# IMBEDDED MARKOV CHAIN ANALYSIS OF SINGLE SERVER BULK QUEUES

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## Summary

In this paper results from Fluctuation Theory are used to analyse the imbedded Markov chains of two single server bulk-queueing systems, (i) with Poisson arrivals and arbitrary service time distribution and (ii) with arbitrary inter-arrival time distribution and negative exponential service time. The discrete time transition probabilities and the equilibrium behaviour of the queue lengths of the systems have been obtained along with distributions concerning the busy periods. From the general results several special cases have been derived.

## 0. Introduction

The general bulk queue is described as follows: Groups of customers arrive at service points and get served in batches. The sizes of the arriving groups and those of the batches for service are random variables having independent distributions. The time intervals between successive group arrivals are independent and identically distributed random variables; so also are the service times of the different batches. We shall call the maximum size of a service group as the "capacity" for that service and assume that this capacity is independent of the queue length at that time. Following Kendall [9] we use the notation $GI^{(x)}|G^{(y)}|1$ to represent the general single server bulk queue, the exponents $x$ and $y$ denoting the sizes of the arriving groups and service capacity respectively. We shall suppress these exponents when they are equal to one. Further, we shall assume that the queue-discipline is "first come, first served" and that when the arrivals are in groups, the units will be ordered for the purpose of service.

The object of this paper is to obtain the discrete time behaviour of the bulk queues (i) $M^{(x)}|G^{(y)}|1$ (Poisson arrivals and arbitrary service time) and (ii) $GI^{(x)}|M^{(y)}|1$ (arbitrary inter-arrival time distribution and negative exponential service time). This is done by analysing the Markov chains imbedded in them. Some aspects of these systems have been studied

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by Miller [10]; and several special cases have been considered by Bailey [1], Jaiswal [6], Takács [13,14], Foster [4], Foster and Nyunt [5], Keilson [7] and Boudreau, Griffin and Kac [2].

In our discussion we make use of known results in Fluctuation Theory for the sums of independent and identical random variables, to obtain the behaviour of $Q_n$, the queue length at $t_n$ (arrival or departure epoch, whichever is convenient). For the study of $W_n$ the virtual waiting time at an instant of arrival $t_n$, these results have already been used by Spitzer [12] and Kemperman [8].

The paper is divided into four sections. In section 1 certain basic results from Fluctuation Theory are described; section 2 deals with the system $M^{(x)}|G^{(y)}|1$ and section 3 with the system $GI^{(x)}|M^{(y)}|1$. Finally some special cases of these queues have been considered in the last section.

## 1. Basic results from fluctuation theory

The following are the special cases of more general results derived by Spitzer [12], Feller [3] and Kemperman [8], for sums of independent and identical random variables (r.v.).

Let $\{Z_n\}$ ($n = 1, 2 \cdots$) be a sequence of mutually independent and identical r.v.'s assuming integral values and $S_n = Z_1 + Z_2 + \cdots + Z_n$ ($n = 1, 2 \cdots$), $S_0 = 0$ be the partial sums of $\{Z_n\}$. Let

$$\begin{aligned} &\Pr\{Z_n = j\} = k_j \qquad (j = \cdots -1, 0, 1, \cdots) \\ &\phi(\theta) = E(\theta^{Z_n}), \qquad 0 < \phi'(1) < \infty \end{aligned} \tag{1.1}$$

and

$$k_j^{(n)} = \Pr\{S_n = j\} \quad (n \geqq 1), \quad k_j^{(1)} = k_j, \quad k_j^{(0)} = 0 \quad (j \neq 0), \; k_0^{(0)} = 1. \tag{1.2}$$

We define two functions $M^-(\theta, z)$ and $M^+(\theta, z)$ as follows.

$$M^-(\theta, z) = \exp\left\{-\sum_1^\infty \frac{z^n}{n} \sum_{-\infty}^{-1} \theta^j k_j^{(n)}\right\} \quad (|z|) < 1, \quad |\theta| \geqq 1) \tag{1.3}$$

$$M^+(\theta, z) = \exp\left\{\sum_1^\infty \frac{z^n}{n} \sum_0^\infty \theta^j k_j^{(n)}\right\} \quad (|z|) < 1, \quad |\theta| \leqq 1) \tag{1.4}$$

such that they are related by the property

$$[1 - z\phi(\theta)]M^+(\theta, z) = M^-(\theta, z) \tag{1.5}$$

(Kemperman [8] equations (13.4)—(13.8)).

For the partial sums $S_n$, we have the following results.

(i) Let

$$g_n^* = \Pr\{S_1 > 0, \; S_2 > 0 \cdots S_{n-1} > 0, \; S_n \leqq 0\};$$

then

$$g^*(z) = \sum_1^\infty g_n^* z^n = 1 - \exp\left\{-\sum_1^\infty \frac{z^n}{n} \sum_{-\infty}^0 k_j^{(n)}\right\} \tag{1.6}$$

$$= 1 - M^-(1, z)\exp\left\{-\sum_1^\infty \frac{z^n}{n} \Pr\{S_n = 0\}\right\}.$$

(ii) Let

$$\pi_n^*(j) = \Pr\{S_1 > 0,\ S_2 > 0, \cdots S_{n-1} > 0,\ S_n = j\} \quad (j > 0);$$

then

$$\pi^*(\theta, z) = \sum_{n=0}^\infty \sum_{j=1}^\infty \pi_n^*(j) z^n \theta^j = \exp\left\{\sum_1^\infty \frac{z^n}{n} \sum_1^\infty \theta^j k_j^{(n)}\right\}$$

$$= M^+(\theta, z)\exp\left\{-\sum_1^\infty \frac{z^n}{n} \Pr\{S_n = 0\}\right\}. \tag{1.7}$$

(iii) Let

$$g_n = \Pr\{S_1 \geqq 0,\ S_2 \geqq 0 \cdots S_{n-1} \geqq 0,\ S_n < 0\},$$

then

$$g(z) = \sum_1^\infty g_n z^n = 1 - \exp\left\{-\sum_1^\infty \frac{z^n}{n} \sum_{-\infty}^{-1} k_j^{(n)}\right\} \tag{1.8}$$

$$= 1 - M^-(1, z).$$

(iv) Let

$$\pi_n(i, j) = \Pr\{i+S_1 \geqq 0,\ i+S_2 \geqq 0 \cdots i+S_{n-1} \geqq 0,\ i+S_n = j\} \quad (i, j \geqq 0);$$

then $\pi(\omega, \theta, z) = \sum_{n=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^\infty \pi_n(i, j) z^n \omega^i \theta^j$

$$= \frac{1}{1-\omega\theta} \exp\left\{\sum_1^\infty \frac{z^n}{n} \sum_0^\infty \theta^j k_j^{(n)} + \sum_1^\infty \frac{z^n}{n} \sum_{-\infty}^{-1} \omega^{-j} k_j^{(n)}\right\} \tag{1.9}$$

$$= \frac{1}{1-\omega\theta} \frac{M^+(\theta, z)}{M^-(\omega^{-1}, z)}.$$

[Spitzer [12]; also Feller [3] equations (9.8), (9.13) and (7.10). For (1.9) here, see Kemperman [8] equation (16.13).]

