

LOWER ESTIMATES FOR THE GROWTH OF THE FOURTH AND THE SECOND PAINLEVÉ TRANSCENDENTS

SHUN SHIMOMURA

*Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku,
Yokohama 223-8522, Japan (shimomur@math.keio.ac.jp)*

(Received 28 May 2003)

Abstract Let $w(z)$ be an arbitrary transcendental solution of the fourth (respectively, second) Painlevé equation. Concerning the frequency of poles in $|z| \leq r$, it is shown that $n(r, w) \gg r^2$ (respectively, $n(r, w) \gg r^{3/2}$), from which the growth estimate $T(r, w) \gg r^2$ (respectively, $T(r, w) \gg r^{3/2}$) immediately follows.

Keywords: Painlevé transcendents; growth order; characteristic function; elliptic functions

2000 *Mathematics subject classification:* Primary 34M55; 34M10

1. Introduction

For a meromorphic function $f(z)$ in \mathbb{C} , denote by $n(r, f)$ the number of poles of $f(z)$ in $|z| \leq r$, each counted according to its multiplicity, and put

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi, \quad \log^+ x := \max\{\log x, 0\},$$
$$N(r, f) := \int_0^r (n(\rho, f) - n(0, f)) \frac{d\rho}{\rho} + n(0, f) \log r.$$

The growth of $f(z)$ is measured by the characteristic function $T(r, f) := m(r, f) + N(r, f)$ or the order $\rho(f) := \limsup_{r \rightarrow \infty} (\log T(r, f) / \log r)$ (for basic facts of the value distribution theory, see, for example, [5] and [9]).

All the solutions of the Painlevé equations

$$w'' = 6w^2 + z, \tag{I}$$

$$w'' = 2w^3 + zw + \alpha, \tag{II}$$

$$w'' = \frac{(w')^2}{2w} + (3/2)w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \tag{IV}$$

(where a prime denotes differentiation with respect to z , and $\alpha, \beta \in \mathbb{C}$) are meromorphic in the whole complex plane (see [6, 11, 14, 15, 19]; see also [4]). It is an interesting and

basic problem to estimate the growth of these solutions. For each solution $w(z)$ of (I), (II) and (IV), the growth is evaluated as $T(r, w) \ll r^{5/2}$, $T(r, w) \ll r^3$ and $T(r, w) \ll r^4$, respectively (see [12, 16]; see also [4]). (We write $\chi(r) \ll \psi(r)$ or $\psi(r) \gg \chi(r)$ if $\chi(r) = O(\psi(r))$ as $r \rightarrow \infty$, and $\chi(r) \asymp \psi(r)$ if $\chi(r) \ll \psi(r) \ll \chi(r)$.) On the other hand, for the first Painlevé transcendents, i.e. the solutions of (I), the lower estimate $\rho(w) \geq 5/2$ was given by [10], and more precise results on $T(r, w)$ have been obtained recently (see [13, 17]). Furthermore, the author [13] proved that, *under the condition* $2\alpha \in \mathbb{Z}$, *every transcendental solution of (II) satisfies* $\rho(w) \geq 3/2$, and Steinmetz [17] obtained the following results.

- (1) *For most classes of transcendental solutions of (II), $\rho(w) \geq 3/2$.*
- (2) *For several classes of transcendental solutions of (IV), $\rho(w) \geq 2$.*

The purpose of this paper is to give an improvement on these results for (II) and (IV), which is stated as follows.

Theorem 1.1. *Let $w(z)$ be an arbitrary transcendental solution of (IV) (respectively, (II)). Then, we have $n(r, w) \gg r^2$ (respectively, $n(r, w) \gg r^{3/2}$).*

Remark 1.2. This theorem implies that every transcendental solution of (IV) (respectively, (II)) satisfies $T(r, w) \gg r^2$ (respectively, $T(r, w) \gg r^{3/2}$).

Remark 1.3. For special values of (α, β) (respectively, α), equation (IV) (respectively, (II)) admits a one-parameter family of solutions such that $T(r, w) \asymp r^2$ (respectively, $T(r, w) \asymp r^{3/2}$) (see, for example, [4, §§ 21, 25]).

In the proof, we employ a method different from those of [13] and [17]. The basic idea is, roughly speaking, to combine properties of $w(z)$ near each pole with asymptotics of Boutroux type [1, 2, 8]; consequently, we get a chain of poles, from which our result immediately follows. Several necessary lemmas are reviewed or proved in § 2. Among them, Lemmas 2.3 and 2.4 are ones concerning a kind of uniformity of a family of elliptic functions, which are essential in constructing a chain of poles. Other lemmas are mainly concerned with the behaviour of $w(z)$ of (IV) around each pole. In § 3, we prove Theorem 1.1 for (IV). To do so, we treat another version of (IV) due to Boutroux [1, 2], whose solution $v(t)$ corresponding to $w(z)$ is asymptotic to an elliptic function in a suitable domain. Using this fact, we find a pair of poles of $v(t)$ that generates a chain of poles of $w(z)$. In the final section, we give an outline of the proof for (II), which is very similar to that for (IV).

Quite recently, after the submission of the original version of this paper, the author was informed that other approaches to our problem are also possible: the same result as above can be obtained through the rescaling method [18]; and the inequalities $\rho(w) \geq 3/2$ (for (II)) and $\rho(w) \geq 2$ (for (IV)) are derived by using certain estimates for logarithmic derivatives [7].

2. Lemmas

2.1. Regular perturbation

Denote by $B(r) \subset \mathbb{C}$ the disc defined by $|z| < r$. Let $F(x, u, \tilde{u}, \varepsilon)$ be a function analytic for $(x, u, \tilde{u}, \varepsilon) \in B(\xi_0) \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times B(\xi_1)$, where ξ_0 and ξ_1 are positive numbers. Consider a differential equation of the form

$$\ddot{u} = F(x, u, \dot{u}, \varepsilon) \tag{2.1}$$

(where the superposed dot denotes differentiation with respect to x). The following lemma is easily derived from a well-known fact on regular perturbation of differential equations (see, for example, [3, Chapter 1, Theorem 8.3]).

Lemma 2.1. *Let $L, R_{0,\pm}$ ($R_{0,-} < R_{0,+}$) and R_1 be given positive numbers. Let γ be an arbitrary curve starting from $x = 0$ with the properties*

- (i) $\gamma \subset B(\xi_0)$;
- (ii) *the length of γ does not exceed L .*

Suppose that (2.1) with $\varepsilon = 0$ admits a solution $\varphi_0(x)$ satisfying $\varphi_0(\gamma) \subset B(R_{0,+}) \setminus B(R_{0,-})$ and $\dot{\varphi}_0(\gamma) \subset B(R_1)$. Let $\varphi(\varepsilon, x)$ be the solution of (2.1) with the initial condition $\varphi(\varepsilon, 0) = \varphi_0(0)$, $\dot{\varphi}(\varepsilon, 0) = \dot{\varphi}_0(0)$. Then, for any $\eta > 0$, there exists a positive number $\varepsilon_0 = \varepsilon_0(L, R_{0,\pm}, R_1, \eta)$ independent of γ such that

$$\sup_{x \in \gamma, |\varepsilon| \leq \varepsilon_0} |\varphi(\varepsilon, x) - \varphi_0(x)| \leq \eta. \tag{2.2}$$

Remark 2.2. In the case where $F(x, u, \tilde{u}, \varepsilon)$ is analytic for $(x, u, \tilde{u}, \varepsilon) \in B(\xi_0) \times \mathbb{C} \times \mathbb{C} \times B(\xi_1)$, supposing $\varphi_0(\gamma) \subset B(R_0)$ and $\dot{\varphi}_0(\gamma) \subset B(R_1)$, we have the same conclusion as above.

2.2. A certain family of elliptic functions

Consider a family of equations

$$\dot{u}^2 = P(\kappa, u) := u^4 + 4u^3 + 4u^2 + \kappa u, \quad \kappa \in \mathbb{C}. \tag{2.3}$$

Let $u = \varpi(\kappa, x)$ be an elliptic (or degenerate elliptic) function satisfying (2.3), and let $J (> 1)$ be a large positive number. Naturally, $\varpi(\kappa, x)$ is not a constant function. Then we have the following lemma.

