Bull. Aust. Math. Soc. **106** (2022), 209–214
doi:10.1017/S000497272100126X

ON A k-ADDITIVE UNIQUENESS SET FOR MULTIPLICATIVE FUNCTIONS

ELCHIN HASANALIZADE

(Received 10 November 2021; accepted 29 November 2021; first published online 8 February 2022)

Abstract

Let $k \geq 2$ be an integer. We prove that the 2-automatic sequence of odious numbers $\mathcal{O}$ is a k-additive uniqueness set for multiplicative functions: if a multiplicative function f satisfies a multivariate Cauchy's functional equation $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_1, \ldots, x_k \in \mathcal{O}$, then f is the identity function $f(n) = n$ for all $n \in \mathbb{N}$.

2020 *Mathematics subject classification*: primary 11A25; secondary 11B85.

Keywords and phrases: additive uniqueness, multiplicative function, odious numbers.

1. Introduction

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Let $\mathcal{M}$ denote the set of complex-valued multiplicative functions.

A set $E \subseteq \mathbb{N}$ is an *additive uniqueness set* of a set of arithmetic functions $\mathcal{F}$ if there is exactly one element $f \in \mathcal{F}$ that satisfies

$$f(m + n) = f(m) + f(n) \quad \text{for all } m, n \in E.$$

For example, $\mathbb{N}$ and $\{1\} \cup 2\mathbb{N}$ are trivially additive uniqueness sets of $\mathcal{M}$.

This concept was introduced by Spiro [13] in 1992. She proved that the set of primes is an additive uniqueness set of $\mathcal{M}_0 = \{f \in \mathcal{M} \mid f(p_0) \neq 0 \text{ for some prime } p_0\}$ and asked whether other interesting sets were additive uniqueness sets for multiplicative functions. Spiro's work has been extended in many directions.

Let $k \geq 2$ be a fixed integer. If there is only one function $f \in \mathcal{F}$ which satisfies $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_i \in E$, $i \in \{1, 2, \ldots, k\}$, then E is called a *k-additive uniqueness set* of $\mathcal{F}$.

In 2010, Fang [5] proved that the set of primes is a 3-additive uniqueness set of $\mathcal{M}_0$. In 2013, Dubickas and Šarka [4] generalised Fang's result to sums of arbitrary primes.

This research was supported by NSERC Discovery grants RGPIN-2020-06731 to Habiba Kadiri and RGPIN-2020-06032 to Nathan Ng.

In 1999, Chung and Phong [3] showed that the set of positive triangular numbers $T_n = \frac{1}{2}n(n+1)$, $n \in \mathbb{N}$, and the set of positive tetrahedral numbers $Te_n = \frac{1}{6}n(n+1)(n+2)$, $n \in \mathbb{N}$, were new additive uniqueness sets for $\mathcal{M}$. Park [11] extended their work to sums of k triangular numbers, $k \geq 3$.

In 2018, Kim *et al.* [7] proved that the set of generalised pentagonal numbers $P_n = \frac{1}{2}n(3n-1)$, $n \in \mathbb{Z}$, is an additive uniqueness set for $\mathcal{M}$. Recently, they showed that the set of positive pentagonal numbers and the set of positive hexagonal numbers $H_n = n(2n-1)$, $n \in \mathbb{N}$, are new additive uniqueness sets for the collection of multiplicative functions [8]. They also conjectured that among the sets of s-gonal numbers, only the sets of triangular, pentagonal and hexagonal numbers are additive uniqueness sets for $\mathcal{M}$.

Park [9] proved that the set of nonzero squares is a k-additive uniqueness set of $\mathcal{M}$ for every $k \geq 3$, although it is not a 2-additive uniqueness set [2]. In 2020, he showed that $\{p-1 \mid p \text{ is a prime}\}$ is an additive uniqueness set for $\mathcal{M}$ [10].

Recently, the author [6] proved that the set of practical numbers is a k-additive uniqueness set of $\mathcal{M}$ for every $k \geq 2$.