Finally we shall define

$$w_n = \max(0, S_1, S_2 \cdots S_n);$$

then

$$\sum_0^\infty z^n E(\theta^{w_n}) = \exp\left\{\sum_1^\infty \frac{z^n}{n} \sum_{-\infty}^0 k_j^{(n)} + \sum_1^\infty \frac{z^n}{n} \sum_1^\infty \theta^j k_j^{(n)}\right\} \tag{1.10}$$

$$= M^+(\theta, z)[M^-(1, z)]^{-1}.$$

When $n \to \infty$, writing $\lim_{n\to\infty} w_n = w_\infty$, we have

(a) if $E(Z_n) \geqq 0$, $w_\infty = \infty$ with probability one;

(b) if $E(Z_n) < 0$, $w_\infty < \infty$ with probability one and is given by

$$E(\theta^{w_\infty}) = \exp\left\{-\sum_1^\infty \frac{1}{n} \sum_1^\infty (1-\theta^j) k_j^{(n)}\right\} \tag{1.11}$$

(Spitzer [12]).

## 2. The queue $M^{(x)}|G^{(y)}|1$

*Description*: The queueing system considered here has the following description.

(i) The arrivals are in a Poisson process with parameter $\lambda t$ in groups of size $\{C_n\}$ having the distribution

$$\Pr\{C_n = r\} = c_r \qquad (r = 0,\ 1,\ 2 \cdots);$$

let

$$c(\theta) = E(\theta^{C_n}).\quad |\theta| \leqq 1,\ 0 < c'(1) < \infty. \tag{2.1}$$

The probability that $j$ customers arrive in time interval $(0, T)$ is given by

$$a_j(T) = \sum_{k=0}^{j} e^{-\lambda T} \frac{(\lambda T)^k}{k!} c_j^{(k)} \tag{2.2}$$

where $\{c_j^{(k)}\}$ is the $k$-fold convolution of $\{c_j\}$ with itself. It should be noted that the compound Poisson process (2.2) has the property that the number of arrivals in non-overlapping time intervals are independent r.v.'s.

(ii) The customers are served in batches of variable capacity. Let the successive departures take place at the instants $t_1, t_2, \cdots$, and denote by $v_n$ the service time of the batch departing at $t_n$. We assume that $\{v_n\}$ $(n = 1, 2 \cdots)$ is a sequence of identically distributed independent r.v.'s with a common distribution function $H(x) = \Pr\{v_n \leqq x\}$. Let

$$\psi(\sigma) = \int_0^\infty e^{-\sigma x} dH(x) \qquad \mathrm{Re}(\sigma) \geqq 0 \tag{2.3}$$

and $0 < -\psi'(0) < \infty$.

Let $X_n$ be the number of customers arrived during a service period; then we have

$$\Pr\{X_n = j\} = a_j = \int_0^\infty \sum_{k=0}^{j} e^{-\lambda t} \frac{(\lambda t)^k}{k!} c_j^{(k)}\, dH(t); \tag{2.4}$$

and

$$K(\theta) = E(\theta^{X_n}) = \psi(\lambda - \lambda c(\theta)). \tag{2.5}$$

(iii) If $Y_n$ is the capacity for service ending at $t_{n+1}$ ($n = 1, 2, 3 \cdots$), we assume that the r.v.'s $Y_n$ are identically distributed and mutually independent and also independent of the $X_n$; let

$$\begin{aligned} &\Pr\{Y_n = j\} = b_j \qquad (j = 0, 1, 2 \cdots); \\ &B(\theta) = E(\theta^{Y_n}). \quad |\theta| \leqq 1, \ 0 < B'(1) < \infty. \end{aligned} \tag{2.6}$$

The relative traffic intensity of the system is defined by

$$\rho = \frac{E(X_n)}{E(Y_n)} = \frac{-\lambda\psi'(0)c'(1)}{B'(1)}, \quad (0 < \rho < \infty). \tag{2.7}$$

Further, we define $Q_n = Q(t_n+0)$ where $Q(t)$ is the queue length at time $t$ (number of customers in the system, including those who are being served); $\{Q_n\}$ is a Markov chain imbedded in the process $Q(t)$. Let $t_n$ and $t_{n+m}$ be such that $Q_{n-1} < Y_{n-1}$, $Q_r \geqq Y_r$, ($r = n, n+1, \cdots n+m-1$), $Q_{n+m} < Y_{n+m}$. During this period $(t_n, t_{n+m})$ full service capacity has been utilized and we shall call such a period "capacity busy period". If at some epoch $t_r$, $Q_r < Y_r$, the server is said to be "slack"; then the following different possibilities are open to him at that time: (a) $Y_n > Q_n \geqq 0$, he may wait until his maximum capacity $Y_n$ is reached; (b) $Y_n > Q_n > 0$, he may take the available customers into service; (c) $Q_n = 0$, he may either wait for the first customer to arrive or proceed for service with no customers. An example of the last service mechanism is an elevator or a transport service which is kept in operation even when there are no customers to be served. The basic processes corresponding to these various possibilities are all different, but the imbedded Markov chains are essentially the same. In every case the chain $\{Q_n\}$ satisfies the recurrence relations

$$Q_{n+1} = \begin{cases} Q_n - Y_n + X_{n+1} & Q_n - Y_n > 0 \\ X_{n+1} & Q_n - Y_n \leqq 0 \end{cases}. \tag{2.8}$$

We assume that the process starts with the commencement of service of the first batch of customers at $t_0+0$. Let $Q_0 = i (\geqq 0)$ be the number waiting at this instant. From (2.8) we obtain

$$\begin{aligned} Q_1 &= i + X_1 \\ Q_n &= \max\,[X_n,\ (X_n + X_{n-1} - Y_{n-1}), \cdots (X_n + \cdots + X_2 - Y_{n-1} \cdots - Y_2) \\ &\qquad (i + X_n + \cdots + X_1 - Y_{n-1} \cdots - Y_1)] \qquad (n \geqq 2). \end{aligned} \tag{2.9}$$

Let $X_n - Y_n = Z_n$, and $Z_1 + Z_2 + \cdots + Z_n = S_n$ ($n = 1, 2 \cdots$), $S_0 = 0$. Thus

$$\begin{aligned} k_j &= \Pr\{Z_n = j\} \\ \phi(\theta) &= E(\theta^{Z_n}) = B(\theta^{-1})\psi(\lambda - \lambda c(\theta)) \\ &= B(\theta^{-1})K(\theta); \end{aligned} \tag{2.10}$$

and

(2.11)
$$k_j^{(n)} = \Pr\{S_n = j\} \qquad (j = \cdots -1,\ 0,\ 1, \cdots)$$
$$E(\theta^{S_n}) = [\phi(\theta)]^n.$$

Now, using $S_n$ $(n = 0,\ 1,\ 2 \cdots)$ (2.9) can be expressed as follows:

(2.12) $Q_n \sim \max\,[X_n+S_r\ \ (r = 0,\ 1,\ 2 \cdots n-2),\ i+X_n+S_{n-1}]$.

