

Linear monads

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A monad $T = (T, \mu, \eta)$ on a category C is said to be *linear* with respect to a dense functor $N : A \rightarrow C$ if the operator T is the epimorphic image of a certain colimit of its values on A . The main aim of the article is to relate the concept of a linear monad to that of a monad with *rank*. A comparison is then made between linear monads and algebraic theories.

Introduction

In Section 1 we commence with a dense functor $N : A \rightarrow C$ and a monad $T = (T, \mu, \eta)$ on C such that the canonical transformation

$\int^A C(NA, C) \cdot TNA \rightarrow TC$ is an epimorphism. Such a monad is called linear

or, more precisely, N -linear. We prove that the free algebras on the values NA form a dense full subcategory of the Eilenberg-Moore category C^T . The terminology follows that of Day [3], Section 5.

Once the foregoing result is established it allows a comparison to be made between C^T as a full subcategory of a functor category $[B, V]$ and the category C^t of algebras in $[B, V]$ derived from the resultant algebraic theory of T (*cf.* Diers [5]). Conditions on C^T to be a Birkhoff subcategory of C^t are examined in Section 2.

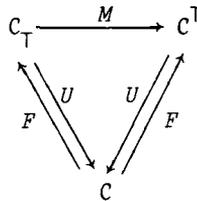
We note here that all categorical algebra is *relative* to a fixed complete and cocomplete symmetric monoidal closed ground category

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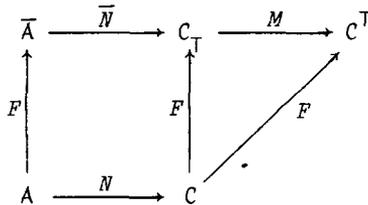
$V = (V, \vee, \otimes, I, \dots)$ unless otherwise indicated. The terminology and notation are basically derived from Eilenberg and Kelly [6] and Mac Lane [9].

1. Linear monads and rank

Throughout this section we suppose that $T = (T, \mu, \eta)$ is a given monad on a category C and that $N : A \rightarrow C$ is a fully faithful dense functor. The standard resolution of T into a Kleisli category and an Eilenberg-Moore category is denoted by



where M is the dense comparison functor. Furthermore, we let \bar{A} denote the full image of $FN : A \rightarrow C_T$ and let $\bar{N} : \bar{A} \rightarrow C_T$ denote the induced functor such that $FN = \bar{N}F$:



If we now suppose that A is small and C^T is cocomplete then, by Day and Kelly [4], (7.1), we have:

LEMMA 1.1. *The composite $M\bar{N} : \bar{A} \rightarrow C^T$ is dense iff each natural transformation α_B from $C^T(M\bar{N}B, C)$ to $C^T(M\bar{N}B, D)$ is of the form $C^T(1, f)$ for a unique T -homomorphism f from C to D . //*

THEOREM 1.2 (The representation theorem for monads). *The comparison functor $M : C_T \rightarrow C^T$ is dense and, for each algebra $(C, \zeta) \in C^T$, the natural transformations from $C^T(M-, C)$ to a prealgebra $G : C_T^{OP} \rightarrow V$*

correspond to the elements in the equaliser of

$$VGC \begin{array}{c} \xrightarrow{VG\mu} \\ \xrightarrow{VGT\zeta} \end{array} VGTC ,$$

where μ and $T\zeta$ are regarded as morphisms in C_T .

For the proof see Day [2], Proposition 8.2. //

THEOREM 1.3. *The composite $M\bar{N} : \bar{A} \rightarrow C^T$ is dense if the canonical transformation*

$$C(TC, D) \rightarrow \int_A [C(NA, C), C(TNA, D)]$$

is a monomorphism for all $C \in C$ and $D \in C^T$.