A set $S \subseteq \mathbb{N}$ is called an *additive basis* (respectively, an *asymptotic additive basis*) of order j for $\mathbb{N}$ if there is a constant j such that every natural number (respectively, every sufficiently large natural number) can be written as a sum of at most j members of S. For example, the classical Lagrange theorem asserts that the set of squares is an additive basis of order 4, and Gauss (1796) proved that the triangular numbers form an additive basis of order 3. The famous binary Goldbach conjecture is equivalent to the assertion that the set of primes is an asymptotic additive basis of order 3.

A set $S \subseteq \mathbb{N}$ is called *k-automatic* if there exists a deterministic finite automaton M that recognises the language of base k representations of elements of S [1].

A number is *odious* if the number of ones in its base 2 representation is odd. The set of odious numbers is 2-automatic. Let $\mathcal{O}$ be the set of odious numbers, that is,

$$\mathcal{O} = \{1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, \ldots\}.$$

Using automata theory, Rajasekaran *et al.* [12] proved the following result.

THEOREM 1.1 (Rajasekaran *et al.*, 2020). *A natural number is the sum of exactly two odious numbers if and only if it is not of the form* $2 \cdot 4^i - 1$ *for* $i \geq 0$.

The next theorem, also from Rajasekaran *et al.* [12], shows that the set of odious numbers is an asymptotic additive basis of order 3.

THEOREM 1.2 (Rajasekaran *et al.*, 2020). *Every natural number* $N > 15$ *is the sum of three distinct odious numbers.*

We prove the following theorem showing that the set of odious numbers is an additive uniqueness set of $\mathcal{M}$.

THEOREM 1.3. *Fix $k \geq 2$. The set $\mathcal{O}$ of odious numbers is a k-additive uniqueness set of $\mathcal{M}$: if a multiplicative function f satisfies*

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$$

for arbitrary $x_1, \ldots, x_k \in \mathcal{O}$, then f is the identity function.

It would be interesting to see whether a result similar to Theorem 1.3 holds for other classes of automatic sets.

2. Proof of Theorem 1.3

The proof consists of four parts.

Case I: $k = 2$. It is easy to show by induction that $f(2^k) = 2^k$ for all $k \in \mathbb{N}$, because $f(2) = f(1 + 1) = 2$ and $f(2^{k+1}) = f(2 \cdot 2^k) = 2f(2^k)$. Suppose that N is an integer such that $f(n) = n$ for all $n \leq N$. We show that $f(N + 1) = N + 1$. If $N + 1 \neq 2 \cdot 4^i - 1$ for $i \geq 1$, then by Theorem 1.1 there are two distinct odious numbers x, y such that $N + 1 = x + y$ and $x, y \leq N$. Thus, $f(N + 1) = f(x) + f(y)$ so that $f(N + 1) = N + 1$. If $N + 1 = 2 \cdot 4^i - 1$ for some $i \geq 1$, then

$$2^{2i+1} = f(2^{2i+1}) = f(2 \cdot 4^i - 1 + 1) = f(N + 1) + 1,$$

since $2 \cdot 4^i - 1 = 2^{2i+1} - 1 = (\underbrace{11\ldots1}_{2i+1})_2 \in \mathcal{O}$. Therefore, $f(N + 1) = N + 1$. Note that in this case we do not use the multiplicativity of f.

Case II: $k = 3$. Clearly, $f(3) = 3$ and $f(10) = f(2)f(5) = f(2)[2f(2) + 1]$. On the other hand, $f(10) = f(4 + 4 + 2) = 2f(4) + f(2)$ and $f(4) = f(2 + 1 + 1) = f(2) + 2$. Hence, $f^2(2) - f(2) - 2 = 0$ with two solutions $f(2) = -1$ and $f(2) = 2$. The first solution yields $f(4) = 1$, which leads to the contradiction

$$\begin{aligned} f(6) &= f(4 + 1 + 1) = f(4) + 2 = 3 \\ &= 3f(2) = -3. \end{aligned}$$

Therefore, we conclude that $f(2) = 2$. From this, it is easy to check that $f(n) = n$ for $1 \leq n \leq 15$. Assume that $f(n) = n$ for all $n \leq N$. We have $N \geq 15$. We show that $f(N + 1) = N + 1$. By Theorem 1.2, there exist distinct odious numbers x, y and z such that $N + 1 = x + y + z$ where $x, y, z < N$. Hence, the assumption $f(n) = n$ for all $n \leq N$ yields $f(N + 1) = f(x + y + z) = f(x) + f(y) + f(z) = x + y + z = N + 1$.