When $i = 0$, we write

(2.13)
$$Q_n \sim X_n + \max\,(0, S_1, S_2 \ldots S_{n-1}).$$

*Transition probabilities of* $\{Q_n\}$: Let the transition probabilities of $\{Q_n\}$ be denoted by

(2.14)
$$P_{ij}^{(n)} = \Pr\{Q_n = j | Q_0 = i\}.$$

We have

THEOREM 1. *For* $|z| < 1$, $|\omega| \leqq 1$ *and* $|\theta| \leqq 1$,

(2.15)
$$\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij}^{(n)} z^n \omega^i \theta^j = \frac{(1-\theta)zK(\theta)}{1-\omega}\,[\pi(1,\theta,z)-\omega\pi(\omega,\theta,z)],$$

*where* $\pi(\omega,\ \theta,\ z)$ *is the transform given by* (1.9).

PROOF. Using the expression (2.12) for $Q_n$ we have

(2.16)
$$\Pr\{Q_n \leqq j | Q_0 = i\} = \Pr\{X_n + S_r \leqq j\ \ (r = 0,\ 1, \cdots n-2),\ i+X_n+S_{n-1} \leqq j\}$$
$$= \sum_{\nu=0}^{j} \Pr\{X_n = \nu\} \Pr\{X_n+S_r \leqq j\ (r = 0, 1, \cdots n-2), i+X_n+S_{n-1} \leqq j | X_n = \nu\}$$
$$= \sum_{l=0}^{\infty}\sum_{\nu=0}^{j} a_\nu \Pr\{\nu+S_r \leqq j\ \ (r = 0,\ 1 \cdots n-2),\ i+\nu+S_{n-1} = j-l\}$$
$$= \sum_{l=0}^{\infty}\sum_{\nu=0}^{j} a_\nu \Pr\{i+\nu+S_{n-1}-(\nu+S_r) = i+S_{n-1-r} \geqq -l\ \ (r = 0,\ 1, \cdots n-2)$$
$$i+\nu+S_{n-1} = j-l\}$$
$$= \sum_{l=0}^{\infty}\sum_{\nu=0}^{j} a_\nu \Pr\{l+i+S_r \geqq 0\ \ (r = 1,\ 2 \cdots n-2),\ l+i+S_{n-1} = j-\nu\}$$
$$= \sum_{l=i}^{\infty}\sum_{\nu=0}^{j} a_\nu \pi_{n-1}\ (l,\ j-\nu)$$
$$= \sum_{l=i}^{\infty}\sum_{\nu=0}^{j} a_{j-\nu} \pi_{n-1}\ (l,\ \nu),$$

where $\pi_n(i,\ j)$ is the probability defined in (1.9). In particular we get

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(2.17) $$P_{i0}^{(n)} = \sum_{l=i}^{\infty} a_0 \pi_{n-1}(l, 0).$$

Further, taking transforms of (2.16) we have

$$(1-\theta)^{-1} \sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij}^{(n)} z^n \omega^i \theta^j = \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{\nu=0}^{\infty} \pi_{n-1}(l, \nu) z^n \theta^\nu \sum_{i=0}^{l} \omega^i \sum_{j=\nu}^{\infty} \theta^{j-\nu} a_{j-\nu}$$
$$= \frac{1}{1-\omega} z\psi(\lambda-\lambda c(\theta)) \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{\nu=0}^{\infty} \pi_n(l, \nu) z^n [1-\omega^{l+1}]\theta^\nu$$

which gives (2.15).

It should be noted here that when $i = 0$, the use of the expression (2.13) for $Q_n$ and the subsequent result

(2.18) $$\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} P_{0j}^{(n)} z^n \theta^j = zK(\theta) \exp\left\{\sum_{1}^{\infty} \frac{z^n}{n} \sum_{-\infty}^{0} k_j^{(n)} + \sum_{1}^{\infty} \frac{z^n}{n} \sum_{1}^{\infty} \theta^j k_j^{(n)}\right\}$$
$$= zK(\theta) M^+(\theta, z)[M^-(1, z)]^{-1}$$

would seem to be more useful. This is obtained by using the result (1.10) for max $(0, S_1, S_2 \cdots S_{n-1})$.

The "capacity busy period" $\tau_i$ initiated by $i$ customers is given by

(2.19) $$\tau_i = \min\{n | Q_n - Y_n < 0\},\ Q_0 = i.$$

Let

(2.20) $$G_{ij}^{(n)} = \Pr\{Q_n = j,\ \tau_i \geqq n\}. \qquad (i, j \geqq 0)$$

We have

THEOREM 2. *For* $|z| < 1$, $|\omega| \leqq 1$ *and* $|\theta| \leqq 1$,

(2.21) $$\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} G_{ij}^{(n)} z^n \omega^i \theta^j = zK(\theta)\pi(\omega, \theta, z),$$

where $\pi(\omega, \theta, z)$ is the transform given by (1.9).

PROOF. From the recurrence relations (2.8) we have

(2.22) $$\begin{aligned} G_{ij}^{(n)} &= \Pr\{i+S_r \geqq 0\ (r = 1, 2 \cdots n-1),\ i+S_{n-1}+X_n = j\} \\ &= \sum_{\nu=0}^{j} \Pr\{X_n = \nu\} \Pr\{i+S_r \geqq 0\ (r = 1, 2 \cdots n-1),\ i+\nu+S_{n-1} = j\} \\ &= \sum_{\nu=0}^{j} a_\nu \Pr\{i+S_r \geqq 0\ (r = 1, 2 \cdots n-2),\ i+S_{n-1} = j-\nu\} \\ &= \sum_{\nu=0}^{j} a_\nu \pi_{n-1}(i, j-\nu). \end{aligned}$$

Forming transforms of (2.22) we get

$$\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} G_{ij}^{(n)} z^n\omega^i\theta^j = z\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} z^n\omega^i \sum_{\nu=0}^{\infty} a_\nu \sum_{j=\nu}^{\infty}\theta^j\pi_n(i, j-\nu)$$

which leads to (2.21).

