operator $v(T)$ is quasinilpotent. By the spectral mapping theorem it follows that

$$v\big(\sigma(T)\big) = \sigma\big(v(T)\big) = 0.$$

If $\sigma(T)$ is a finite set, then T is algebraic by the Lemma. If $\sigma(T)$ is an infinite set, then $v \equiv 0$. Hence u is a homeomorphism, so that $\lim_{z\to\infty} |u(z)| = \infty$, and u has a pole at infinity by the classical result of the behaviour of a function in the neighbourhood of an essential singularity. Therefore u is a polynomial. Since it is also a homeomorphism, it follows that u is a linear function. Hence, $T^* = aT + bI$ for some complex numbers a and b. If $a \neq -1$, then let us define a hermitian operator H by $H = T + T^*$. It follows that $T = \big(1/(a+1)\big)H - \big(b/(a+1)\big)I$. If $a = -1$, then $H = iT - iT^*$ is a hermitian operator, which satisfies the equation $T = \big(1/(2i)\big)H + (b/2)I$. In both cases we have proved that T is a linear function of some hermitian operator, and therefore 5 holds.

Since 2 obviously implies 4, we only have to prove that 5 implies 2. Suppose first that T is normal and algebraic. Then its spectrum is a finite set of complex numbers, say $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. By the Lemma it follows that

$$T = \sum_{k=1}^{n} \lambda_k P_k \quad \text{and} \quad T^* = \sum_{k=1}^{n} \bar{\lambda}_k P_k,$$

where $\{P_k\}$ are the spectral projections corresponding to $\{\lambda_k\}$. Therefore $T^* = p(T)$ for any polynomial p with $p(\lambda_k) = \bar{\lambda}_k$ for all $k = 1, 2, \ldots, n$. If $T = aH + bI$ with H hermitian, then $T^* = \bar{a}H + \bar{b}I$. If $a \neq 0$ then $H = (1/a)T - (b/a)I$, so that $T^* = p(T)$ for linear polynomial $p(x) = \bar{a}x/a + (\bar{b} - \bar{a}b/a)$. If $a = 0$ then $T^* = (\bar{b}/b)T$ (if $b = 0$ as well, then $T^* = T = 0$). Therefore 2 holds.

By the Corollary the condition 5 is equivalent to the symmetric condition obtained by interchanging T and T^*, and this observation completes the proof of the theorem. ■

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