Proof. The notation U will sometimes be omitted. Suppose $\alpha : C^T(FNA, C) \rightarrow C^T(FNA, D)$ is a transformation which is natural in $FNA \in \bar{A}$. An extension $\bar{\alpha}$ is defined by commutativity of

$$\begin{array}{ccc} C^T(FB, C) & \xrightarrow{\bar{\alpha}} & C^T(FB, D) \\ \parallel \wr & & \parallel \wr \\ \int_A [C(NA, B), C^T(FNA, C)] & \xrightarrow{[1, \alpha]} & \int_A [C(NA, B), C^T(FNA, D)] \end{array} .$$

First we note that $\bar{\alpha}|_{\bar{A}} = \alpha$: the diagram

$$\begin{array}{ccc} C^T(TNA', C) & \xrightarrow{\alpha} & C^T(TNA', D) \\ \parallel \wr & & \parallel \wr \\ \int_A [C(NA, NA'), C^T(TNA, C)] & \xrightarrow{[1, \alpha]} & \int_A [C(NA, NA'), C^T(TNA, D)] \end{array}$$

transforms to

$$\begin{array}{ccc}
 C(NA, NA') \otimes C^\top(TNA', C) & \xrightarrow{1 \otimes \alpha} & C(NA, NA') \otimes C^\top(TNA', D) \\
 \downarrow & & \downarrow \\
 C^\top(TNA, C) & \xrightarrow{\alpha} & C^\top(TNA, D) ,
 \end{array}$$

which commutes by naturality of α . Secondly, if β is a transformation from $C^\top(FB, C)$ to $C^\top(FB, D)$ which is natural in $FB \in C_T$ then $\beta = \bar{\beta}$. This follows from the diagram:

$$\begin{array}{ccc}
 C^\top(TB, C) & \xrightarrow{\beta} & C^\top(TB, D) \\
 \Downarrow \cong & & \Downarrow \cong \\
 \int_A [C(NA, B), C^\top(TNA, C)] & \xrightarrow{[1, \beta]} & \int_A [C(NA, B), C^\top(TNA, D)] ,
 \end{array}$$

which commutes by naturality of β . By Theorem 1.2 it is now required to show that $\bar{\alpha}$ corresponds to the element $\bar{\alpha}(\zeta) \in VGC = C_0^\top(FC, D)$ in the equaliser of:

