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Linear monads

B.J. Day

A monad $\mathrm{T} = (T, \mu, \eta)$ on a category C is said to be *linear* with respect to a dense functor $N : A \to C$ if the operator T is the epimorphic image of a certain colimit of its values on A. The main aim of the article is to relate the concept of a linear monad to that of a monad with *rank*. A comparison is then made between linear monads and algebraic theories.

Introduction

In Section 1 we commence with a dense functor $N : A \to C$ and a monad $\mathrm{T} = (T, \mu, \eta)$ on C such that the canonical transformation $\int^{A} C(NA, C) \cdot TNA \to TC$ is an epimorphism. Such a monad is called linear or, more precisely, N-linear. We prove that the free algebras on the values NA form a dense full subcategory of the Eilenberg-Moore category C^{T}. The terminology follows that of Day [3], Section 5.

Once the foregoing result is established it allows a comparison to be made between C^{T} as a full subcategory of a functor category $[B, V]$ and the category C^{t} of algebras in $[B, V]$ derived from the resultant algebraic theory of T (*cf.* Diers [5]). Conditions on C^{T} to be a Birkhoff subcategory of C^{t} are examined in Section 2.

We note here that all categorical algebra is *relative* to a fixed complete and cocomplete symmetric monoidal closed ground category

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$V = (V, V, \otimes, I, \ldots)$ unless otherwise indicated. The terminology and notation are basically derived from Eilenberg and Kelly [6] and Mac Lane [9].

1. Linear monads and rank

Throughout this section we suppose that $\mathrm{T} = (T, \mu, \eta)$ is a given monad on a category C and that $N : A \to C$ is a fully faithful dense functor. The standard resolution of T into a Kleisli category and an Eilenberg-Moore category is denoted by

$$\begin{tikzcd} C_{\mathrm{T}} \arrow[rr, "M"] \arrow[dr, shift left, "U"] & & C^{\mathrm{T}} \arrow[dl, shift right, "U"'] \\ & C \arrow[ul, shift left, "F"] \arrow[ur, shift right, "F"'] & \end{tikzcd}$$

where M is the dense comparison functor. Furthermore, we let $\bar{A}$ denote the full image of $FN : A \to C_{\mathrm{T}}$ and let $\bar{N} : \bar{A} \to C_{\mathrm{T}}$ denote the induced functor such that $FN = \bar{N}F$:

$$\begin{tikzcd} \bar{A} \arrow[r, "\bar{N}"] & C_{\mathrm{T}} \arrow[r, "M"] & C^{\mathrm{T}} \\ A \arrow[u, "F"] \arrow[r, "N"] & C \arrow[u, "F"] \arrow[ur, "F"'] & \end{tikzcd}$$

If we now suppose that A is small and C^{T} is cocomplete then, by Day and Kelly [4], (7.1), we have:

LEMMA 1.1. *The composite* $M\bar{N} : \bar{A} \to C^{\mathrm{T}}$ *is dense iff each natural transformation* α_B *from* $C^{\mathrm{T}}(M\bar{N}B, C)$ *to* $C^{\mathrm{T}}(M\bar{N}B, D)$ *is of the form* $C^{\mathrm{T}}(1, f)$ *for a unique* T*-homomorphism* f *from* C *to* D . //

THEOREM 1.2 (The representation theorem for monads). *The compariso functor* $M : C_{\mathrm{T}} \to C^{\mathrm{T}}$ *is dense and, for each algebra* $(C, \zeta) \in C^{\mathrm{T}}$, *the natural transformations from* $C^{\mathrm{T}}(M-, C)$ *to a prealgebra* $G : C_{\mathrm{T}}^{\mathrm{op}} \to V$

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correspond to the elements in the equaliser of

$$VGC \underset{VGT\zeta}{\overset{VG\mu}{\rightrightarrows}} VGTC \ ,$$

where μ *and* $T\zeta$ *are regarded as morphisms in* C_T .

For the proof see Day [2], Proposition 8.2. //

THEOREM 1.3. *The composite* $M\bar{N} : \bar{A} \to C^T$ *is dense if the canonical transformation*

$$C(TC,\ D) \to \int_A [C(NA,\ C),\ C(TNA,\ D)]$$

is a monomorphism for all $C \in C$ *and* $D \in C^T$.