For the distribution $G^{(n)}$ of the number of batches served in a capacity busy period we have

THEOREM 3. *For* $|z| < 1$,

$$\sum_{n=1}^{\infty} G^{(n)}z^n = 1-\exp\left\{-\sum_{1}^{\infty}\frac{z^n}{n}\sum_{-\infty}^{-1} k_j^{(n)}\right\} = 1-M^-(1, z). \tag{2.23}$$

PROOF. Clearly

$$\begin{aligned} G^{(n)} &= \Pr\{\tau_0 = n\} \\ &= \Pr\{S_r \geqq 0 \ \ (r = 1, 2\cdots n-1),\ S_n < 0\} \end{aligned} \tag{2.24}$$

as can be obtained from the recurrence relations (2.8). Now the theorem follows by using the result (1.8).

*Limiting behaviour of* $Q_n$: Let $Q_\infty = \lim_{n\to\infty} Q_n$; from the expressions (2.12) and (2.13) we may write this as

$$Q_\infty = X_n+\max(0, S_1, S_2\cdots). \tag{2.25}$$

The limiting distribution of $Q_n$ is therefore given by the following

THEOREM 4. *With probability one* $Q_\infty = \infty$ *if* $\rho \geqq 1$ *and* $Q_\infty < \infty$ *if* $\rho < 1$. *In the latter case for* $|\theta| \leqq 1$

$$E(\theta^{Q_\infty}) = K(\theta)\exp\left\{-\sum_{1}^{\infty}\frac{1}{n}\sum_{1}^{\infty}(1-\theta^j)k_j^{(n)}\right\}. \tag{2.26}$$

PROOF. It is clear that the behaviour of $Q_\infty$ follows that of $w_\infty = \max(0, S_1, S_2\cdots)$ as given in section 1 and that the conditions $\rho \geqq 1$ and $< 1$ are equivalent to $E(Z_n) \geqq 0$ and $< 0$ respectively. Further, when $\rho < 1$

$$E(\theta^{Q_\infty}) = E(\theta^{X_n})E(\theta^{w_\infty}).$$

The theorem now follows by substituting for $E(\theta^{w_\infty})$ from (1.11).

*A modified queueing scheme*: Consider the queue $M^{(x)}|G^{(y)}|1$ with the following modification in its service mechanism: if, at an instant $t_n$ $(n = 1, 2\cdots)$ $Q_n < Y_n$, the server takes the available customers into service and those arriving before $t_{n+1}$, join this batch with probability $p$ until the capacity is filled — without effecting the service time. With this mechanism the arriving customers fall into two categories, (i) those who would opt for immediate service with the batch being served and (ii) those who would prefer to wait for the next full service; the corresponding probabilities are $p$ and $q$ $(= 1-p)$ respectively. Let $C_n^{(1)}$ and $C_n^{(2)}$ be the respective number of customers in the above two categories, arriving in the $n$th group such that

$$\text{(2.27)} \qquad \begin{aligned} \Pr\{C_n^{(1)} = k\} &= \sum_{r=k}^{\infty} c_r \binom{r}{k} p^k q^{r-k} \\ \Pr\{C_n^{(2)} = l\} &= \sum_{r=l}^{\infty} c_r \binom{r}{l} p^{r-l} q^l \end{aligned}$$

and

$$\text{(2.28)} \qquad \begin{aligned} c^{(1)}(\theta) &= E(\theta^{C_n^{(1)}}) = c(q+p\theta) \\ c^{(2)}(\theta) &= E(\theta^{C_n^{(2)}}) = c(p+q\theta) \end{aligned}$$

where $c(\theta)$ is the probability generating function defined in (2.1). During a service period $v_n$, let $U_{n-1}$ be the number of customers who would opt for immediate service and $X_n$, of those who would prefer to wait. Then we have

$$\text{(2.29)} \qquad U(\theta) = E(\theta^{U_n}) = \psi(\lambda - \lambda c(q+p\theta))$$

$$\text{(2.30)} \qquad K(\theta) = E(\theta^{X_n}) = \psi(\lambda - \lambda c(p+q\theta)).$$

Consider the imbedded Markov chain $\{Q_n\}$ and the transitions between $Q_n$ and $Q_{n+1}$ $(n = 0, 1, 2 \cdots)$. As the $U_n$ customers arriving during $(t_n, t_{n+1})$ are prepared to go into service immediately, the sum $Q_n + U_n$ can now be treated as the queue length at $t_n + 0$. Thus we have the recurrence relations

$$\text{(2.31)} \qquad Q_{n+1} = \begin{cases} Q_n + U_n - Y_n + X_{n+1} & Q_n + U_n - Y_n > 0 \\ X_{n+1} & Q_n + U_n - Y_n \leqq 0 \end{cases}$$

As before we assume that $Q_0 = i$ $(\geqq 0)$ represents the number of customers waiting after the commencement of the first service. From (2.31) we obtain

$$\begin{aligned} Q_1 &= i + U_0 + X_1 \\ Q_n &= \max [X_n, (X_n + X_{n-1} + U_{n-1} - Y_{n-1}), (X_n + X_{n-1} + X_{n-2} + U_{n-1} + U_{n-2} \\ &\qquad - Y_{n-1} - Y_{n-2}), \cdots (i + U_0 + X_n + X_{n-1} \cdots + X_1 + U_{n-1} \\ &\qquad + \cdots + U_1 - Y_{n-1} \cdots Y_1)]. \end{aligned}$$

Now, define $Z_n = X_n + U_n - Y_n$ and $S_n = Z_1 + \cdots + Z_n$, $S_0 = 0$ and hence

$$k_j = \Pr\{Z_n = j\}$$

and

$$\text{(2.32)} \qquad \phi(\theta) = E(\theta^{Z_n}) = B(\theta^{-1}) K(\theta) U(\theta).$$

In terms of the partial sums $S_n$, we have

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$$(2.33)\qquad Q_n = \max\,[X_n+S_r\ \ (r = 0,\ 1,\cdots n-2),\ i+U_0+X_n+S_{n-1}],$$

an expression similar to (2.12). The analysis follows as before with necessary modifications.

As a particular case of the above scheme, set $p = 1$.

Then $\Pr\{X_n = 0\} = 1$, and hence

$$(2.34)\qquad Q_{n+1} = \begin{cases} Q_n+U_n-Y_n & Q_n+U_n-Y_n > 0 \\ 0 & Q_n+U_n-Y_n \leqq 0. \end{cases}$$

This is a Markov chain different from the one which we have studied so far; we propose to investigate the behaviour of this chain in section 3, in connection with the system $GI^{(x)}|M^{(y)}|1$.

## 3. The queue $GI^{(x)}|M^{(y)}1$

*Description*: In this section the following single server queueing model will be considered.

(i) Customers arrive at the instants $t_0,\ t_1,\ t_2\cdots$ in groups; let $X_n$ be the size of the group arriving at $t_{n-1}$ and have the distribution

$$(3.1)\qquad \begin{array}{ll} \Pr\{X_n = j\} = b_j & (j = 0,\ 1,\ 2\cdots); \\ B(\theta) = E(\theta^{X_n}). & |\theta| \leqq 1,\ 0 < B'(1) < \infty. \end{array}$$

Also, let the inter-arrival times $t_{n+1}-t_n > 0$ $(n = 0,\ 1,\ 2\cdots)$ form a sequence of identically distributed independent r.v.'s with a common distribution function $H(x)$. Let

$$(3.2)\qquad \psi(\sigma) = \int_0^\infty e^{-\sigma x}dH(x) \qquad \mathrm{Re}\,(\sigma) \geqq 0$$

and $0 < -\psi'(0) < \infty$.