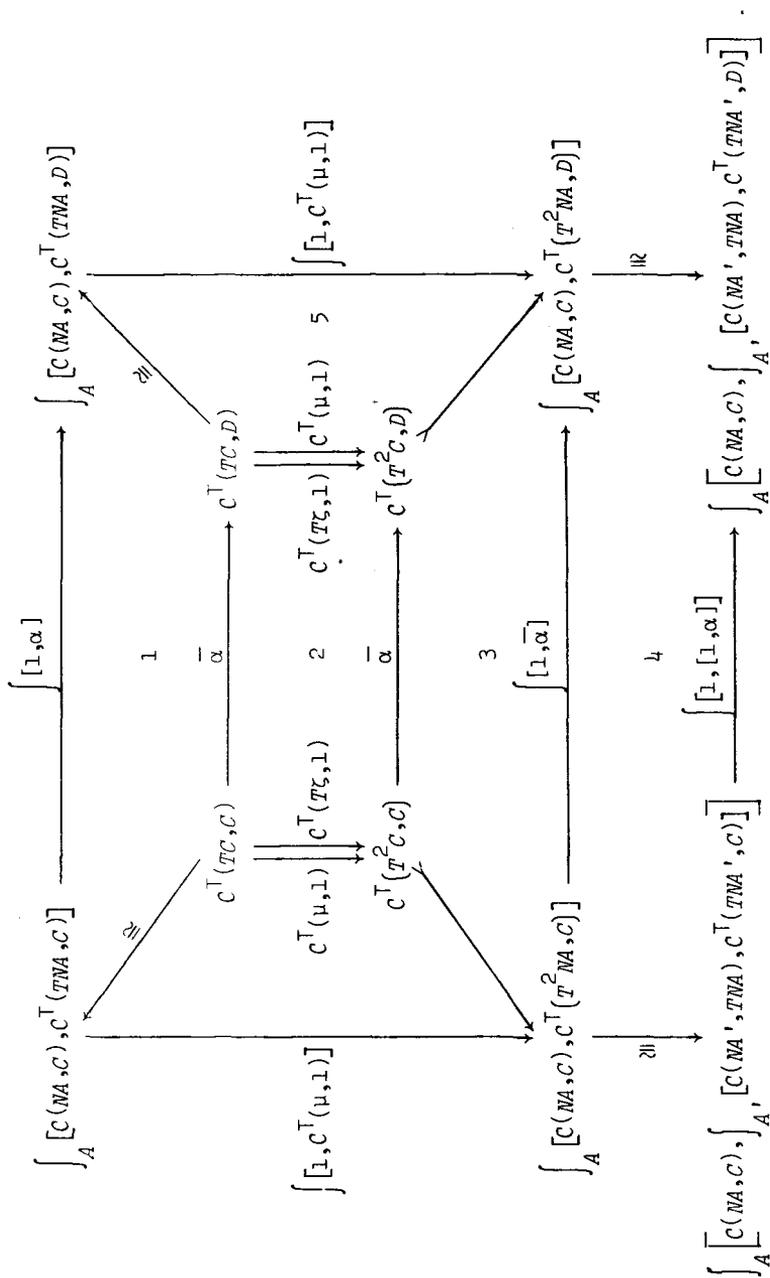
$$VGC \begin{array}{c} \xrightarrow{VG\mu} \\ \xrightarrow{VGT\zeta} \end{array} VGTC .$$

Firstly, because $\bar{\alpha} = \{\bar{\alpha}_{FB}\}$ is natural in $B \in C$, we see that the family $\bar{\alpha}_{FB}$ is derived from $\bar{\alpha}_{FC}(\zeta) : I \rightarrow C^\top(FC, D)$ by the (ordinary) representation theorem:

$$\begin{array}{ccc}
 C^\top(FB, C) & \xrightarrow{\bar{\alpha}_{FB}} & C^\top(FB, D) \\
 \Downarrow \cong & \nearrow & \\
 C(B, UC) & &
 \end{array}$$

Thus it remains to verify that $\bar{\alpha}(\zeta)$ is in the equaliser of $(VG\mu, VGT\zeta)$. Consider Diagram 1.4; subdiagrams 1 and 4 commute by definition of $\bar{\alpha}$ so

DIAGRAM 1.4



it remains to show that subdiagrams 2, 3, and 5, and the exterior commute.

Diagram 2 becomes Diagram 1.5 which clearly commutes. Diagram 3 becomes Diagram 1.6; thus it suffices to show that subdiagram 3' commutes. This follows by applying the representation theorem to $D \in \mathcal{C}^T$ because both legs are natural in D ; this diagram then becomes Diagram 1.7.