Proof. The notation U will sometimes be omitted. Suppose $\alpha : C^T(FNA,\ C) \to C^T(FNA,\ D)$ is a transformation which is natural in $FNA \in \bar{A}$. An extension $\bar{\alpha}$ is defined by commutativity of

$$\begin{tikzcd}
C^T(FB,C) \arrow[r, "\bar{\alpha}"] \arrow[d, "\wr\shortparallel"'] & C^T(FB,D) \arrow[d, "\shortparallel\wr"] \\
\int_A [C(NA,B),C^T(FNA,C)] \arrow[r, "\int {[1,\alpha]}"'] & \int_A [C(NA,B),C^T(FNA,D)]
\end{tikzcd} .$$

First we note that $\bar{\alpha}|\bar{A} = \alpha$: the diagram

$$\begin{tikzcd}
C^T(TNA',C) \arrow[r, "\alpha"] \arrow[d, "\wr\shortparallel"'] & C^T(TNA',D) \arrow[d, "\shortparallel\wr"] \\
\int_A [C(NA,NA'),C^T(TNA,C)] \arrow[r, "\int {[1,\alpha]}"'] & \int_A [C(NA,NA'),C^T(TNA,D)]
\end{tikzcd}$$

transforms to

$$\begin{tikzcd}
C(NA,NA') \otimes C^{T}(TNA',C) \arrow[r, "1 \otimes \alpha"] \arrow[d] & C(NA,NA') \otimes C^{T}(TNA',D) \arrow[d] \\
C^{T}(TNA,C) \arrow[r, "\alpha"'] & C^{T}(TNA,D)
\end{tikzcd} ,$$

which commutes by naturality of α . Secondly, if β is a transformation from $C^{T}(FB, C)$ to $C^{T}(FB, D)$ which is natural in $FB \in C_{T}$ then $\beta = \overline{\beta}$. This follows from the diagram:

$$\begin{tikzcd}
C^{T}(TB,C) \arrow[r, "\beta"] \arrow[d, "\wr|"'] & C^{T}(TB,D) \arrow[d, "|\wr"] \\
\int_{A} [C(NA,B), C^{T}(TNA,C)] \arrow[r, "\int {[1,\beta]}"'] & \int_{A} [C(NA,B), C^{T}(FNA,D)]
\end{tikzcd} ,$$

which commutes by naturality of β . By Theorem 1.2 it is now required to show that $\overline{\alpha}$ corresponds to the element $\overline{\alpha}(\zeta) \in VGC = C_{0}^{T}(FC, D)$ in the equaliser of:

$$VGC \underset{VGT\zeta}{\overset{VG\mu}{\rightrightarrows}} VGTC \; .$$

Firstly, because $\overline{\alpha} = \{\overline{\alpha}_{FB}\}$ is natural in $B \in C$, we see that the family $\overline{\alpha}_{FB}$ is derived from $\overline{\alpha}_{FC}(\zeta) : I \to C^{T}(FC, D)$ by the (ordinary) representation theorem:

$$\begin{tikzcd}
C^{T}(FB,C) \arrow[r, "\overline{\alpha}_{FB}"] \arrow[d, "\wr|"'] & C^{T}(FB,D) \\
C(B,UC) \arrow[ur, dashed] &
\end{tikzcd}$$

Thus it remains to verify that $\overline{\alpha}(\zeta)$ is in the equaliser of $(VG\mu, VGT\zeta)$. Consider Diagram 1.4; subdiagrams 1 and 4 commute by definition of $\overline{\alpha}$ so