Lemma 2.3. *For each $q \in \mathbb{Z}$, and for each κ , $|\kappa| \leq J$, the function $\varpi(\kappa, x)$ admits a period $\omega(\kappa)$ satisfying $c^- \leq |\omega(\kappa)| \leq c^+$ and $|\arg \omega(\kappa) - (2q - 1)\pi/4| < \pi/4 + \pi/200$, where $c^\pm = c^\pm(J)$ are some positive numbers independent of κ and q .*

Lemma 2.4. *Suppose that $|\kappa| \leq J$, and that $\zeta_0 = \varpi(\kappa, 0)$, $|\zeta_0| > 10$. Then there exists a curve $\Gamma_\kappa(\zeta_0)$ joining 0 to $\omega(\kappa)$ with the following properties.*

- (i) The length of $\Gamma_\kappa(\zeta_0)$ does not exceed $L(J, |\zeta_0|)$, where $L(J, |\zeta_0|)$ is some positive number independent of κ .
- (ii) $\Lambda_-(J) < |\varpi(\kappa, x)| < \Lambda_+(J, |\zeta_0|)$ along $\Gamma_\kappa(\zeta_0)$, where $\Lambda_-(J)$ and $\Lambda_+(J, |\zeta_0|)$ are some positive numbers independent of κ .

To prove these lemmas, we examine the Riemann surface \mathfrak{R}_κ of $\sqrt{P(\kappa, u)}$ and its cycles, especially in the generic case. It is easy to see that $P(\kappa, u)$ admits multiple zeros if and only if $\kappa = 0$ or $\kappa = 32/27$; then

$$P(0, u) = u^2(u + 2)^2, \quad P(32/27, u) = u(u + 2/3)^2(u + 8/3).$$

We may regard κ as a point in the set

$$E_J := \{\kappa \in \mathcal{R}_0 \mid |\kappa| \leq J, \ |\arg \kappa - \pi| \leq \pi\},$$

where \mathcal{R}_0 denotes the universal covering of $\mathbb{C} \setminus \{0\}$.

(A) Generic case. At first we suppose that

$$\kappa \in E^{(\mu)} := \{\kappa \in \mathcal{R}_0 \mid |\kappa| \geq \mu, \ |\kappa - 32/27| \geq \mu, \ |\arg \kappa - \pi| \leq \pi\}, \quad (2.4)$$

where $\mu = \mu(J)$ is a small positive number which will be fixed later. Then all the zeros of $P(\kappa, u)$ are distinct, and $\varpi(\kappa, x)$ is doubly periodic. The distinct zeros are given by $u = 0, u_j(\kappa)$ ($j = 0, 1, 2$), where

- (i) the $u_j(\kappa)$ are continuous for $\kappa \in E^{(\mu)}$; and
- (ii) $u_j(\kappa) = (-\kappa)^{1/3} e^{2j\pi i/3} (1 + o(1))$ as $|\kappa| \rightarrow \infty$.

To define the Riemann surface and its cycles, suppose, at first, that $-\kappa > 0$, and that $|\kappa|$ is large. Let X_κ^l ($l = 1, 2$) be two copies of $P^1(\mathbb{C}) \setminus (\Sigma_0 \cup \Sigma_1)$ ($P^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$) cut along the segments $\Sigma_0 := [0, u_0(\kappa)]$, $\Sigma_1 := [u_1(\kappa), u_2(\kappa)]$; and denote by $\Sigma_0^{l,-}, \Sigma_0^{l,+}$ (respectively, $\Sigma_1^{l,-}, \Sigma_1^{l,+}$) the lower and the upper (respectively, the left and the right) edges of the cut along Σ_0 (respectively, Σ_1) in each X_κ^l . Gluing $\Sigma_0^{1,\pm}$ to $\Sigma_0^{2,\mp}$, and $\Sigma_1^{1,\pm}$ to $\Sigma_1^{2,\mp}$, we get, from X_κ^l ($l = 1, 2$), the Riemann surface \mathfrak{R}_κ of $\sqrt{P(\kappa, u)}$. Then, define its cycles γ_κ^1 and γ_κ^2 in the standard way.

- (i) γ_κ^1 is a loop lying in X_κ^1 , surrounding the cut $\Sigma_0^{1,\pm}$ in the positive sense, and not surrounding the other one $\Sigma_1^{1,\pm}$.
- (ii) γ_κ^2 is a loop lying in $X_\kappa^1 \cup X_\kappa^2$ such that
 - (a) γ_κ^2 surrounds the branch points 0 and $u_1(\kappa)$ in the positive sense, and
 - (b) $\gamma_\kappa^2 = \gamma_\kappa^{2,1} \cup \gamma_\kappa^{2,2}$, where $\gamma_\kappa^{2,1}$ is a curve in X_κ^1 joining $u_0(\kappa)/2 \in \Sigma_0^{1,+}$ to $(u_1(\kappa) + u_2(\kappa))/2 \in \Sigma_1^{1,-}$, and $\gamma_\kappa^{2,2}$ is a curve in X_κ^2 joining $(u_1(\kappa) + u_2(\kappa))/2 \in \Sigma_1^{2,+}$ to $u_0(\kappa)/2 \in \Sigma_0^{2,-}$.

Furthermore, for every $\kappa \in E_J^{(\mu)} := E_J \cap E^{(\mu)}$, we can choose the Riemann surface \mathfrak{R}_κ with its cycles $\gamma_\kappa^1, \gamma_\kappa^2$ in such a way that $\mathfrak{R}_\kappa, \gamma_\kappa^1$ and γ_κ^2 are consequences of continuous deformation (with respect to κ) of those for special values of κ defined above. Then $\varpi(\kappa, x)$ admits the periods

$$\omega_1(\kappa) := \int_{\gamma_\kappa^1} \frac{du}{\sqrt{P(\kappa, u)}}, \quad \omega_2(\kappa) := \int_{\gamma_\kappa^2} \frac{du}{\sqrt{P(\kappa, u)}}, \tag{2.5}$$

which are continuous for $\kappa \in E_J^{(\mu)}$, and satisfy $\text{Im}(\omega_2(\kappa)/\omega_1(\kappa)) \neq 0$.

(B) Degenerate and almost degenerate cases. If $\kappa = 0$ (respectively, $\kappa = 32/27$), then $\varpi(\kappa, x)$ is simply periodic, and admits the period

$$\omega(0) = 2\pi i \text{Res}(0) = \pi i \quad (\text{respectively, } \omega(32/27) = 2\pi i \text{Res}(-2/3) = \sqrt{3}\pi), \tag{2.6}$$

where $\text{Res}(u_0)$ denotes the residue of $1/\sqrt{P(\kappa, u)}$ at the pole $u = u_0$. Now choose the positive number $\mu = \mu(J)$ in (2.4) so small that, for every $\kappa \in E_J$ satisfying $|\kappa| < \mu$ (respectively, $|\kappa - 32/27| < \mu$), all zeros of $P(\kappa, u)$ remain in either of the discs $|u| < 1/10$, $|u + 2| < 1/10$ (respectively, $|u| < 1/10$, $|u + 2/3| < 1/10$, $|u + 8/3| < 1/10$), and that, for κ with the same condition, a period of $\varpi(\kappa, x)$ given by

$$\omega(\kappa) := \int_{|u|=1} \frac{du}{\sqrt{P(\kappa, u)}} \quad \left(\text{respectively, } \omega(\kappa) := \int_{|u+2/3|=1/3} \frac{du}{\sqrt{P(\kappa, u)}} \right) \tag{2.7}$$

satisfies $|\omega(\kappa) - \omega(0)| < 1/100$ (respectively, $|\omega(\kappa) - \omega(32/27)| < 1/100$).

Proof of Lemma 2.3. We prove this lemma for $q = 1$; the other cases are treated similarly. For $\kappa \in E_J$ satisfying $|\kappa| < \mu$ or $|\kappa - 32/27| < \mu$, the period given by (2.6) and (2.7) satisfies $\pi/2 < |\omega(\kappa)| < 2\pi$, $|\arg \omega(\kappa) - \pi/4| < \pi/4 + \pi/200$. Next suppose that $\kappa \in E_J^{(\mu)} = E_J \cap E^{(\mu)}$. Since $E_J^{(\mu)}$ is compact and simply connected, the periods given by (2.5) have the following properties:

$$\begin{aligned} M_J^- &:= \inf_{\kappa \in E_J^{(\mu)}} \min\{|\omega_1(\kappa)|, |\omega_2(\kappa)|\} > 0, \\ M_J^+ &:= \sup_{\kappa \in E_J^{(\mu)}} \max\{|\omega_1(\kappa)|, |\omega_2(\kappa)|\} < +\infty, \\ \eta_J &:= \inf_{\kappa \in E_J^{(\mu)}} |\sin(\arg \omega_2(\kappa) - \arg \omega_1(\kappa))| > 0. \end{aligned}$$