DIAGRAM 1.5

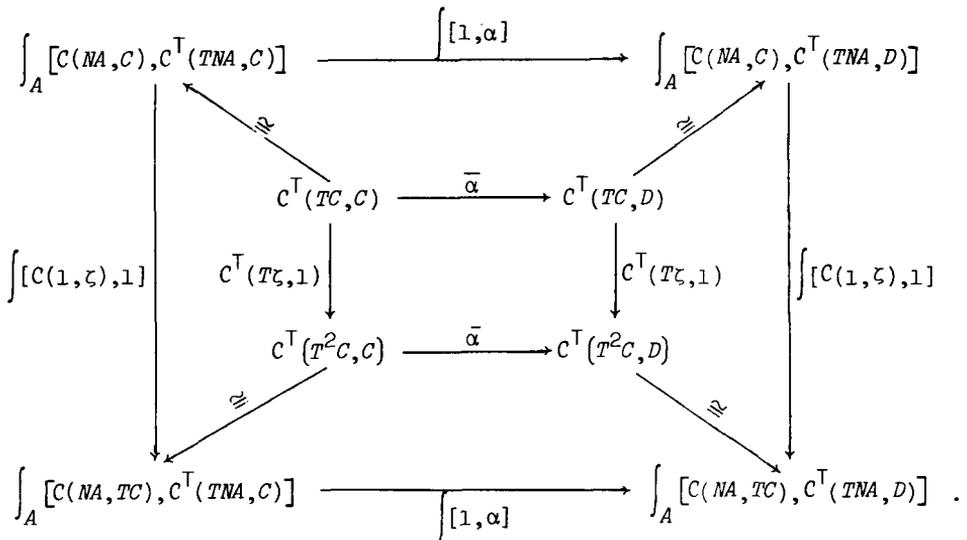


DIAGRAM 1.6

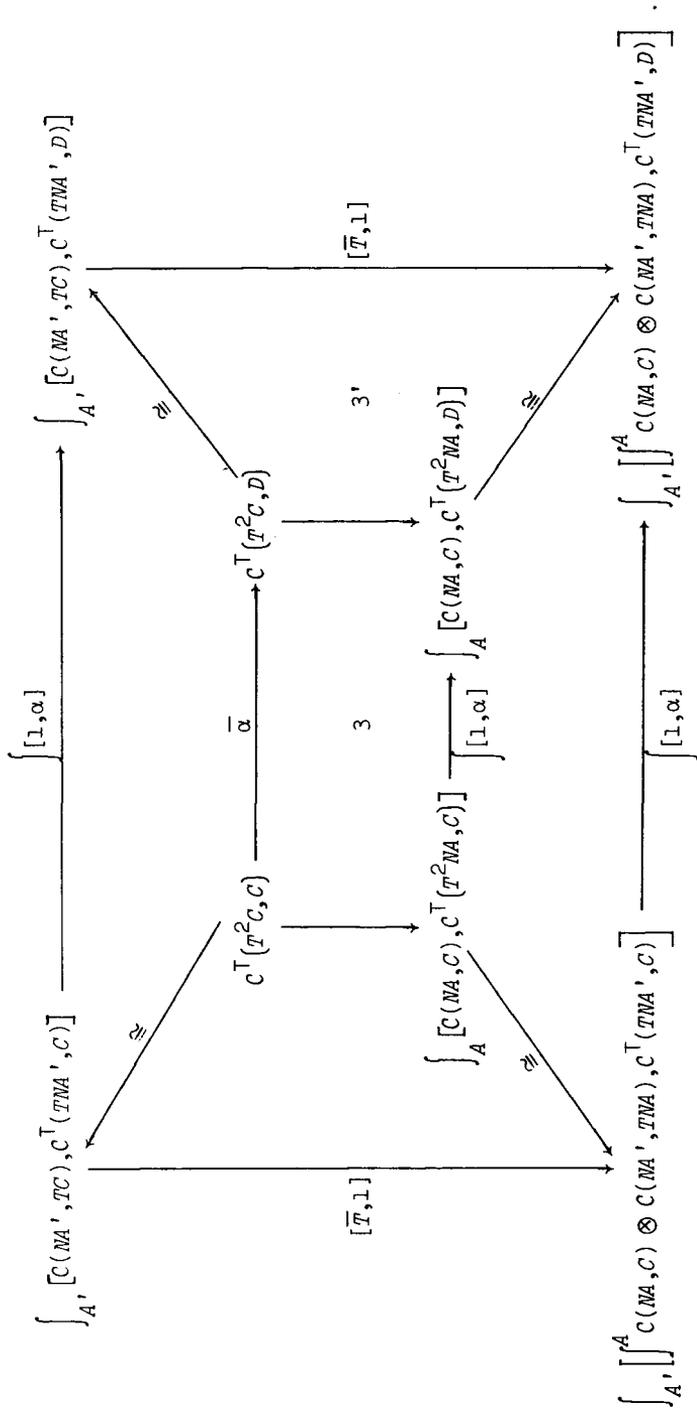


DIAGRAM 1.7

$$\begin{array}{ccc}
 C(NA, C) \otimes C(NA', TNA) & \xrightarrow{T^2 \otimes 1} & C^T(T^2NA, T^2C) \otimes C(NA', TNA) \\
 \downarrow T \otimes 1 & & \downarrow 1 \otimes T \\
 C(TNA, TC) \otimes C(NA', TNA) & & C^T(T^2NA, T^2C) \otimes C^T(TNA', T^2NA) \\
 \downarrow \text{compn.} & & \downarrow \text{compn.} \\
 C(NA', TC) & \xrightarrow{T} & C^T(TNA', T^2C) .
 \end{array}$$

Again, this diagram commutes by the representation theorem applied to $C \in \mathcal{C}$. Next consider Diagram 5, which transforms to

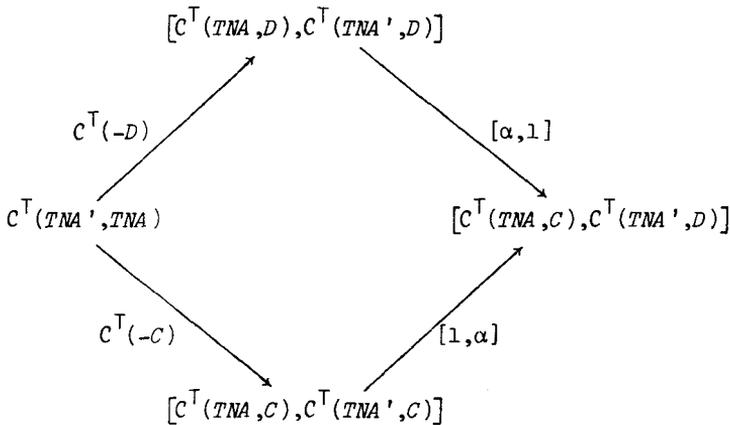
$$\begin{array}{ccc}
 C(NA, C) & \xrightarrow{C^T(T-, D)} & \int_A [C(TC, D), C(TNA, D)] \\
 \downarrow C^T(T^2-, D) & & \downarrow [C^T(\mu, 1), 1] \\
 \int_A [C^T(T^2C, D), C(T^2NA, D)] & \xrightarrow{[1, C^T(\mu, 1)]} & \int_A [C^T(T^2C, D), C^T(TNA, D)] .