Case III: $k = 4$. By Theorem 1.2 and straightforward calculations, every integer ≥ 4 can be written as a sum of four odious numbers.

Note that $f(4) = 4$, $f(6) = f(2)f(3) = f(2 + 2 + 1 + 1) = 2f(2) + 2$ and $f(12) = 4f(3) = f(4 + 4 + 2 + 2) = 8 + 2f(2)$. For convenience, let $a = f(2)$, $b = f(3)$.

https://doi.org/10.1017/S000497272100126X Published online by Cambridge University Press

This gives the system of equations

$$\begin{cases} ab = 2a + 2 \\ 2b = a + 4. \end{cases}$$

We obtain the two solutions $f(2) = -2$, $f(3) = 1$ and $f(2) = 2$, $f(3) = 3$. The first solution yields $f(5) = f(2 + 1 + 1 + 1) = 1$, which leads to the contradiction

$$\begin{aligned} f(10) &= f(4 + 4 + 1 + 1) = 10 \\ &= f(2)f(5) = -2. \end{aligned}$$

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$. So, $f(n) = n$ for $n \leq 4$, and f must be the identity function by induction.

Case IV: $k \geq 5$. In this case we follow closely Park's argument in [11]. It is clear that the sum of k odious numbers can represent k but cannot represent any number from 1 to $k - 1$. Since sums of four odious numbers represent all integers ≥ 4 as in Case III, the sum

$$\underbrace{1 + \cdots + 1}_{k-4 \text{ times}} + x + y + z + w, \tag{2.1}$$

where $x, y, z, w \in \mathcal{O}$, can represent all integers $\geq k$.

Let $k \geq 5$. Note that

$$\begin{aligned} (k-2) + 8 &= (k-2) \cdot 1 + 4 + 4 \\ &= (k-2) \cdot 1 + 7 + 1, \\ (k-3) + 18 &= (k-3) \cdot 1 + 14 + 2 + 2 \\ &= (k-3) \cdot 1 + 7 + 7 + 4, \\ (k-4) + 33 &= (k-4) \cdot 1 + 28 + 2 + 2 + 1 \\ &= (k-4) \cdot 1 + 14 + 14 + 4 + 1. \end{aligned}$$

Let $a = f(2)$, $b = f(4)$, $c = f(7)$. The above equalities give rise to the system of equations

$$\begin{cases} 2b = c + 1 \\ ac + 2a = 2c + b \\ bc + 2a = 2ac + b. \end{cases}$$

The solutions are

$$\begin{gathered} f(2) = \tfrac{1}{4},\ f(4) = \tfrac{1}{2},\ f(7) = 0 \\ f(2) = f(4) = f(7) = 1 \\ f(2) = 2,\ f(4) = 4,\ f(7) = 7. \end{gathered}$$

Observe that $f(k + 1) = k - 1 + f(2)$, $f(k + 4) = k - 2 + f(4) + f(2)$ and $f(k + 6) = k - 4 + f(4) + 3f(2)$.

https://doi.org/10.1017/S000497272100126X Published online by Cambridge University Press

If $\gcd(4, k+1) = 1$, the equalities

$$f(4(k+1)) = f(\underbrace{4 + \cdots 4}_{k-3 \text{ times}} + 7 + 7 + 2) = f(4)(k-3) + 2f(7) + f(2)$$
$$= f(4)f(k+1) = f(4)(k-1+f(2))$$

exclude the first set of solutions $f(2) = \frac{1}{4}$, $f(4) = \frac{1}{2}$, $f(7) = 0$.

If $4 \nmid k+1$ but $2 \mid k+1$, then $\gcd(4, k+4) = 1$, and the equalities

$$f(4(k+4)) = f(\underbrace{4 + \cdots 4}_{k-3 \text{ times}} + 14 + 7 + 7) = f(4)(k-3) + f(2)f(7) + 2f(7)$$
$$= f(4)f(k+4) = f(4)(k-2+f(4)+f(2))$$

exclude the first set of solutions.