Consider the set of lattice points $\Omega_\kappa = \{k_1\omega_1(\kappa) + k_2\omega_2(\kappa) \mid (k_1, k_2) \in \mathbb{Z}^2\}$. Since

$$\Omega_\kappa^J := \Omega_\kappa \cap \{x \mid 0 < \text{Re } x < 2M_J^+, 0 < \text{Im } x < 2M_J^+\} \neq \emptyset,$$

we can choose $\omega(\kappa) = k_1^\kappa \omega_1(\kappa) + k_2^\kappa \omega_2(\kappa) \in \Omega_\kappa^J$ satisfying $|\arg \omega(\kappa) - \pi/4| < \pi/4$ and $\eta_J M_J^- \leq |\omega(\kappa)| \leq 2\sqrt{2}M_J^+$. Here $|k_1^\kappa| + |k_2^\kappa| \leq h_J := 8M_J^+ / (\eta_J M_J^-)$ for every $\kappa \in E_J^{(\mu)}$; indeed, even if neither of $\pm\omega_1(\kappa), \pm\omega_2(\kappa)$ is located in Ω_κ^J , the set Ω_κ^J is contained in the parallelogram with sides $[0, \epsilon_1 h_J \omega_1(\kappa)/2], [0, \epsilon_2 h_J \omega_2(\kappa)/2]$ for suitably chosen signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Therefore, for each κ satisfying $|\kappa| \leq J, 0 \leq \arg \kappa < 2\pi$, there exists a period $\omega(\kappa)$ with the desired property. \square

Proof of Lemma 2.4. First consider the case where $\kappa \in E_J$, $|\kappa| < \mu$. To the circle $|u| = 1$, add the segment $[1, |\zeta_0|]$ (from 1 to $|\zeta_0|$), the arc $u = |\zeta_0|e^{i\phi}$, $0 \leq \phi \leq \arg \zeta_0 < 2\pi$ (from $|\zeta_0|$ to ζ_0), and their inverses. Then we obtain a new loop $\gamma_\kappa(\zeta_0)$ starting from and returning to $u = \zeta_0$. The function $y = \varpi(\kappa, x)$ with $\varpi(\kappa, 0) = \zeta_0$ satisfies

$$x = \int_{\gamma_\kappa(\zeta_0)(y)} \frac{du}{\sqrt{P(\kappa, u)}}, \quad \text{in particular, } \omega(\kappa) = \int_{\gamma_\kappa(\zeta_0)} \frac{du}{\sqrt{P(\kappa, u)}} \tag{2.8}$$

(cf. (2.7)), where $\gamma_\kappa(\zeta_0)(y)$ denotes the part of $\gamma_\kappa(\zeta_0)$ from the starting point ζ_0 to $y \in \gamma_\kappa(\zeta_0)$. Let $\Gamma_\kappa(\zeta_0)$ be the image of the loop $\gamma_\kappa(\zeta_0)$ under mapping (2.8). Then, $1 \leq |\varpi(\kappa, x)| \leq |\zeta_0|$ along $\Gamma_\kappa(\zeta_0)$, and the length of $\Gamma_\kappa(\zeta_0)$ does not exceed

$$\int_{\gamma_\kappa(\zeta_0)} \frac{|du|}{|\sqrt{P(\kappa, u)}|} \leq (10/9)^2 \int_{\gamma_\kappa(\zeta_0)} |du| < 10|\zeta_0|,$$

because $|\zeta_0| > 10$, and because the zeros of $P(\kappa, u)$ remain in the disc $|u| < 1/10$ or $|u + 2| < 1/10$ if $|\kappa| < \mu$. For $\kappa \in E_J$, $|\kappa - 32/27| < \mu$, we can choose $\Gamma_\kappa(\zeta_0)$ with the same properties. Finally, suppose that $\kappa \in E_J^{(\mu)} = E_J \cap E^{(\mu)}$, and recall the cycles γ_κ^ι ($\iota = 1, 2$). Note that

$$M_J^* := \inf_{\kappa \in E_J^{(\mu)}} \min\{|u_j(\kappa)|, |u_j(\kappa) - u_{j'}(\kappa)| \mid 0 \leq j \leq 2, 0 \leq j' \leq 2, j \neq j'\} > 0,$$

and that $|u_j(\kappa)| \leq 2J^{1/3}$ ($j = 0, 1, 2$) for $|\kappa| \leq J$, since J is large. Hence, we may suppose that

$$A'_-(J) < |u| < 2J, \quad |u - u_j(\kappa)| > A'_-(J) \quad (j = 0, 1, 2) \quad \text{for } u \in \gamma_\kappa^\iota, \tag{2.9}$$

where $A'_-(J) := M_J^*/4$. Since the length of γ_κ^ι is a continuous function of $\kappa \in E_J^{(\mu)}$, it does not exceed some positive number $L'(J)$ independent of κ . Let $\gamma_\kappa = k_1^\kappa \gamma_\kappa^1 + k_2^\kappa \gamma_\kappa^2$ be the composite cycle corresponding to the choice of $\omega(\kappa)$, where $|k_1^\kappa| + |k_2^\kappa| \leq h_J$ (cf. the proof of Lemma 2.3). The length of γ_κ does not exceed $h_J L'(J)$. Recall the cut planes X_κ^l ($l = 1, 2$). For each l , denote by $\zeta_0^{(l)} \in X_\kappa^l$ the point corresponding to ζ_0 ; in the case where $\zeta_0 \in \Sigma_0$ (respectively, Σ_1), we regard $\zeta_0^{(l)}$ as a point in $\Sigma_0^{l,-}$ (respectively, $\Sigma_1^{l,-}$). For each κ , let $R_\kappa^l(\zeta_0) \subset \mathfrak{R}_\kappa$ be the ray starting from 0 and passing through $\zeta_0^{(l)}$ which is drawn in such a way that, when traversing a cut, the ray transfers to the other plane, and that $R_\kappa^1(\zeta_0) \cap R_\kappa^2(\zeta_0) \subset \{0, u_j(\kappa) \mid j = 0, 1, 2\}$. Then, either of $R_\kappa^l(\zeta_0)$ ($l = 1, 2$) intersects γ_κ . Suppose, for example, that $R_\kappa^1(\zeta_0) \cap \gamma_\kappa \ni \zeta^\kappa$, and take the segment $[\zeta_0, \zeta^\kappa] \subset R_\kappa^1(\zeta_0)$ starting from $\zeta_0^{(1)}$ and ending at $\zeta^\kappa \in \gamma_\kappa$. Adding $[\zeta_0, \zeta^\kappa]$ and its inverse to γ_κ , we obtain a loop $\gamma_\kappa(\zeta_0)$. Then, for each $\kappa \in E_J^{(\mu)}$, it has the following properties.

- (i) $\gamma_\kappa(\zeta_0)$ is homotopic to γ_κ in \mathfrak{R}_κ .
- (ii) $\gamma_\kappa(\zeta_0)$ starts from and returns to $\zeta_0^{(1)} \in \mathfrak{R}_\kappa$.
- (iii) $A'_-(J) < |u| < A'_+(J, |\zeta_0|)$ for $u \in \gamma_\kappa(\zeta_0)$, where $A'_+(J, |\zeta_0|) := 2J + |\zeta_0|$.

Consider (2.8) corresponding to $\gamma_\kappa(\zeta_0)$ of this case, and let $\Gamma_\kappa(\zeta_0)$ be the image of $\gamma_\kappa(\zeta_0)$ under it. Then, by (iii), we have $A'_-(J) < |\varpi(\kappa, x)| < A'_+(J, |\zeta_0|)$ along $\Gamma_\kappa(\zeta_0)$. Furthermore, the length of $\Gamma_\kappa(\zeta_0)$ does not exceed

$$\int_{\gamma_\kappa(\zeta_0)} \frac{|du|}{|\sqrt{P(\kappa, u)}|} = L_1 + 2L_2,$$

where

$$L_1 = \int_{\gamma_\kappa} \frac{|du|}{|\sqrt{P(\kappa, u)}|}, \quad L_2 = \int_{[\zeta_0, \zeta^\kappa]} \frac{|du|}{|\sqrt{P(\kappa, u)}|}.$$

By (2.9), we have $L_1 \leq h_J L'(J) A'_-(J)^{-2}$. If $[\zeta_0, \zeta^\kappa] \cap \{u \mid |u| \leq A'_-(J)\} = \emptyset$, and if $[\zeta_0, \zeta^\kappa] \cap \{u \mid |u - u_j(\kappa)| \leq A'_-(J)\} = \emptyset$ ($j = 0, 1, 2$), then $L_2 \leq |\zeta_0 - \zeta^\kappa| A'_-(J)^{-2} \leq (|\zeta_0| + 2J) A'_-(J)^{-2}$. To treat the complementary case, suppose, for example, that $Z(\zeta_0, \kappa) := [\zeta_0, \zeta^\kappa] \cap \{u \mid |u| \leq A'_-(J)\} \neq \emptyset$. Observing that $|u_j(\kappa)| \geq 4A'_-(J)$ ($j = 0, 1, 2$), we have

$$\int_{Z(\zeta_0, \kappa)} \frac{|du|}{|\sqrt{P(\kappa, u)}|} \leq \frac{2}{A'_-(J)^{3/2}} \int_0^{A'_-(J)} \frac{d\lambda}{\lambda^{1/2}} = \frac{4}{A'_-(J)}.$$