 \end{array}$$

This diagram commutes by naturality of $\mu : T^2 \rightarrow T$. It remains to check that the diagram

$$\begin{array}{ccc}
 * & \xrightarrow{\alpha} & * \\
 C^T(\mu, 1) \downarrow & & \downarrow C^T(\mu, 1) \\
 * & \xrightarrow{\bar{\alpha}} & *
 \end{array}$$

commutes.

This diagram transforms to



composed with

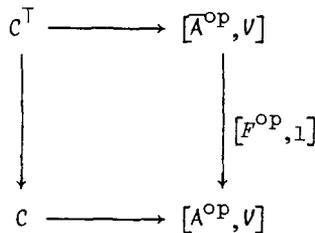
$$C(NA', TNA) \xrightarrow{T} C^\top(TNA', T^2NA) \xrightarrow{C^\top(1, \mu)} C^\top(TNA', TNA) ;$$

thus it commutes by naturality of α . //

In view of this result we make the following definition with respect to a fully faithful dense functor $N : A \rightarrow C$.

DEFINITION 1.8. A monad $T = (T, \mu, \eta)$ on C is called *linear* (or *N-linear*) if $C(NA, C) \circ TNA$ exists for all $C \in C$ and $C(NA, C) \circ TNA \rightarrow TC$ is an epimorphism. The monad is called *strictly linear* if this transformation is an isomorphism.