(ii) The customers are served in batches of variable capacity $\{C_n\}$. We assume that the r.v.'s $C_n$ are identically distributed and mutually independent and also independent of the $X_n$; let

$$(3.3)\qquad \begin{array}{ll} \Pr\{C_n = r\} = c_r\ \ (r = 0,\ 1,\ 2\cdots) & \\ c(\theta) = E(\theta^{C_n}). & |\theta| \leqq 1,\ 0 < c'(1) < \infty. \end{array}$$

Further, the service times have the negative exponential distribution $\lambda e^{-\lambda t}dt (0 < t < \infty)$. Let $Y_n$ be the total capacity of the batches that would be served during the period $(t_{n-1},\ t_n)$; we have

$$(3.4)\qquad \Pr\{Y_n = j\} = \int_0^\infty \sum_{k=0}^{j} e^{-\lambda t}\,\frac{(\lambda t)^k}{k!}\,c_j^{(k)}\,dH(t)$$

and

$$K(\theta) = E(\theta^{Y_n}) = \psi(\lambda - \lambda c(\theta)). \tag{3.5}$$

(iii) The service mechanism is such that when there is vacancy in the group being served, the arriving customers will join the group immediately, until its capacity is reached. The rest of the group of arrivals will wait for the next service.

The relative traffic intensity in this system is given by

$$\rho_2 = \frac{B'(1)}{-\lambda\psi'(0)c'(1)} = \rho^{-1} \tag{3.6}$$

where $\rho$ is the relative traffic intensity of the dual queueing system $M^{(x)}|G^{(y)}|1$ considered in section 2.

We define $Q_n = Q(t_n - 0)$, where $Q(t)$ is the queue length at time $t$ including those who are being served. $\{Q_n\}$ is a Markov chain imbedded in the process $Q(t)$. For this chain we have the recurrence relations

$$\begin{aligned} Q_{n+1} &= \begin{cases} Q_n + X_{n+1} - Y_{n+1} & Q_n + X_{n+1} - Y_{n+1} > 0 \\ 0 & Q_n + X_{n+1} - Y_{n+1} \leqq 0 \end{cases} \\ &= \max\,(0,\ Q_n + Z_{n+1}) \end{aligned} \tag{3.7}$$

where $Z_n = X_n - Y_n$.
From (i) and (ii) we have

$$\begin{aligned} \phi(\theta) = E(\theta^{Z_n}) &= B(\theta)\psi(\lambda - \lambda c(\theta^{-1})) \\ &= B(\theta)K(\theta^{-1}). \end{aligned} \tag{3.8}$$

Let $Q_0 = i$; from (3.7) we obtain

$$\begin{aligned} Q_n &= \max\,[0, Z_n, (Z_n + Z_{n-1}), \cdots (Z_n + \cdots + Z_2), (i + Z_n + \cdots + Z_1)] \\ &\sim \max\,(0,\ S_1,\ S_2, \cdots S_{n-1},\ i + S_n), \end{aligned} \tag{3.9}$$

where $S_n = Z_1 + Z_2 \cdots + Z_n$.

*Transition probabilities of* $\{Q_n\}$: For the transition probabilities $P_{ij}^{(n)}$ of the chain $\{Q_n\}$ we have

THEOREM 5. *For* $|z| < 1$, $|\omega| < 1$ *and* $|\theta| \leqq 1$

$$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij}^{(n)} z^n \omega^i \theta^j = \frac{1-\theta}{1-\omega}\,[\pi(1,\ \theta,\ z) - \omega\pi(\omega,\ \theta,\ z)] \tag{3.10}$$

*where* $\pi(\omega,\ \theta,\ z)$ *is the transform given by* (1.9).

PROOF. We have

(3.11)

$$\begin{aligned}\Pr\{Q_n \leqq j \mid Q_0 = i\} &= \Pr\{S_r \leqq j \ (r = 0, 1, 2\cdots n-1), i+S_n \leqq j\}\\ &= \sum_{\nu=0}^{\infty} \Pr\{S_r \leqq j \ \ (r = 0,\ 1,\ 2\cdots n-1),\ i+S_n = j-\nu\}\\ &= \sum_{\nu=0}^{\infty} \Pr\{i+S_n-S_r = i+S_{n-r} \geqq -\nu \ (r = 1, 2\cdots n-1), i+S_n = j-\nu\}\\ &= \sum_{\nu=0}^{\infty} \Pr\{i+\nu+S_r \geqq 0 \ \ (r = 1,\ 2\cdots n-1),\ i+\nu+S_n = j\}\\ &= \sum_{\nu=i}^{\infty} \pi_n(\nu,\ j)\end{aligned}$$

where $\pi_n(\nu, j)$ is the probability defined in (1.9). In particular we get

$$P_{i0}^{(n)} = \sum_{\nu=i}^{\infty} \pi_n(\nu,\ 0). \tag{3.12}$$

Further, taking transforms of (3.11), we have

$$\begin{aligned}(1-\theta)^{-1} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij}^{(n)} z^n \omega^i \theta^j &= \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{j=0}^{\infty} \pi_n(\nu,\ j) z^n \theta^j \sum_{i=0}^{\nu} \omega^i\\ &= \frac{1}{1-\omega} \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \sum_{j=0}^{\infty} \pi_n(\nu, j) z^n (1-\omega^{\nu+1}) \theta^j\end{aligned}$$

which gives (3.10).

When $i = 0$, the following simplified form of the result (3.10) would be useful. This is directly obtained from the distribution of $\max(0, S_1, S_2 \cdots S_n)$ given by (1.10).

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} P_{0j}^{(n)} z^n \theta^j &= \exp\left\{\sum_{1}^{\infty} \frac{z^n}{n} \sum_{-\infty}^{0} k_j^{(n)} + \sum_{1}^{\infty} \frac{z^n}{n} \sum_{1}^{\infty} \theta^j k_j^{(n)}\right\}\\ &= M^+(\theta,\ z)[M^-(1,\ z)]^{-1}.\end{aligned} \tag{3.13}$$

In this system, we shall define the r.v. $\tau_i$ as

$$\tau_i = \min\{n \mid Q_n = 0\},\ Q_0 = i. \tag{3.14}$$

When $i = 0$, this represents the number of arriving groups in a period marked by the commencement of two consecutive busy periods. Let

$$G_{ij}^{(n)} = \Pr\{Q_n = j,\ \tau_i > n\} \quad (i \geqq 0,\ j > 0). \tag{3.15}$$

For these probabilities we have the following

THEOREM 6. *For* $|z| < 1$, $|\omega| \leqq 1$ *and* $|\theta| \leqq 1$

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$$(3.16)\quad \text{a)}\ \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} G_{0j}^{(n)} z^n \theta^j = \exp\left\{\sum_1^\infty \frac{z^n}{n}\sum_1^\infty \theta^j k_j^{(n)}\right\}$$
$$= M^+(\theta,\ z)\exp\left\{-\sum_1^\infty \frac{z^n}{n}\Pr\{S_n=0\}\right\}$$

$$(3.17)\quad \text{b)}\ \sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} G_{ij}^{(n)}\, z^n\omega^i\theta^j = \omega\theta\pi(\omega,\ \theta,\ z)$$

where $\pi(\omega,\ \theta,\ z)$ is the transform given by (1.9).