Finally, if $4 \mid k+1$, then $\gcd(4, k+6) = 1$, and we consider

$$f(4(k+6)) = f(\underbrace{4 + \cdots 4}_{k-3 \text{ times}} + 28 + 7 + 1) = f(4)(k-3) + f(4)f(7) + f(7) + 1$$
$$= f(4)f(k+6) = f(4)(k-4+f(4)+3f(2)),$$

which excludes the first set of solutions.

Now consider the second solution set $f(2) = f(4) = f(7) = 1$. Arrange the odious numbers into an increasing sequence, and let x_n denote the nth term. Then, $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 1$. As seen in Case III, every x_n with $n \geq 3$ can be written as a sum of four odious numbers. From the equality

$$(k-5) + 1 + 2 + 2 + 2 + x_e = (k-5) + 7 + x_a + x_b + x_c + x_d \tag{2.2}$$

we infer that $f(x_n) = 1$ for all $n \geq 5$ inductively. But for sufficiently large n, x_n can be represented as a sum of k odious numbers by (2.1), so $f(x_n) = k$, which is a contradiction.

Hence, we conclude that $f(2) = 2$, $f(4) = 4$ and $f(7) = 7$. Moreover, (2.2) yields $f(x_n) = x_n$ for every $n \geq 1$.

If N is a sum of k odious numbers, then $f(N) = N$. Otherwise, choose an integer $M \geq k$ such that $\gcd(M, N) = 1$. Then, M and MN can be represented as sums of k odious numbers by (2.1). By the multiplicativity of f,

$$Mf(N) = f(M)f(N) = f(MN) = MN.$$

Therefore, $f(N) = N$, and this completes the proof.

Acknowledgement

The author would like to thank the anonymous referee for helpful suggestions.

References

[1] J.-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations* (Cambridge University Press, Cambridge, 2003).

https://doi.org/10.1017/S000497272100126X Published online by Cambridge University Press

[2] P. V. Chung, 'Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$', *Math. Slovaca* **46**(2–3) (1996), 165–171.

[3] P. V. Chung and B. M. Phong, 'Additive uniqueness sets for multiplicative functions', *Publ. Math. Debrecen* **55**(3–4) (1999), 237–243.

[4] A. Dubickas and P. Šarka, 'On multiplicative functions which are additive on sums of primes', *Aequationes Math.* **86**(1–2) (2013), 81–89.

[5] J.-H. Fang, 'A characterization of the identity function with equation $f(p + q + r) = f(p) + f(q) + f(r)$', *Combinatorica* **31**(6) (2011), 697–701.

[6] E. Hasanalizade, 'On multiplicative functions additive on Goldbach-type sets', *Aequationes Math.* (2021); doi:10.1007/s00010-021-00849-z.

[7] B. Kim, J. Y. Kim, C. G. Lee and P.-S. Park, 'Multiplicative functions additive on generalized pentagonal numbers', *C. R. Math. Acad. Sci. Paris* **356**(2) (2018), 125–128.

[8] B. Kim, J. Y. Kim, C. G. Lee and P.-S. Park, 'Multiplicative functions additive on polygonal numbers', *Aequationes Math.* **95** (2021), 601–621.

[9] P.-S. Park, 'On k-additive uniqueness of the set of squares for multiplicative functions', *Aequationes Math.* **92**(3) (2018), 487–495.

[10] P.-S. Park, 'Additive uniqueness of PRIMES-1 for multiplicative functions', *Int. J. Number Theory* **16**(6) (2020), 1369–1376.

[11] P.-S. Park, 'Multiplicative functions which are additive on triangular numbers', *Bull. Korean Math. Soc.* **58**(3) (2021), 603–608.

[12] A. Rajasekaran, J. Shallit and T. Smith, 'Additive number theory via automata theory', *Theory Comput. Syst.* **64**(3) (2020), 542–567.

[13] C. A. Spiro, 'Additive uniqueness sets for arithmetic functions', *J. Number Theory* **42**(2) (1992), 232–246.

ELCHIN HASANALIZADE, Department of Mathematics and Computer Science,
University of Lethbridge, 4401 University Drive,
Lethbridge, Alberta T1K 3M4, Canada
e-mail: e.hasanalizade@uleth.ca