COROLLARY 1.9. If T on C is *N-linear* then the canonical diagram



commutes (to within a natural isomorphism) where the horizontal functors are fully faithful. //

2. Comparison with algebraic theories

Let $N : A \rightarrow C$ and $T = (T, \mu, \eta)$ be as in Section 1. Then

$t = F^{op} : A^{op} \rightarrow \overline{A}^{op} = T$ is an N -algebraic theory in the sense of Diers [5]. Thus we form the category C^t of t -algebras by means of the pullback

$$\begin{array}{ccc}
 C^t & \longrightarrow & [T, V] \\
 U^t \downarrow & & \downarrow [t, 1] \\
 C & \longrightarrow & [A^{op}, V],
 \end{array}$$

where the horizontal functors are fully faithful.

By Corollary 1.9, C^T is a full reflective subcategory of $[T, V]$ and it lies in C^t . This gives a reflection $S : C^t \rightarrow C^T$:

$$\begin{array}{ccc}
 C^T & \xleftarrow{S} & C^t \\
 & \xrightarrow{H} & \\
 F \swarrow & & \searrow U^t \\
 & C &
 \end{array}$$

THEOREM 2.1. *If T is strictly linear then C^T is category equivalent to C^t .*

Proof. Because $C(NA, C) \circ TNA \cong TC$ we have that T preserves N -absolute colimits. Thus the hypotheses of Diers [5], Theorem 5.1, are satisfied by $F \dashv U$. //

Now suppose that C has canonical factorisations for the system {strong epimorphisms and monomorphisms} (cf. Freyd and Kelly [7]).

PROPOSITION 2.2. *If the transformation*

$$(2.1) \quad \int^A C(NA, C) \otimes C(NA', TNA) \rightarrow C(NA', TC)$$

is a strong epimorphism in V and $C(NA, C) \circ TNA$ exists in C and $C(NA, -) : C \rightarrow V$ preserves strong epimorphisms for all $A \in A$, then the

unit η of the reflection $S \rightarrow H$ is a strong epimorphism.

Proof. On applying $- \circ NA'$ to both sides of (2.1) we see that $C(NA, C) \circ TNA \rightarrow TC$ is a strong epimorphism. Thus Theorem 1.3 applies and also T preserves strong epimorphisms since, if $e : C \rightarrow D$ is a strong epimorphism in C , we have that

$$\begin{array}{ccc} C(NA, C) \circ TNA \rightarrow TC & & \\ \downarrow C(1, e) \circ 1 & & \downarrow Te \\ C(NA, D) \circ TNA \rightarrow TD & & \end{array}$$

commutes. Thus the diagonal is a strong epimorphism so Te is a strong epimorphism. Now consider the factorisation $\eta_C : C \rightarrow D \rightarrow SC$ in C . It is required to show that D has a T -algebra structure. This structure is derived from the following diagram:

$$\begin{array}{ccc} \int^A C(NA, C) \otimes C(NA', TNA) & \xrightarrow{\zeta_C} & C(NA', C) \\ \downarrow a & & \downarrow \\ C(NA', TC) & & \\ \downarrow b & & \\ C(NA', TD) & \dashrightarrow & C(NA', D) \\ \downarrow & & \downarrow \\ C(NA', TSC) & \longrightarrow & C(NA', SC) \end{array}$$

where a and b are both strong epimorphisms and the top morphism ζ_C

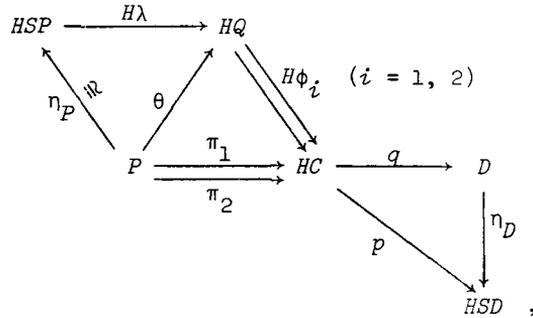
is derived from the C^t -structure on C in the following manner. An algebra $(C, \zeta) \in C^t$ comprises $C \in C$ together with