Proof. Applying the definition (3.14) to the recurrence relations (3.7) we get

$$G_{ij}^{(n)} = \Pr\{i+S_r>0\ \ (r=1, 2\cdots n-1), i+S_n=j\}\ (i\geqq 0,\ j>0)$$
$$(3.18)\quad = \begin{cases}\Pr\{S_r>0\ \ (r=1,\ 2\cdots n-1)\ S_n=j\}\ (j>0)\\ \Pr\{i-1+S_r\geqq 0\ (r=1,2\cdots n-1)\, i-1+S_n=j-1\}\ (i,j>0).\end{cases}$$

These are clearly the probabilities given by (1.7) and (1.9) respectively and hence the theorem follows.

Suppose we are interested in the distribution of the number of arriving groups in a busy period. This is given by

$$(3.19)\qquad G^{(n)} = \Pr\{\tau_0 = n\},$$

for which we have

Theorem 7. *For* $|z|<1$

$$(3.20)\quad \sum_{n=1}^{\infty} G^{(n)}z^n = 1-\exp\left\{-\sum_1^\infty \frac{z^n}{n}\sum_{-\infty}^{0} k_j^{(n)}\right\}$$
$$= 1-M^-(1,\ z)\exp\left\{-\sum_1^\infty \frac{z^n}{n}\Pr\{S_n=0\}\right\}$$

Proof. As in (3.18) we write

$$(3.21)\qquad G^{(n)} = \Pr\{S_r>0\ \ (r=1,\ 2\cdots n-1),\ S_n\leqq 0\},$$

which is precisely the probability given by (1.6). Hence the theorem follows.

*Limiting behaviour of* $Q_n$: As $n\to\infty$, the expression (3.9) can be written as

$$(3.22)\qquad Q_\infty = \lim_{n\to\infty} Q_n = \max(0,\ S_1,\ S_2\cdots).$$

The distribution of $Q_\infty$ is therefore given by the following:

THEOREM 8. *With probability one* $Q_\infty = \infty$ *if* $\rho_2 \geqq 1$ *and* $Q_\infty < \infty$ *if* $\rho_2 < 1$. *In the latter case for* $|\theta| \leqq 1$

$$E(\theta^{Q_\infty}) = \exp\left\{-\sum_1^\infty \frac{1}{n} \sum_1^\infty (1-\theta^j) k_j^{(n)}\right\}. \tag{3.23}$$

The theorem follows by identifying $Q_\infty$ with $w_\infty$ given in section 1 and the conditions $\rho_2 \geqq 1$ and $< 1$ with $E(Z_n) \geqq 0$ and $< 0$ respectively.

## 4. Particular cases

The two models considered in sections 2 and 3 admit several special cases. Clearly the important systems $M|G|1$ and $GI|M|1$ are two of them. So also are the systems $M^{(x)}|G|1$ studied by Gaver [15] for its continuous time transient behaviour, and $GI|M^{(y)}|1$. The imbedded Markov chain analysis of these systems using combinatorial methods has been given by N. U. Prabhu and U. Narayan Bhat [11]. Here we shall consider more general systems.

1. *Bailey's bulk queue* $(M|G^s|1)$:

Here the customers arrive in a Poisson process with parameter $\lambda t$ and get served in batches of size not exceeding $s$. The service times have the distribution (2.4). The imbedded Markov chain is essentially the same as that of the system in which the service is in batches of fixed size $s$; its transient behaviour has been obtained by Takács.

We have

$$Q_n \sim X_n + \max(0, S_1, S_2 \cdots S_{n-2}, i+S_{n-1}) \tag{4.1}$$

where

$$\begin{aligned} \Pr\{X_n = j\} &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} dH(t) \qquad (j = 0, 1 \cdots) \\ K(\theta) &= E(\theta^{X_n}) = \psi(\lambda - \lambda\theta); \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} Z_n = X_n - s,\ S_n &= Z_1 + Z_2 + \cdots + Z_n,\ S_0 = 0, \\ k_j &= \Pr\{Z_n = j\} \quad (j = -s, -s+1, \cdots 0, 1 \cdots) \\ \phi(\theta) &= E(\theta^{Z_n}) = \frac{1}{\theta^s} \psi(\lambda - \lambda\theta). \end{aligned} \tag{4.3}$$

Further, the functions defined in (1.3) and (1.4) are given by

$$M^-(\theta, z) = \prod_{r=1}^s (1 - \xi_r \theta^{-1}) \quad (|z| < 1,\ |\theta| \geqq 1) \tag{4.4}$$

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(4.5) $$M^{+}(\theta, z) = \frac{1}{\theta^s(1-z\phi(\theta))}\prod_{r=1}^{s}(\theta-\xi_r) \quad (|z|<1,\ |\theta|\leqq 1)$$

where $\xi_r = \xi_r(z)$ $(r = 1, 2\cdots s)$ are the $s$ distinct roots of the equation

(4.6) $$\theta^s - z\psi(\lambda-\lambda\theta) = 0$$

in the unit circle $|\theta| < 1$ [see Kemperman [8] equation (20.27)]. When $i = 0$ we therefore have

(4.7) $$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} P_{0j}^{(n)} z^n\theta^j = 1+\frac{zK(\theta)}{\theta^s - zK(\theta)}\prod_{1}^{s}\left(\frac{\theta-\xi_r}{1-\xi_r}\right)$$

and

(4.8) $$\begin{aligned} E(\theta^{Q_\infty}) &= K(\theta)\lim_{z\to 1}(1-z)\left[1+\frac{zK(\theta)}{\theta^s-zK(\theta)}\prod_{1}^{s}\left(\frac{\theta-\xi_r}{1-\xi_r}\right)\right] \\ &= \frac{s(1-\rho)(1-\theta)}{1-\theta^s/K(\theta)}\prod_{1}^{s-1}\left(\frac{\theta-\zeta_r}{1-\zeta_r}\right) \end{aligned}$$

if $\rho < 1$, while $Q_\infty = \infty$ with probability one if $\rho \geqq 1$. Here the relative traffic intensity $\rho = -\lambda\psi'(0)/s$ and $\zeta_r$ $(r = 1, 2\cdots s)$ are the roots of the equation