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DIAGRAM 1.4

$$\begin{tikzcd}
\int_A [C(NA,C),C^{\mathsf{T}}(TNA,C)] \arrow[rrr, "{\int[1,\alpha]}"] \arrow[ddd, "{\int[1,C^{\mathsf{T}}(\mu,1)]}"'] & & & \int_A [C(NA,C),C^{\mathsf{T}}(TNA,D)] \arrow[ddd, "{\int[1,C^{\mathsf{T}}(\mu,1)]}"] \\
& C^{\mathsf{T}}(TC,C) \arrow[r, "\bar{\alpha}"] \arrow[ul, "\cong"'] \arrow[d, shift right, "{C^{\mathsf{T}}(\mu,1)}"'] \arrow[d, shift left, "{C^{\mathsf{T}}(T\zeta,1)}"] \arrow[r, phantom, shift left=6ex, "1"] \arrow[d, phantom, shift left=12ex, "2"] & C^{\mathsf{T}}(TC,D) \arrow[ur, "\cong"] \arrow[d, shift right, "{C^{\mathsf{T}}(T\zeta,1)}"'] \arrow[d, shift left, "{C^{\mathsf{T}}(\mu,1)}"] \arrow[dr, phantom, shift left=4ex, "5"] & \\
& C^{\mathsf{T}}(T^2C,C) \arrow[r, "\bar{\alpha}"] \arrow[dl] \arrow[r, phantom, shift right=6ex, "3"] & C^{\mathsf{T}}(T^2C,D) \arrow[dr] & \\
\int_A [C(NA,C),C^{\mathsf{T}}(T^2NA,C)] \arrow[rrr, "{\int[1,\bar{\alpha}]}"] \arrow[d, "\cong"'] \arrow[rrrd, phantom, "4"] & & & \int_A [C(NA,C),C^{\mathsf{T}}(T^2NA,D)] \arrow[d, "\cong"] \\
\int_A \left[C(NA,C),\int_{A'} [C(NA',TNA),C^{\mathsf{T}}(TNA',C)]\right] \arrow[rrr, "{\int[1,[1,\alpha]]}"] & & & \int_A \left[C(NA,C),\int_{A'} [C(NA',TNA),C^{\mathsf{T}}(TNA',D)]\right] .
\end{tikzcd}$$

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it remains to show that subdiagrams 2, 3, and 5 , and the exterior commute. Diagram 2 becomes Diagram 1.5 which clearly commutes. Diagram 3 becomes Diagram 1.6; thus it suffices to show that subdiagram 3' commutes. This follows by applying the representation theorem to $D \in C^T$ because both legs are natural in D ; this diagram then becomes Diagram 1.7.

DIAGRAM 1.5

$$\begin{tikzcd}
\int_A [C(NA,C),C^T(TNA,C)] \arrow[rrr, "{\int[1,\alpha]}"] \arrow[ddd, "{\int[C(1,\zeta),1]}"'] & & & \int_A [C(NA,C),C^T(TNA,D)] \arrow[ddd, "{\int[C(1,\zeta),1]}"] \\
& C^T(TC,C) \arrow[ul, "\cong"'] \arrow[r, "\bar{\alpha}"] \arrow[d, "{C^T(T\zeta,1)}"'] & C^T(TC,D) \arrow[ur, "\cong"] \arrow[d, "{C^T(T\zeta,1)}"] & \\
& C^T(T^2C,C) \arrow[r, "\bar{\alpha}"] \arrow[dl, "\cong"] & C^T(T^2C,D) \arrow[dr, "\cong"'] & \\
\int_A [C(NA,TC),C^T(TNA,C)] \arrow[rrr, "{\int[1,\alpha]}"'] & & & \int_A [C(NA,TC),C^T(TNA,D)]
\end{tikzcd}$$

.

DIAGRAM 1.6

$$\begin{tikzcd}
\int_{A'}\left[C(NA',TC),C^{\mathsf{T}}(TNA',C)\right] \arrow[rrr, "{\int[1,\alpha]}"] \arrow[ddd, "{[\overline{T},1]}"'] & & & \int_{A'}\left[C(NA',TC),C^{\mathsf{T}}(TNA',D)\right] \arrow[ddd, "{[\overline{T},1]}"] \\
& C^{\mathsf{T}}(T^2C,C) \arrow[r, "\overline{\alpha}"] \arrow[ul, "\cong"'] \arrow[d] \arrow[dr, phantom, "3"] & C^{\mathsf{T}}(T^2C,D) \arrow[ur, "\cong"] \arrow[d] \arrow[dr, phantom, "3'"] & \\
& \int_{A}\left[C(NA,C),C^{\mathsf{T}}(T^2NA,C)\right] \arrow[r, "{\int[1,\alpha]}"'] \arrow[dl, "\cong"] & \int_{A}\left[C(NA,C),C^{\mathsf{T}}(T^2NA,D)\right] \arrow[dr, "\cong"] & \\
\int_{A'}\left[\int^{A} C(NA,C)\otimes C(NA',TNA),C^{\mathsf{T}}(TNA',C)\right] \arrow[rrr, "{\int[1,\alpha]}"'] & & & \int_{A'}\left[\int^{A} C(NA,C)\otimes C(NA',TNA),C^{\mathsf{T}}(TNA',D)\right]
\end{tikzcd}$$