$$\zeta_C : \int^A C(NA', TNA) \otimes C(NA, C) \rightarrow C(NA', C) ;$$

that is, a structure for the monad $[t, 1]\bar{t}$ on $[A^{op}, V]$ where $\bar{t} \rightarrow [t, 1]$. Then, by factorisation, the dashed arrow provides a T -algebra structure on D , using the density of N . //

PROPOSITION 2.3. *Suppose the unit η of the reflection $S \rightarrow H$ is a strong epimorphism and C has kernel pairs. If $U : C^\top \rightarrow C$ reflects kernel pairs then C^\top is closed in C^t under coequalisers.*

Proof. Both U and U^t create kernel pairs and we omit them from the notation. Let $q : HC \rightarrow D$ be a coequaliser in C^t and let $p = \eta_D \cdot q$. Let (π_1, π_2) be the kernel pair of q in C^t and let (ϕ_1, ϕ_2) be the kernel pair of p in C^\top . This gives



where θ is monic, so η_P is monic and thus is an isomorphism. This implies that p is the coequaliser of $(H\phi_1 \cdot H\lambda, H\phi_2 \cdot H\lambda)$ in C^\top and that this latter pair is a kernel pair in C^t . Thus $(\phi_1 \lambda, \phi_2 \lambda)$ is a kernel pair in C^\top , so λ is an isomorphism, so θ is an isomorphism, so η_D is an isomorphism, as required to show that D lies in C^\top . //

COROLLARY 2.4. *If $\int^A C(NA, C) \otimes C(NA', TNA) \rightarrow C(NA', TC)$ is a strong epimorphism in V and $C(NA, C) \circ TNA$ exists in C and $C(NA, -)$ preserves strong epimorphisms for all $A \in A$ and C has kernel pairs reflected by $U : C^\top \rightarrow C$, then C^\top is a Birkhoff reflective subcategory of C^t . //*

PROPOSITION 2.5. *If C^t is cocomplete and $C(NA, -)$ preserves coequalisers of reflective pairs, then C^t is monadic over C iff*

(a) f is a coequaliser in C^t iff $U^t f$ is a coequaliser in C , and

(b) U^t reflects kernel pairs.

Proof. If C^t is cocomplete then $F^t \rightarrow U^t$ exists and $U^t F^t$ preserves coequalisers of reflective pairs since $U^t : C^t \rightarrow C$ creates coequalisers because $C(NA', -)$ preserves them (cf. Diers [5], Proposition 1.1). Thus the result follows from Borceux and Day [1], Corollary 6.2. //

PROPOSITION 2.6. *Suppose C and C^t are cocomplete and let $K : [T, V] \rightarrow C^t$ denote the canonical reflection. If U^t preserves epimorphisms and those unit components of the form $T(tA, -) \rightarrow KT(tA, -)$ are epimorphisms, then $U^t F^t$ generates a linear monad.*

Proof. We have

$$U^t F^t NA = U^t K \left\{ \int^A C(NA', NA) \otimes T(tA', -) \right\} \cong U^t K(T(tA, -))$$

by the representation theorem because N is fully faithful. Also

$$U^t F^t C = U^t K \left\{ \int^A C(NA, C) \otimes T(tA, -) \right\}.$$

Thus, to show that

$$(2.2) \quad \int^A C(NA, C) \cdot U^t F^t NA \rightarrow U^t F^t C$$

is an epimorphism consider the following diagram

$$\begin{array}{ccc}
 \int^A C(NA, C) \cdot U^t T(tA, -) & \longrightarrow & \int^A C(NA, C) \cdot U^t KT(tA, -) \\
 \downarrow \cong & & \downarrow (2.2) \\
 & & U^t K \left\{ \int^A C(NA, C) \otimes T(tA, -) \right\} \\
 & & \uparrow \cong \\
 U^t \left\{ \int^A C(NA, C) \cdot T(tA, -) \right\} & \longrightarrow & U^t \left\{ \int^A C(NA, C) \cdot KT(tA, -) \right\}.
 \end{array}$$