Hence, $L_2 \leq (|\zeta_0| + 2J) A'_-(J)^{-2} + 16A'_-(J)^{-1}$, which completes the proof. □

2.3. Behaviour of a solution of (IV) near a pole

Let $w(z)$ be an arbitrary transcendental solution of (IV), and denote by

$$\Psi(z) := \frac{w'(z)^2}{w(z)} - w(z)^3 - 4zw(z)^2 - 4(z^2 - \alpha)w(z) + \frac{2\beta}{w(z)} \tag{2.10}$$

the auxiliary function satisfying

$$\Psi'(z) = -4w(z)^2 - 8zw(z). \tag{2.11}$$

Take a circle $|z| = r_0 > 10$ on which $w(z) \neq 0, -1, \infty$. Let K be a positive number such that

$$K > 2r_0^{-1}M(w; r_0) + M(\Psi; r_0) + 10^4 + 4|\alpha| + 2|\beta|, \tag{2.12}$$

where $M(f; r) = \max_{|z|=r} |f(z)|$, and put

$$D_K := \{z \mid |w(z)| > K|z|\}, \quad R(\theta) := \{z = re^{i\theta} \mid r \geq r_0\}.$$

Now we recall the following lemma (cf. [12, Proposition 5.3], and see also §2.4).

Lemma 2.5. *Suppose that a , $|a| > r_0$, has the following properties:*

- (i) $|w(a)| = K|a|$; and
- (ii) for a sufficiently small positive number δ_a ,

$$\left. \begin{aligned} R(\arg a) \cap \{z \mid |a| - \delta_a < |z| \leq |a|\} &\subset \mathbb{C} \setminus D_K, \\ R(\arg a) \cap \{z \mid |a| < |z| < |a| + \delta_a\} &\subset D_K; \end{aligned} \right\} \tag{2.13}$$

and

- (iii) $|\Psi(a)| \leq 7K^2|a|^3$.

Then there exist a pole σ_a of $w(z)$ and the disc $\Delta(\sigma_a) : |z - \sigma_a| < (11/10)K^{-1}|\sigma_a|^{-1}$ satisfying

- (a) $|a| - 1/10 < |\sigma_a| < |a| + 1/10$;
- (b) $a \in \Delta(\sigma_a)$;
- (c) $w(z)$ is analytic in $\Delta(\sigma_a) \setminus \{\sigma_a\}$; and
- (d) $|w(z)| < K|z|$ on the circle $\partial\Delta(\sigma_a)$.

By the same argument as in [12, § 3.2] (see also § 2.4), we have the following lemma.

Lemma 2.6. *Let $a', a'' \in R(\arg a') = R(\arg a'')$ be points satisfying (i)–(iii) above, and let $(\sigma_{a'}, \Delta(\sigma_{a'}))$, $(\sigma_{a''}, \Delta(\sigma_{a''}))$ be the corresponding pairs of pole and disc given by Lemma 2.5. Then one of the following cases occurs:*

- (i) $\Delta(\sigma_{a'}) \cap \Delta(\sigma_{a''}) = \emptyset$; or
- (ii) $\Delta(\sigma_{a'}) = \Delta(\sigma_{a''})$.

From these lemmas, we have the following lemma.

Lemma 2.7. *Suppose that $a, |a| > r_0$ satisfies (i) and (ii) of Lemma 2.5. Then there exist a point $a_* \in R(\arg a)$, $|a_*| \leq |a|$, a curve $C_0(a_*) \subset \{z \mid |z| \geq r_0\}$ and a pole σ_{a_*} of $w(z)$ with the properties:*

- (a) $C_0(a_*)$ joins a point z_a , $|z_a| = r_0$, to a_* , and its length does not exceed $\pi|a_*|/2$;
- (b) $|w(z)| \leq K|z|$ along $C_0(a_*)$, and $|w(a_*)| = K|a_*|$;
- (c) $a, a_* \in \Delta(\sigma_{a_*})$;
- (d) $w(z)$ is analytic in $\Delta(\sigma_{a_*}) \setminus \{\sigma_{a_*}\}$; and
- (e) $|w(z)| < K|z|$ on the circle $\partial\Delta(\sigma_{a_*})$.

Proof. Draw the ray $R_a = R(\arg a)$. By (2.12), at the starting point $z_a = r_0 e^{i \arg a}$, we have the inequalities $|w(z_a)| < Kr_0/2$, $|\Psi(z_a)| < K$; and hence $z_a \notin D_K$. Start from z_a and proceed along R_a . There exists a point a_1 satisfying (i) and (ii) of Lemma 2.5 (with a_1 instead of a) and $[z_a, a_1] \subset \mathbb{C} \setminus D_K$. Then, by (2.11),

$$\begin{aligned} |\Psi(a_1)| &\leq |\Psi(z_a)| + 4 \int_{[z_a, a_1]} (|w(s)|^2 + 2|sw(s)|) |ds| \\ &\leq K + 4(K^2|a_1|^2 + 2K|a_1|^2)|a_1| \\ &\leq 4K^2|a_1|^3(1 + 2K^{-1} + K^{-1}|a_1|^{-3}) \\ &\leq 5K^2|a_1|^3, \end{aligned}$$

which implies that (iii) is also fulfilled. By Lemma 2.5, there exist a pole σ_{a_1} and the disc $\Delta(\sigma_{a_1}) \ni a_1$. If $a = a_1$ or if $a \in \Delta(\sigma_{a_1})$, then we get the point $a_* =$

a_1 and the segment $C_0(a_*) = [z_a, a_*]$ satisfying (a)–(e). If $a \neq a_1$, and if $a \notin \Delta(\sigma_{a_1})$, then we take the shorter arc C_1 of the circle $\partial\Delta(\sigma_{a_1})$ from b_1^- to b_1^+ , where $b_1^-, b_1^+ \in \partial\Delta(\sigma_{a_1}) \cap (R_a \cup [0, a])$, $|b_1^-| < |b_1^+|$. Replacing the segment $[b_1^-, b_1^+] \subset R_a$ by C_1 , we get the path $R_a^{(1)} = (R_a \setminus [b_1^-, b_1^+]) \cup C_1$. (When $z_a \in \Delta(\sigma_{a_1})$, we put

$$R_a^{(1)} = ((R_a \setminus [b_1^-, b_1^+]) \cup C_1) \cap \{z \mid |z| \geq r_0\},$$

which starts from a point z'_a , $|z'_a| = r_0$.) Let us restart from b_1^+ and proceed along $R_a^{(1)}$ until we meet a point a_2 satisfying (i) and (ii) of Lemma 2.5 (with a_2). Denote by $R_a^{(1)}[a_2]$ the part of $R_a^{(1)}$ from z_a (or z'_a) to a_2 . Then the length of $R_a^{(1)}[a_2]$ does not exceed $\pi|a_2|/2$, and $|w(z)| \leq K|z| \leq K|a_2|$ on $R_a^{(1)}[a_2]$. Hence

$$\begin{aligned} |\Psi(a_2)| &\leq |\Psi(z_a)| + 4 \int_{R_a^{(1)}[a_2]} (|w(s)|^2 + 2|sw(s)|) |ds| \\ &\leq K + 2(K^2|a_2|^2 + 2K|a_2|^2)\pi|a_2| \\ &\leq 2\pi K^2|a_2|^3(1 + 2K^{-1} + K^{-1}|a_2|^{-3}) \\ &\leq 7K^2|a_2|^3. \end{aligned}$$

Applying Lemma 2.5, we get a pole σ_{a_2} and the disc $\Delta(\sigma_{a_2}) \ni a_2$. By Lemma 2.6, $\Delta(\sigma_{a_1}) \cap \Delta(\sigma_{a_2}) = \emptyset$. If $a \notin \Delta(\sigma_{a_2})$, we make the same replacement as that for $\partial\Delta(\sigma_{a_1})$, and obtain a new path $R_a^{(2)}$, which is a modification of $R_a^{(1)}$. We repeat this procedure. As far as $|a_j| < |a|$, the area of $\Delta(\sigma_{a_j})$ is not less than $\pi K^{-2}|\sigma_{a_j}|^{-2} \geq \pi K^{-2}(|a| + 1)^{-2}$. By this fact and Lemmas 2.5 and 2.6, we arrive at the case of $a = a_p$ or of $a \in \Delta(\sigma_{a_p})$ for some p , and get the point $a_* = a_p \in R_a$ and a curve $C_0(a_*)$ terminating in a_* with length not exceeding $\pi|a_*|/2$. Clearly, $a_*, \sigma_{a_*}, C_0(a_*)$ and $\Delta(\sigma_{a_*})$ fulfil (a)–(e). \square

Lemma 2.8. *Suppose that a , $|a| > r_0$, satisfies $|w(a)| = K|a|$. Then there exist a pole σ_a^\sharp of $w(z)$ and the disc $\Delta(\sigma_a^\sharp)$ such that $a \in \Delta(\sigma_a^\sharp)$.*

Proof. If (2.13) is fulfilled, then, by Lemma 2.7, the conclusion with $\sigma_a^\sharp = \sigma_{a_*}$ immediately follows. If (2.13) is not fulfilled, then there exist two possibilities:

- (1) $R(\arg a) \cap \{z \mid |a| - \delta_a < |z| < |a|\} \subset D_K$; and
- (2) neither (1) nor (2.13) is valid.