(4.9) $$\zeta^s = K(\zeta).$$

Further, $G^{(n)}$, the distribution of the number of batches served in a "capacity busy period" is given by

(4.10) $$\sum_{n=1}^{\infty} G^{(n)}z^n = 1-\prod_{1}^{s}(1-\xi_r);$$

and the probability $G_{ij}^{(n)}$ by

(4.11) $$\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} G_{ij}^{(n)} z^n\omega^i\theta^j = \frac{zK(\theta)}{(1-\omega\theta)(\theta^s - zK(\theta))}\prod_{1}^{s}\left(\frac{\theta-\xi_r}{1-\omega\xi_r}\right)$$

2. *The queue* $E_s|G|1$:

Suppose the customers arrive at the instants $T_0, T_1\cdots$ and the inter-arrival time $\chi_n = T_{n+1}-T_n$ $(n = 0, 1, 2\cdots)$ has the distribution

(4.12) $$dA(x) = \Pr\{x < \chi_n < x+dx\} = e^{-\lambda x}\frac{\lambda^s x^{s-1}}{(s-1)!}dx,$$

while the service time has the distribution (2.4). The relative traffic intensity $\rho = -\lambda\psi'(0)/s < \infty$. Consider an input process which is Poisson with parameter $\lambda t$. The interval between the arrival points of consecutive $s$th customers in such an arriving scheme has the distribution (4.12), and therefore instead of each customer of the system $E_s|G|1$ we can think of

$s$ hypothetical customers who arrive in a Poisson process and get served in a single batch. Consequently the study of the system $E_s|G|1$ is identical with that of $M|G^s|1$.

Let $Q'_n$ be the queue length just after the $n$th departure in the system $E_s|G|1$ and $Q_n$ be that in $M|G^s|1$. $Q'_n$ may now be obtained from $Q_n$ using the relationship

$$Q'_n = [Q_n/s] \tag{4.13}$$

where $[n]$ is the largest integer in the number $n$.

3. *The queue* $M^r|G^s|1$:

This is a modification of Bailey's bulk queue and has the same description, except that the arrivals are in groups of size $r$. Thus we have

$$\Pr\{X_n = j\} = \begin{cases} \int_0^\infty e^{-\lambda t}\dfrac{(\lambda t)^{j/r}}{(j/r)!}dH(t) & \text{for } j = kr \quad (k = 0, 1, 2\cdots) \\ 0 & \text{Otherwise.} \end{cases} \tag{4.14}$$

and

$$\phi(\theta) = E(\theta^{Z_n}) = \frac{1}{\theta^s}\psi(\lambda-\lambda\theta^r). \tag{4.15}$$

The analysis follows on the same lines as that of $M|G^s|1$.

4. *The queue* $M|G^{(v)}|1$:

This is yet another modification of Bailey's bulk queue and has the same description, except that the service is in batches of variable capacity $Y_n$ having the distribution

$$\Pr\{Y_n = j\} = \begin{cases} b_j & (j = 0, 1, 2\cdots s) \\ 0 & (j > s) \end{cases} \tag{4.16}$$

and $B(\theta) = E(\theta^{Y_n})$. Thus the equations (4.1) and (4.2) hold true in this case and instead of (4.3) we have

$$\begin{aligned} &Z_n = X_n - Y_n \quad \text{and} \\ &\phi(\theta) = E(\theta^{Z_n}) = B(\theta^{-1})\psi(\lambda-\lambda\theta). \end{aligned} \tag{4.17}$$

The functions $M^-(\theta, z)$ and $M^+(\theta, z)$ are given by the equations (4.4) and (4.5), where $\xi_r$ $(r = 1, 2\cdots s)$ are now the $s$ distinct roots of the equation

$$\theta^s = zB_s(\theta)K(\theta) \tag{4.18}$$

in the unit circle $|\theta| < 1$. Here $B_s(\theta) = \sum_0^s b_r\theta^{s-r}$. It can be shown that the equation (4.18) has exactly $s$ distinct roots in the unit circle as follows:

$K(\theta)$ is a probability generating function and hence for $|z| < 1$ and

$|\theta| < 1$ we have $|zK(\theta)| < (1-\delta)^s$ if $|\theta| = 1-\delta$ ($\delta > 0$). When this condition holds we also have $B_s(\theta) \leqq \sum_0^s b_r = 1$.

Thus $|zB_s(\theta)K(\theta)| \leqq |zK(\theta)|\,|B_s(\theta)| < (1-\delta)^s$ if $|\theta| = 1-\delta$ ($\delta > 0$) and therefore by Rouché's theorem the equation (4.18) has exactly $s$ roots within the unit circle $|\theta| < 1$.

The required results for this queueing system are then given by the equations (4.7)—(4.11).

5. *The queue* $M|D|(y)$:

Here the customers arrive in a Poisson process with parameter $\lambda t$, the service time is a constant $= b$ and the number of servers is a r.v. $Y_n$ having the distribution (4.16). Consider the instants of time $nb$ ($n = 0, 1, 2 \cdots$) and let $X_n$ be the number of arrivals during the interval $(nb-b+0,\ nb)$. The distribution of $X_n$ is given by

$$\Pr\{X_n = j\} = e^{-\lambda b}\frac{(\lambda b)^j}{j!} \qquad (j = 0,\ 1,\ 2 \cdots) \tag{4.19}$$

$$K(\theta) = E(\theta^{X_n}) = e^{-\lambda b(1-\theta)}.$$

Let $Q_n = Q(nb+0)$; then $Q_n$ satisfies the recurrence relations

$$Q_{n+1} = \begin{cases} Q_n - Y_n + X_{n+1} & Q_n - Y_n > 0 \\ X_{n+1} & Q_n - Y_n \leqq 0 \end{cases}$$

which is essentially the same as (2.8) for the special case $M|D^{(y)}|1$. The required results are therefore given by the equations (4.7)—(4.11), where $\xi_r$ ($r = 1,\ 2 \cdots s$) are now the $s$ roots of the equation

$$1 - zB(\theta^{-1})e^{-\lambda b(1-\theta)} = 0 \tag{4.21}$$

in the unit circle $|\theta| < 1$.

Obviously when $Y_n = s$ with probability one, we have the system $M|D|s$.

6. *The queue* $GI^s|M|1$:

The customers arrive in groups of size $s$, the inter-arrival times having the distribution (3.3). The service times have the negative exponential distribution $\lambda e^{-\lambda t}$ ($0 < t < \infty$) and the service is given individually. The transient behaviour of this system has been obtained by Takács [14].