The bottom arrow is an epimorphism by hypothesis, so (2.2) is an epimorphism, as required. //

3. Example

Suppose the ground category V has canonical $E - M$ factorisations for the system $E - M = \{\text{strong epimorphisms and monomorphisms}\}$ (see Freyd and Kelly [7]). Suppose also that V has arbitrary cointersections of E -quotients and that finite powers preserve strong epimorphisms.

DEFINITION 3.1. A functor $G : A \rightarrow V$ from a category A with finite products to V is said to E -preserve finite products if the canonical morphism $G(A \times A') \rightarrow GA \times GA'$ is a strong epimorphism for all $A, A' \in A$.

DEFINITION 3.2. Let $M : A \rightarrow B$ be a functor between categories with finite products. Then V is said to satisfy *axiom* $E(\pi)$ if the left Kan extension of a functor $G : A \rightarrow V$ which E -preserves finite products along M again E -preserves finite products.

One then obtains results precisely analogous to those obtained for axiom π in Borceux and Day [1], Sections 1 and 2.

DEFINITION 3.3. If T is a finitary algebraic theory (see Borceux and Day [1], Definition 3.1) then a functor $G : T \rightarrow V$ which E -preserves finite products is called an E -algebra (of T).

Now let T^q denote the category of E -algebras for T , regarded as a full subcategory of $[T, V]$. Let T^b denote the ordinary category of algebras of T ; namely, the full subcategory of $[T, V]$ defined by the finite-product-preserving functors. Then there are inclusions $T^b \subset T^q \subset [T, V]$. The second embedding is *coreflective* and the coreflection maps G to the union of the E -algebras which are M -subfunctors of G ; the coreflection counit lies in M . The first embedding is *reflective* and the reflection maps $A \in T^q$ to the largest T^b E -quotient of A ; the reflection unit is in E . Thus we have

$$\begin{array}{ccccc}
 T^b & \rightleftarrows & T^a & \rightleftarrows & [T, V] \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 V & \rightleftarrows & (V_f^{op})^a & \rightleftarrows & [V_f^{op}, V]
 \end{array}$$

where V_f is the initial finitary theory and where the centre adjunction is a strictly linear monadic situation.

THEOREM 3.4. *If V satisfies the hypotheses of this section (and satisfies axiom $E(\pi)$) then a monad T on V generates a Birkhoff subcategory of an algebraic category T^b iff*

- (a) V^T has coequalisers,
- (b) $\int^m [m, X] \otimes [n, Tm] \rightarrow [n, TX]$ is a strong epimorphism, and
- (c) $U : V^T \rightarrow V$ reflects kernel pairs.

Proof. Because V^T has coequalisers iff V^T is cocomplete (see Linton [8]) the conditions are sufficient by Corollary 2.4. Necessity of (a) is clear since T^b is always cocomplete. Moreover, if V^T is a Birkhoff subcategory of T^b then the unit of the composite reflection $T^a \rightarrow T^b \rightarrow V^T$ is a strong epimorphism. This implies that (b) is necessary. Finally, the functor $U^t : T^b \rightarrow V$ reflects kernel pairs and the Birkhoff property implies that the embedding $V^T \subset T^b$ reflects kernel pairs, so (c) is necessary. //

An example of a monad which satisfies (a) and (b) but not (c) is the reflection to Hausdorff k -spaces from non-Hausdorff k -spaces.

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