Note that $r_0 e^{i \arg a} \in R(\arg a) \setminus (D_K \cup \partial D_K)$. In case (1), we can take the segment $[a', a] \subset R(\arg a)$, $r_0 < |a'| < |a|$ such that $|w(z)| \geq K|z|$ along $[a', a]$, and such that (2.13) with a' (instead of a) is valid. In case (2), we can take an arc $\gamma(a, a'') \subset \partial D_K$ joining a to a'' such that (2.13) with a'' is valid. In both cases, by Lemma 2.7, there exist a point a_*^\sharp , a pole $\sigma_a^\sharp := \sigma_{a_*^\sharp}$ and the disc $\Delta(\sigma_a^\sharp)$ such that $a' \in \Delta(\sigma_a^\sharp)$ (respectively, $a'' \in \Delta(\sigma_a^\sharp)$). By (e) of Lemma 2.7, we have $[a', a] \subset \Delta(\sigma_a^\sharp)$ (respectively, $\gamma(a, a'') \subset \Delta(\sigma_a^\sharp)$), and hence $a \in \Delta(\sigma_a^\sharp)$, which completes the proof. \square

Lemma 2.9. *Let σ be an arbitrary pole of $w(z)$ such that $|\sigma| > 2r_0$. Then there exist a point $a^\sigma \in \Delta(\sigma)$ and a curve $C_0(a^\sigma) \subset \{z \mid |z| \geq r_0\}$ with the following properties:*

Lemma 2.10. Let $\gamma(z) \subset U(a) := \{z \mid |z - a| < 2K^{-1}|a|^{-1}\}$ be a path starting from a and ending at z with the following properties:

- (i) $\|\gamma(z)\| := \int_{\gamma(z)} |ds| \leq 1/10$; and
- (ii) $|w(s)|$ is monotone increasing along $\gamma(z)$.

Then $|\Phi(z)| < 9K^2|z|^3$.

To prove this lemma, we first note that $|w(s)| \geq |w(a)| = K|a| \geq 10^5$ for $s \in \gamma(z)$. By the condition $\|\gamma(z)\| \leq 1/10$, we have $0.99|a| \leq |s| \leq 1.01|a|$ and $|E(a, s)^{\pm 1}| \leq \exp(10^{-4}) < 1.0002$ along $\gamma(z)$. Using these inequalities and (2.16), we can derive the conclusion $|\Phi(z)| < 9K^2|z|^3$ of this lemma.

Putting $y(z) = 1/w(z)$, we write (2.15) in the form

$$y'(z)^2 - \frac{4y(z)^2 y'(z)}{y(z) + 1} - 1 - 4zy(z) - 4(z^2 - \alpha)y(z)^2 + 2\beta y(z)^4 - y(z)^3 \Phi(z) = 0,$$

which implies that

$$y'(z) = \pm(1 + F(z))^{1/2} + \frac{2y(z)^2}{y(z) + 1} \tag{2.17}$$

with

$$F(z) := 4zy(z) + 4(z^2 - \alpha)y(z)^2 - 2\beta y(z)^4 + y(z)^3 \Phi(z) + \frac{4y(z)^4}{(y(z) + 1)^2}. \tag{2.18}$$

By [12, Lemma 3.4], we choose a steepest descent path γ_0 starting from a and ending at a_* in such a way that a_* satisfies $y(a_*) = 0$ or $a_* \in \partial U(a)$, and that, along γ_0 , $d|y(z)|/d\lambda = -|dy(z)/d\lambda|$ with $\lambda := \|\gamma_0(z)\|$. Here $\gamma_0(z)$ denotes the part of γ_0 from a to z . As long as $\|\gamma_0(z)\| \leq 1/10$, by Lemma 2.10, we have $|F(z)| \leq 2 \times 10^{-3}$, which implies that $|y'(z) - 1| < 3 \times 10^{-3}$ or that $|y'(z) + 1| < 3 \times 10^{-3}$. Using these facts, by the same argument as in the proof of [12, Lemma 3.5], we obtain the following.

Lemma 2.11. There exists a pole σ_a of $w(z)$ satisfying $|\sigma_a - a| \leq 1.01K^{-1}|a|^{-1} < 1/10$ and $|\Phi(\sigma_a)| < 9K^2|\sigma_a|^3$.

Instead of [12, Lemma 3.6], we have the following.

Lemma 2.12. In the disc $|z - \sigma_a| < (6/5)K^{-1}|\sigma_a|^{-1}$,

$$|\Phi(z)| < 10K^2|z|^3, \tag{2.19}$$

$$|y'(z) - 1| < 10^{-2} \quad \text{or} \quad |y'(z) + 1| < 10^{-2}, \tag{2.20}$$

$$0.99|z - \sigma_a| \leq |y(z)| \leq 1.01|z - \sigma_a|. \tag{2.21}$$

By Lemma 2.11, we have $|a| - 1/10 < |\sigma_a| < |a| + 1/10$, implying (a) of Lemma 2.5. Observing that $|\sigma_a/a| < (|a| + 1/10)/|a| \leq 1.01$, we have $|\sigma_a - a| \leq 1.01K^{-1}|a|^{-1} \leq 1.01^2K^{-1}|\sigma_a|^{-1} < (11/10)K^{-1}|\sigma_a|^{-1}$, implying (b). Assertion (c) immediately follows from (2.21). Note that $|z/\sigma_a - 1| < 10^{-2}$ on $\partial\Delta(\sigma_a)$. Hence, by (2.21), $|y(z)| \geq 0.99 \times 1.1K^{-1}|\sigma_a|^{-1} \geq 1.07K^{-1}|z|^{-1}$ on $\partial\Delta(\sigma_a)$, which implies (d). In this way we have proved Lemma 2.5.

B. To prove Lemma 2.6, suppose that $\Delta(\sigma_{a'}) \cap \Delta(\sigma_{a''}) \neq \emptyset$. Then either of the two inequalities of (2.20) is valid in $\Delta(\sigma_{a'}) \cup \Delta(\sigma_{a''})$; and, at the point $z_* \in \Delta(\sigma_{a'}) \cap \Delta(\sigma_{a''}) \cap [\sigma_{a'}, \sigma_{a''}]$, we have, for example,

$$|y(z_*) - (z_* - \sigma_{a'})| \leq 10^{-2}|z_* - \sigma_{a'}|, \quad |y(z_*) - (z_* - \sigma_{a''})| \leq 10^{-2}|z_* - \sigma_{a''}|.$$

This implies

$$|\sigma_{a'} - \sigma_{a''}| \leq 10^{-2}(|\sigma_{a''} - z_*| + |z_* - \sigma_{a'}|) = 10^{-2}|\sigma_{a'} - \sigma_{a''}|,$$

and hence $\sigma_{a'} = \sigma_{a''}$, from which Lemma 2.6 immediately follows.

3. Proof of Theorem 1.1 for (IV)

Let $w(z)$ be an arbitrary transcendental solution of (IV), and let $\Psi(z)$ be the related auxiliary function defined by (2.10). Recall the circle $|z| = r_0 > 10$ on which $w(z) \neq 0, -1, \infty$. In what follows, we suppose that

$$K - 1 > 2r_0^{-1}M(w; r_0) + M(\Psi; r_0) + 10^4 + 4|\alpha| + 2|\beta| \quad (3.1)$$

(instead of (2.12)), which means that K itself also satisfies (2.12).