We have

$$Q_n \sim \max(0,\ S_1,\ S_2 \cdots S_{n-1},\ i+S_n) \tag{4.22}$$

where

$$\Pr\{Y_n = j\} = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^j}{j!}\,dH(t) \tag{4.23}$$

$$K(\theta) = E(\theta^{Y_n}) = \psi(\lambda - \lambda\theta); \quad |\theta| < 1$$

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and

(4.24)
$$Z_n = s - Y_n, \; S_n = Z_1 + Z_2 \cdots + Z_n, \; S_0 = 0$$
$$k_j = \Pr\{Z_n = j\} \qquad (j = \cdots -1, 0, 1, \cdots s)$$
$$\phi(\theta) = E(\theta^{Z_n}) = \theta^s \psi(\lambda - \lambda\theta^{-1}).$$

Clearly, if we denote the corresponding partial sum in the system $M|G^s|1$ by $S'_n$, we have $-S_n = S'_n$. Noting this duality relationship we get

(4.25)
$$M^-(\theta, z) = \frac{[1 - z\phi(\theta)] \exp\left\{\sum_1^\infty \frac{z^n}{n} k_{ns}^{(n)}\right\}}{\prod_1^s (1-\theta\xi_r)} \qquad (|z| < 1, \; |\theta| \geqq 1)$$

(4.26)
$$M^+(\theta, z) = \frac{\exp\left\{\sum_1^\infty \frac{z^n}{n} k_{ns}^{(n)}\right\}}{\prod_1^s (1-\theta\xi_r)} \qquad (|z| < 1, \; |\theta| \leqq 1)$$

where $k_{ns}^{(n)} = \Pr\{Y_1 + \cdots + Y_n = ns\}$ and $\xi_r = \xi_r(z)$ $(r = 1, \; 2 \cdots s)$ are the $s$ roots of the equation (4.6).

When $i = 0$, we have

(4.27)
$$\sum_{n=0}^\infty \sum_{j=0}^\infty P_{0j}^{(n)} z^n \theta^n = \frac{1}{1-z} \prod_1^s \left(\frac{1-\xi_r}{1-\theta\xi_r}\right)$$

and

(4.28)
$$E(\theta^{Q_\infty}) = \prod_1^s \left(\frac{1-\zeta_r}{1-\theta\zeta_r}\right)$$

if $\rho_2 < 1$, while no such distribution exists for $\rho_2 \geqq 1$. Here the relative traffic intensity $\rho_2 = \rho^{-1} = s/-\lambda\psi'(0)$. Further, the distribution $G^{(n)}$ of the number of arriving groups in a busy period is given by

(4.29)
$$\sum_{n=1}^\infty G^{(n)} z^n = 1 - \frac{1-z}{\prod_1^s (1-\xi_r)}$$

and the probability $G_{ij}^{(n)}$ defined in (3.16) is obtained as

(4.30)
$$\sum_{n=0}^\infty \sum_{j=1}^\infty G_{0j}^{(n)} z^{(n)} \theta^j = \frac{1}{\prod_1^s (1-\theta\xi_r)}$$

(4.31)
$$\sum_{n=0}^\infty \sum_{i=1}^\infty \sum_{j=1}^\infty G_{ij}^{(n)} z^n \omega^i \theta^j = \frac{\omega\theta}{(1-\omega\theta)[\omega^s - zK(\omega)]} \prod_1^s \left(\frac{\omega-\xi_r}{1-\theta\xi_r}\right).$$

7. *The queue* $GI|E_s|1$:

The inter-arrival times have the distribution (3.3) and the service times have the distribution $dA(x)$ given by (4.12). The relative traffic intensity $\rho_2 = s/(-\lambda\psi'(0)) < \infty$. The study of this system is identical with that of $GI^s|M|1$ above, if we consider the service time of each arriving customer in $GI|E_s|1$ as consisting of $s$ phases each with a negative exponential distribution $\lambda e^{-\lambda t}dt(0 < t < \infty)$. Suppose $Q'_n$ is the queue length just before the arrival of a customer in $GI|E_s|1$ and $Q_n$, in $GI^s|M|1$. We have

$$Q'_n = \left[\frac{Q_n+s-1}{s}\right]. \tag{4.32}$$

8 *and* 9. *The queues* $GI^{(x)}|M|1$ *and* $GI^r|M^s|1$:

The transition probabilities of the chain $\{Q_n\}$ in these systems are given by the equations (4.29)—(4.31), where $\xi_r$ $(r = 1, 2 \cdots s)$ are the roots of the relevant form of the equation

$$1-z\phi(\theta) = 0.$$

The discussion follows as in the case of $M|G^{(v)}|1$ and $M^r|G^s|1$ respectively.

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## References

[1] Bailey, N. T. J., On Queueing Processes with Bulk Service, *J. Roy. Stat. Soc.* B16 (1954), 80—87.

[2] Boudreau, P. E., Griffin, J. S., Jr., and Mark Kac, An Elementary Queueing Problem, *Amer. Math. Monthly* 68 (1962) 713—724.

[3] Feller, W., On Combinatorial Methods in Fluctuation Theory, *Probability and Statistics, Harald Cramer Volume*, John Wiley and Sons, (1959), 75—91.

[4] Foster F. G., Queues with Batch Arrivals I, *Acta Math. Acad. Sci. Hung.* 12 (1961), 1—10.

[5] Foster F. G. and Nyunt, K. M., Queues with Batch Departures I, *Ann. Math. Stat.* 32 (1961), 1324—1332.

[6] Jaiswal, N. K., A Bulk-Service Queueing Problem with Variable Capacity, *J. Roy. Stat. Soc.* B23 (1961), 143—148.

[7] Keilson, J., The General Bulk Queue as a Hilbert Problem, *J. Roy. Stat. Soc.* B24 (1962), 344—358.

[8] Kemperman, J. H. B., *The Passage Problem of a Stationary Markov Chain*, The University of Chicago Press (1961).

[9] Kendall, D. G., Stochastic Processes occurring in the Theory of Queues and Their Analysis by the Method of Imbedded Markov Chain, *Ann. Math. Stat.* 24 (1953), 338—354.

[10] Miller, R. G., Jr., A contribution to the Theory of Bulk Queues, *J. Roy. Stat. Soc.* B21 (1959), 320—337.

[11] Prabhu, N.U., and Narayan Bhat U., Some First Passage Problems and Their Application to Queues, *Sankhya* A25 (1963), 281—292.

[12] Spitzer, F., A Combinatorial Lemma and its Application to Probability Theory, *Trans. Amer. Math. Soc.* 82 (1956), 323—339.
[13] Takács, L., Transient Behaviour of Single Server Queueing Processes with Erlang Input, *Trans. Amer. Math. Soc.* 100 (1961), 1—28.
[14] Takács, L., Transient Behaviour of a Single Server Queueing Process with Recurrent Input and Gamma Service Time, *Ann. Math. Stat.* 32 (1961), 1286—1298.
[15] Gaver, D. P., Jr., Imbedded Markov Chain Analysis of a Waiting Line Process in Continuous Time, *Ann. Math. Stat.* 30 (1959), 698—720.

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