3.1. Equation of Boutroux type

Make the change of variables:

$$t = z^2/2, \quad v(t) = z^{-1}w(z). \quad (3.2)$$

Then $v = v(t)$ satisfies an equation of the form

$$2v_{tt} = \frac{v_t^2}{v} + 3v^3 + 8v^2 + 4v - 2t^{-1}v_t - 2\alpha t^{-1}v + (1/4)t^{-2}v + \frac{\beta t^{-2}}{2v} \quad (3.3)$$

(where the subscript 't' denotes differentiation with respect to t). Multiplying both sides by v_t/v , we see that the function

$$V(t) := \frac{v_t(t)^2}{v(t)} - v(t)^3 - 4v(t)^2 - 4v(t) + 2\alpha t^{-1}v(t) - (1/4)t^{-2}v(t) + \frac{\beta t^{-2}}{2v(t)}$$

satisfies the first-order equation

$$V_t + 2t^{-1}V = 2t^{-1}(-v(t)^3 - 4v(t)^2 - 4v(t) + \alpha t^{-1}v(t)).$$

By this fact, we have

$$V(t) = t_0^2 V(t_0) t^{-2} - 2t^{-2} \int_{C(t_0, t)} s(v(s)^3 + 4v(s)^2 + 4v(s) - \alpha s^{-1}v(s)) ds, \quad (3.4)$$

where $C(t_0, t)$ denotes a path joining t_0 to t .

Lemma 3.1. *Let σ be an arbitrary pole of $w(z)$ such that $|\sigma| > 2r_0$, and let a^σ be the point given by Lemma 2.9. Then the corresponding point $t = b^\sigma = (a^\sigma)^2/2$ satisfies $|b^\sigma - \sigma^2/2| < 2/K$, $|v(b^\sigma)| = K$ and*

$$|V(b^\sigma)| \leq J_0, \quad |v_t(b^\sigma)^2/v(b^\sigma) - v(b^\sigma)^3 - 4v(b^\sigma)^2 - 4v(b^\sigma)| \leq J_0, \quad (3.5)$$

where $J_0 = J_0(K)$ is some positive number independent of σ .

Proof. Let $\tilde{C}_0(b^\sigma)$ be the image of $C_0(a^\sigma)$ (cf. Lemma 2.9) under the mapping $t = z^2/2$, and let $b_0, |b_0| = r_0^2/2$, be the starting point of $\tilde{C}_0(b^\sigma)$. By Lemma 2.9 and (3.4) with $C(t_0, t) = \tilde{C}_0(b^\sigma)$, observing that $|v(t)| \leq K$ along $\tilde{C}_0(b^\sigma)$, we have

$$\begin{aligned} |V(b^\sigma)| &\leq |V(b_0)| + 2|b^\sigma|^{-2} \left| \int_{\tilde{C}_0(b^\sigma)} s(v(s)^3 + 4v(s)^2 + 4v(s) - \alpha s^{-1}v(s)) ds \right| \\ &\leq |V(b_0)| + 2|b^\sigma|^{-2} \int_{\tilde{C}_0(b^\sigma)} (K^3 + 4K^2 + 4K + |\alpha|K)|s| ds \\ &\leq |V(b_0)| + |b^\sigma|^{-2}(2 + |\alpha|)K^3 \int_{C_0(a^\sigma)} |z|^3 |dz| \\ &\leq |V(b_0)| + 4|a^\sigma|^{-4}(2 + |\alpha|)K^3(\pi|a^\sigma|/2)^4 \\ &\leq J'_0(K) := |V(b_0)| + \pi^4(2 + |\alpha|)K^3. \end{aligned}$$

Since $|V(b_0)|$ depends only on the initial values of $w(z)$ on the circle $|z| = r_0$, the constant $J'_0(K)$ is independent of σ . Taking $J_0 = J_0(K) = J'_0(K) + (2|\alpha| + 1)K + |\beta|$, we get (3.5). Since $a^\sigma \in \Delta(\sigma)$, we have

$$|b^\sigma - \sigma^2/2| = |(\sigma^2 - (a^\sigma)^2)/2| = |\sigma + a^\sigma| |\sigma - a^\sigma|/2 < (11/10)(|\sigma| + 1)/(|\sigma|K) < 2/K,$$

which completes the proof. □

Lemma 3.2. *Suppose that $b, |b| > r_0^2$, satisfies $|v(b)| = K$. Then there exists a pole τ_b of $v(t)$ such that $|b - \tau_b| < 2/K$.*

Proof. Let $z = a$ be a point satisfying $a^2/2 = b, |w(a)| = K|a|$. By Lemma 2.8, there exists a pole σ of $w(z)$ such that $a \in \Delta(\sigma)$. Then, $t = \tau_b = \sigma^2/2$ is a pole of $v(t)$ such that $|b - \tau_b| = |a^2 - \sigma^2|/2 = |a - \sigma| |a + \sigma|/2 < (11/10)(|\sigma| + 1)/(K|\sigma|) < 2/K$, which completes the proof. □

3.2. Construction of a pair of poles

Consider the sectors $S_q : |\arg z - (2q - 1)\pi/8| < \pi/8 + \delta$ ($-3 \leq q \leq 4, \delta > 0$), the union of which covers the full neighbourhood of $z = \infty$. The image of S_q under the mapping $t = z^2/2$ is the sector $T_q : |\arg t - (2q - 1)\pi/4| < \pi/4 + 2\delta$. Note that every non-zero pole of $w(z)$ belongs to at least one of the sectors S_q ($-3 \leq q \leq 4$).

Lemma 3.3. *Let τ be an arbitrary pole of $v(t)$ satisfying $|\arg \tau - \pi/4| < \pi/4 + \pi/100$ and $|\tau| > \rho_\infty > 4r_0^2$, where ρ_∞ is a sufficiently large positive number. Then there exists another pole τ' of $v(t)$ such that*

$$0 < |\tau' - \tau| < c_*, \quad |\arg(\tau' - \tau) - \pi/4| < \pi/4 + \pi/100, \tag{3.6}$$

where c_* is some positive number independent of τ and τ' .

Remark 3.4. This lemma is concerned with a pole of $v(t)$ in T_1 , whose corresponding pole of $w(z)$ belongs to S_1 . For poles in the other sectors T_q ($q \neq 1$), we can show analogous facts.

Proof. Suppose that τ satisfies $|\arg \tau - \pi/4| < \pi/4 + \pi/100$ and $|\tau| > \rho_\infty$, where ρ_∞ will be chosen afterwards. Let $\sigma = (2\tau)^{1/2}$ be the corresponding pole of $w(z)$, which satisfies $|\sigma| > 2\sqrt{2}r_0$. By Lemmas 2.9 and 3.1, there exists a point $b := b^{(2\tau)^{1/2}}$ satisfying

$$|b - \tau| < 2/K, \quad |v(b)| = K, \quad |\kappa_b| \leq J_0 = J_0(K), \tag{3.7}$$

where

$$\kappa_b := v_t(b)^2/v(b) - v(b)^3 - 4v(b)^2 - 4v(b). \tag{3.8}$$

In (3.3), we put $t = b + x$. Then $u = u(b, x) = v(b + x)$ is a solution of

$$2\ddot{u} = \frac{\dot{u}^2}{u} + 3u^3 + 8u^2 + 4u - \frac{2b^{-1}(\dot{u} + \alpha u)}{1 + b^{-1}x} + \frac{b^{-2}(u + 2\beta/u)}{4(1 + b^{-1}x)^2} \tag{3.9}$$

(where the superposed dot denotes differentiation with respect to x), with the initial condition $u(b, 0) = v(b)$, $\dot{u}(b, 0) = v_t(b)$. If $|b|$ is large, (3.9) is regarded as a perturbed equation of

$$2\ddot{u} = \frac{\dot{u}^2}{u} + 3u^3 + 8u^2 + 4u. \tag{3.10}$$

Let $u_0(x)$ be a solution of (3.10) satisfying $u_0(0) = u(b, 0) = v(b)$, $\dot{u}_0(0) = \dot{u}(b, 0) = v_t(b)$. Since

$$v_t(t) = -z^{-2}(z^{-1}w(z)) - (z^{-1}w(z))^2(1/w(z))' = -z^{-2}v(t) - v(t)^2y'(z),$$

by (2.20), we have $|v_t(b)| \geq K^2/2 > 0$, guaranteeing that $u_0(x)$ is not a constant function. This is also a solution of (2.3) with $\kappa = \kappa_b$ (cf. (3.8)), and we write $u_0(x) = \varpi(\kappa_b, x)$. To apply Lemma 2.1, we choose constants successively by the following process.

- (i) Let J_0 be the constant of (3.7) given by Lemma 3.1. In Lemma 2.3, we put $J = J_0$. Then, by (3.7) and Lemma 2.3, the elliptic function $u_0(x) = \varpi(\kappa_b, x)$ admits a period $\omega(\kappa_b)$ such that

$$c_0^- \leq |\omega(\kappa_b)| \leq c_0^+, \quad |\arg \omega(\kappa_b) - \pi/4| < \pi/4 + \pi/200, \tag{3.11}$$

with $c_0^\pm := c^\pm(J_0)$. We put

$$c_* := c_0^+ \nu_0 + 1, \tag{3.12}$$

where ν_0 is a positive integer such that

$$6/K < c_0^- \nu_0 \sin(\pi/200). \tag{3.13}$$

(ii) In Lemma 2.4, put $J = J_0$, $\zeta_0 = \varpi(\kappa_b, 0) = u_0(0) = v(b)$ implying $|\zeta_0| = K$ (cf. (3.7)). Then, by Lemma 2.4 and the periodicity of $\varpi(\kappa_b, x)$, there exists a curve $\Gamma_*(\kappa_b)$ joining 0 to $\nu_0\omega(\kappa_b)$ such that the length of it does not exceed ν_0L_0 and such that

$$\Lambda_{0,-} < |\varpi(\kappa_b, x)| < \Lambda_{0,+} \quad \text{along } \Gamma_*(\kappa_b) \tag{3.14}$$

for $L_0 := L(J_0, K)$, $\Lambda_{0,-} := \Lambda_-(J_0)$ and $\Lambda_{0,+} := \Lambda_+(J_0, K)$.

(iii) Put $\rho_\infty = 2r_0^2 + \nu_0L_0 + 2$. Since $|b| > |\tau| - 1 > \rho_\infty - 1$ (cf. (3.7)),

$$\{t \mid |t - b| < \nu_0L_0 + 1\} \subset \{t \mid |t| > 2r_0^2\}. \tag{3.15}$$

We apply Lemma 2.1 with $\varepsilon = b^{-1}$ to (3.9). In its application, we put $\varphi_0(x) = u_0(x) = \varpi(\kappa_b, x)$, $\varphi(\varepsilon, x) = u(b, x)$, and choose the curve and the constants as follows:

$$\gamma = \Gamma_*(\kappa_b), \quad L = \nu_0L_0, \quad \xi_0 = \nu_0L_0 + 1, \quad R_{0,\pm} = \Lambda_{0,\pm}, \quad \xi_1 = (\nu_0L_0 + 1)^{-1/2}$$

and

$$R_1 = A_1 := 2(J_0 + \Lambda_{0,+}^3 + 4\Lambda_{0,+}^2 + 4\Lambda_{0,+})^{1/2}A_{0,+}^{1/2},$$

because $|\dot{\varpi}(\kappa_b, x)| = |P(\kappa_b, \varpi(\kappa_b, x))|^{1/2} < A_1$ along $\Gamma_*(\kappa_b)$ (cf. (3.14)). Note that all the constants above are independent of κ_b , $|\kappa_b| \leq J_0$, and that all the suppositions of Lemma 2.1 are fulfilled. Then, for $|b| > \varepsilon_0^{-1}$, we have

$$|u(b, \nu_0\omega(\kappa_b)) - \varpi(\kappa_b, \nu_0\omega(\kappa_b))| = |v(b + \nu_0\omega(\kappa_b)) - v(b)| \leq 1, \tag{3.16}$$

where $\varepsilon_0 = \varepsilon_0(\nu_0L_0, \Lambda_{0,\pm}, A_1, 1)$ is independent of τ ; if necessary, we re-choose ρ_∞ so large that $\rho_\infty > \varepsilon_0^{-1} + 1$. By (3.16), $K' = |v(b + \nu_0\omega(\kappa_b))| \geq |v(b)| - 1 = K - 1$. By (3.1), (3.15) and Lemma 3.2 with K' , we see that the point $b' := b + \nu_0\omega(\kappa_b)$ belongs to the domain $|t| > 2r_0^2$, and that there exists a pole τ' of $v(t)$ satisfying $|b' - \tau'| < 2/(K - 1) < 3/K$; this, together with (3.7), yields $|b - \tau| + |b' - \tau'| < 5/K$. Hence, by (3.11) and (3.13),

$$\begin{aligned} |\arg(\tau' - \tau) - \pi/4| &= |\arg(b' - b + (\tau' - b') - (\tau - b)) - \pi/4| \\ &\leq |\arg(b' - b) - \pi/4| + \sin^{-1}((|\tau' - b'| + |\tau - b|)/|b' - b|) \\ &\leq \pi/4 + \pi/200 + \sin^{-1}((5/K)/(c_0^- \nu_0)) \\ &< \pi/4 + \pi/100, \end{aligned}$$

and, by (3.12) and (3.13),

$$\begin{aligned} |\tau' - \tau| &\leq |\nu_0\omega(\kappa_b)| + |b - \tau| + |b' - \tau'| < c_0^+ \nu_0 + 5/K < c_*, \\ |\tau' - \tau| &\geq |b - b'| - |\tau' - b'| - |\tau - b| \geq c_0^- \nu_0 - 5/K > 6/K - 5/K = 1/K. \end{aligned}$$

This completes the proof of Lemma 3.3. □

3.3. Completion of the proof of Theorem 1.1

Fix the positive number ρ_∞ of Lemma 3.3. Consider the domains

$$D_k := \{t \mid |\arg(t - \rho_\infty e^{(1+2k)\pi i/4}) - (1 + 2k)\pi/4| < \pi/4 + \pi/100\} \subset \{t \mid |t| > \rho_\infty\}$$

($k = 0, 1, 2, 3$) in the t -plane, the union of which covers the full neighbourhood of $t = \infty$. By the transcendency of $w(z)$, there exists a pole σ of $w(z)$ such that the corresponding pole $\tau_0 = \sigma^2/2$ of $v(t)$ belongs to $\bigcup_{k=0}^3 D_k$ (see [4, §10]); without loss of generality, we may suppose that $\tau_0 \in D_0$ (cf. Remark 3.4). Note that $D_0 \subset \{t \mid |\arg t - \pi/4| < \pi/4 + \pi/100, |t| > \rho_\infty\}$. By Lemma 3.3 with $\tau = \tau_0$, there exists a pole τ_1 satisfying

$$0 < |\tau_1 - \tau_0| < c_*, \quad |\arg(\tau_1 - \tau_0) - \pi/4| < \pi/4 + \pi/100.$$

This implies that $\tau_1 \in D_0$ and that

$$D_0^{(1)} := \{t \mid |\arg(t - \tau_1) - \pi/4| < \pi/4 + \pi/100\} \subset D_0.$$

Therefore, Lemma 3.3 is applicable to $\tau = \tau_1$ as well, and there exists another pole τ_2 satisfying the relations

$$\tau_2 \in D_0^{(1)} \subset D_0$$

and

$$D_0^{(2)} := \{t \mid |\arg(t - \tau_2) - \pi/4| < \pi/4 + \pi/100\} \subset D_0^{(1)} \subset D_0.$$

We can repeat this procedure and we get an infinite sequence of poles $\{\tau_\nu\}_{\nu \geq 0}$ of $v(t)$ such that

$$|\arg(\tau_{\nu+1} - \tau_\nu) - \pi/4| < \pi/4 + \pi/100, \quad 0 < |\tau_{\nu+1} - \tau_\nu| < c_*.$$

This implies that $n(\rho, v) \geq \rho/c_* + O(1)$, and hence $n(r, w) \gg n(r^2/2, v) \gg r^2$, which completes the proof of Theorem 1.1 for (IV).

4. Outline of the proof of Theorem 1.1 for (II)

Theorem 1.1 for (II) is proved by an argument parallel to that used for (IV).

A. Let $w(z)$ be an arbitrary transcendental solution of (II). Then $v = v(t) = z^{-1/2}w(z)$ with $t = (2/3)z^{3/2}$ is a solution of

$$v_{tt} = 2v^3 + v - t^{-1}v_t + (2/3)\alpha t^{-1} + (1/9)t^{-2}v. \tag{4.1}$$

The function

$$V(t) := v_t(t)^2 - v(t)^4 - v(t)^2 - (4/3)\alpha t^{-1}v(t) - (1/9)t^{-2}v(t)^2$$

satisfies

$$V_t + 2t^{-1}V = -2t^{-1}v(t)^4 - 2t^{-1}v(t)^2 - (4/3)\alpha t^{-2}v(t),$$

from which we derive

$$V(t) = t_0^2 V(t_0) t^{-2} - 2t^{-2} \int_{C(t_0, t)} (sv(s)^4 + sv(s)^2 + (2/3)\alpha v(s)) ds,$$

where $C(t_0, t)$ is a path joining t_0 to t . The corresponding elliptic function $\varpi_*(\kappa, x)$, $\kappa \in \mathbb{C}$, satisfies

$$\dot{u}^2 = \hat{P}(\kappa, u) = u^4 + u^2 + \kappa. \tag{4.2}$$

For each $q \in \mathbb{Z}$, and for each κ , $|\kappa| \leq J$, we find a period $\omega(\kappa)$ of $\varpi_*(\kappa, x)$ with the same property as in Lemma 2.3. Furthermore, if $|\kappa| \leq J$, and if $\varpi_*(\kappa, 0) = \zeta_0$, $|\zeta_0| > 10$, then there exists a curve $\Gamma_\kappa(\zeta_0)$ joining 0 to $\omega(\kappa)$ with the property (i) of Lemma 2.4 and

(ii') $|\varpi_*(\kappa, x)| < \Lambda(J, |\zeta_0|)$ along $\Gamma_\kappa(\zeta_0)$ for some $\Lambda(J, |\zeta_0|)$ independent of κ .

B. Instead of (2.10), consider the auxiliary function

$$\Psi(z) := w'(z)^2 - w(z)^4 - zw(z)^2 - 2\alpha w(z).$$

Take a circle $|z| = r_0 > 10$ on which $w(z) \neq 0, \infty$. For K satisfying $K > 2r_0^{-1/2}M(w; r_0) + M(\Psi; r_0) + 10^2 + 2|\alpha|$, we put $D_K := \{z \mid |w(z)| > K|z|^{1/2}\}$. Starting from [12, Proposition 5.1], by the same argument as in § 2.3, we obtain the (II) versions of Lemmas 2.8 and 2.9.

Lemma 4.1. *Suppose that a , $|a| > r_0$, satisfies $|w(a)| = K|a|^{1/2}$. Then there exist a pole $\sigma_a^\#$ of $w(z)$ and the disc*

$$\hat{\Delta}(\sigma_a^\#) : |z - \sigma_a^\#| < (11/10)K^{-1}|\sigma_a^\#|^{-1/2}$$

such that $a \in \hat{\Delta}(\sigma_a^\#)$.

Lemma 4.2. *Let σ be an arbitrary pole of $w(z)$ such that $|\sigma| > 2r_0$. Then, there exist a point $a^\sigma \in \hat{\Delta}(\sigma)$ and a curve $\hat{C}_0(a^\sigma) \subset \{z \mid |z| \geq r_0\}$ with the following properties:*

- (i) $\hat{C}_0(a^\sigma)$ joins a point z^σ , $|z^\sigma| = r_0$, to a^σ , and the length of it does not exceed $\pi|a^\sigma|/2$; and
- (ii) $|w(z)| \leq K|z|^{1/2}$ along $\hat{C}_0(a^\sigma)$, and $|w(a^\sigma)| = K|a^\sigma|^{1/2}$.

C. Make the replacement $K \rightarrow K - 1$ in the supposition on K (cf. (3.1)). Concerning $v(t)$, instead of Lemmas 3.1 and 3.2, we have the following.

Lemma 4.3. *Let σ be an arbitrary pole of $w(z)$ such that $|\sigma| > 2r_0$, and let a^σ be the point given by Lemma 4.2. Then the corresponding point $b^\sigma = (2/3)(a^\sigma)^{3/2}$ satisfies $|b^\sigma - (2/3)\sigma^{3/2}| < 2/K$, $|v(b^\sigma)| = K$, $|V(b^\sigma)| \leq J_0$, and $|v_t(b^\sigma)^2 - v(b^\sigma)^4 - v(b^\sigma)^2| \leq J_0$ for some $J_0 = J_0(K)$ independent of σ .*

Lemma 4.4. *Suppose that b , $|b| > r_0^{3/2}$, satisfies $|v(b)| = K$. Then there exists a pole τ_b of $v(t)$ such that $|b - \tau_b| < 2/K$.*

D. The union of the sectors $\hat{S}_q : |\arg z - (2q - 1)\pi/6| < \pi/6 + \delta$ ($-2 \leq q \leq 3$) covers the full neighbourhood of $z = \infty$. The image of \hat{S}_q under the mapping $t = (2/3)z^{3/2}$ is $\hat{T}_q : |\arg t - (2q - 1)\pi/4| < \pi/4 + 3\delta/2$. For an arbitrary pole $\tau \in \hat{T}_1$ (with $3\delta/2 = \pi/100$) of $v(t)$ with sufficiently large modulus, we can find another pole τ' with the same property as (3.6). To show this fact, in place of (3.9), we treat the perturbed equation

$$\ddot{u} = 2u^3 + u - \frac{b^{-1}(\dot{u} - 2\alpha/3)}{1 + b^{-1}x} + \frac{b^{-2}u}{9(1 + b^{-1}x)^2} \quad (4.3)$$

for b close to τ . This is obtained from (4.1) by the change of variables $t = b + x$, $u(b, x) = v(b + x)$. Let $\varpi_*(\kappa_b, x)$ be a solution of (4.2) with $\kappa = \kappa_b := v_t(b)^2 - v(b)^4 - v(b)^2$ satisfying $\varpi_*(\kappa_b, 0) = v(b)$, $\dot{\varpi}_*(\kappa_b, 0) = v_t(b)$. The solution of (4.3) with the initial condition $u(b, 0) = v(b)$, $\dot{u}(b, 0) = v_t(b)$ is asymptotic to $\varpi_*(\kappa_b, x)$ in a suitable domain. By the same argument as used in § 3.3 together with Remark 2.2, we construct a sequence of poles of $v(t)$ satisfying $n(\rho, v) \gg \rho$, which implies the estimate $n(r, v) \gg r^{3/2}$.

Acknowledgements. The author is grateful to the referee for an important and helpful comment concerning a crucial part of the proof (in § 3.3) of the main result, and also to Professor Nalini Joshi for a useful discussion on Boutroux asymptotics.

References

1. P. BOUTROUX, Recherches sur les transcendants de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre, *Annls Sci. Ec. Norm. Super.* **30** (1913), 255–375.
2. P. BOUTROUX, Recherches sur les transcendants de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre (suite), *Annls Sci. Ec. Norm. Super.* **31** (1914), 99–159.
3. E. A. CODDINGTON AND N. LEVINSON, *Theory of ordinary differential equations* (McGraw-Hill, New York, 1955).
4. V. I. GROMAK, I. LAINE AND S. SHIMOMURA, *Painlevé differential equations in the complex plane* (Walter de Gruyter, Berlin, New York, 2002).
5. W. K. HAYMAN, *Meromorphic functions* (Clarendon, Oxford, 1964).
6. A. HINKKANEN AND I. LAINE, Solutions of the first and second Painlevé equations are meromorphic, *J. Analyse Math.* **79** (1999), 345–377.
7. A. HINKKANEN AND I. LAINE, Growth results for Painlevé transcendants, *Math. Proc. Camb. Phil. Soc.*, in press.
8. N. JOSHI, Asymptotic studies of the Painlevé equations, *The Painlevé property, one century later* (ed. R. Conte), pp. 181–227 (Springer, 1999).
9. I. LAINE, *Nevanlinna theory and complex differential equations* (Walter de Gruyter, Berlin, New York, 1993).
10. E. MUES AND R. REDHEFFER, On the growth of the logarithmic derivatives, *J. Lond. Math. Soc.* **8** (1974), 412–425.
11. K. OKAMOTO AND K. TAKANO, The proof of the Painlevé property by Masuo Hukuhara, *Funkcial. Ekvac.* **44** (2001), 201–217.
12. S. SHIMOMURA, Growth of the first, the second and the fourth Painlevé transcendants, *Math. Proc. Camb. Phil. Soc.* **134** (2003), 259–269.
13. S. SHIMOMURA, Lower estimates for the growth of Painlevé transcendants, *Funkcial. Ekvac.* **46** (2003), 287–295.

14. S. SHIMOMURA, Proofs of the Painlevé property for all Painlevé equations, *Jpn. J. Math.* **29** (2003), 159–180.
15. N. STEINMETZ, On Painlevé's equations I, II and IV, *J. Analyse Math.* **82** (2000), 363–377.
16. N. STEINMETZ, Value distribution of the Painlevé transcendents, *Isr. J. Math.* **128** (2002), 29–52.
17. N. STEINMETZ, Global properties of the Painlevé transcendents, submitted.
18. N. STEINMETZ, Boutroux's method vs. re-scaling, Lower estimates for the orders of growth of the second and fourth Painlevé transcendents, submitted.
19. K. TAKANO, Defining manifolds for Painlevé equations, *Toward the exact WKB analysis of differential equations, linear or nonlinear* (ed. C. J. Howls, T. Kawai and Y. Takei), pp. 261–269 (Kyoto University Press, Kyoto, 2000).