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DIAGRAM 1.7

$$\begin{tikzcd}
C(NA,C) \otimes C(NA',TNA) \arrow[r, "T^2 \otimes 1"] \arrow[d, "T \otimes 1"'] & C^{T}(T^2NA,T^2C) \otimes C(NA',TNA) \arrow[d, "1 \otimes T"] \\
C(TNA,TC) \otimes C(NA',TNA) \arrow[d, "\text{compn.}"'] & C^{T}(T^2NA,T^2C) \otimes C^{T}(TNA',T^2NA) \arrow[d, "\text{compn.}"] \\
C(NA',TC) \arrow[r, "T"] & C^{T}(TNA',T^2C)
\end{tikzcd} .$$

Again, this diagram commutes by the representation theorem applied to $C \in \mathcal{C}$. Next consider Diagram 5, which transforms to

$$\begin{tikzcd}
C(NA,C) \arrow[r, "C^{T}(T-{,}D)"] \arrow[d, "C^{T}(T^2-{,}D)"] & \int_A [C(TC,D),C(TNA,D)] \arrow[d, "{[C^{T}(\mu,1),1]}"'] \\
\int_A [C^{T}(T^2C,D),C(T^2NA,D)] \arrow[r, "\int{[1,C^{T}(\mu,1)]}"'] & \int_A [C^{T}(T^2C,D),C^{T}(TNA,D)]
\end{tikzcd} .$$

This diagram commutes by naturality of $\mu : T^2 \to T$. It remains to check that the diagram

$$\begin{tikzcd}
* \arrow[r, "\alpha"] \arrow[d, "C^{T}(\mu,1)"'] & * \arrow[d, "C^{T}(\mu,1)"] \\
* \arrow[r, "\bar{\alpha}"'] & *
\end{tikzcd}$$

commutes.

This diagram transforms to

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$$\begin{tikzcd}[column sep=small]
 & {[C^T(TNA,D),C^T(TNA',D)]} \arrow[dr, "{[\alpha,1]}"] & \\
C^T(TNA',TNA) \arrow[ur, "C^T(-D)"] \arrow[dr, "C^T(-C)"'] & & {[C^T(TNA,C),C^T(TNA',D)]} \\
 & {[C^T(TNA,C),C^T(TNA',C)]} \arrow[ur, "{[1,\alpha]}"'] &
\end{tikzcd}$$

composed with

$$C(NA',\ TNA) \xrightarrow{T} C^T(TNA',\ T^2NA) \xrightarrow{C^T(1,\mu)} C^T(TNA',\ TNA)\ ;$$

thus it commutes by naturality of α . //

In view of this result we make the following definition with respect to a fully faithful dense functor $N : A \to C$.

DEFINITION 1.8. A monad $T = (T,\ \mu,\ \eta)$ on C is called *linear* (or N-linear) if $C(NA,\ C) \circ TNA$ exists for all $C \in C$ and $C(NA,\ C) \circ TNA \to TC$ is an epimorphism. The monad is called *strictly linear* if this transformation is an isomorphism.

COROLLARY 1.9. *If* T *on* C *is* N*-linear then the canonical diagram*

$$\begin{tikzcd}
C^T \arrow[r] \arrow[d] & {[A^{op},V]} \arrow[d, "{[F^{op},1]}"] \\
C \arrow[r] & {[A^{op},V]}
\end{tikzcd}$$

commutes (to within a natural isomorphism) where the horizontal functors are fully faithful. //

2. Comparison with algebraic theories

Let $N : A \to C$ and $T = (T,\ \mu,\ \eta)$ be as in Section 1. Then

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$t = F^{op} : A^{op} \to \bar{A}^{op} = T$ is an N-algebraic theory in the sense of Diers [5]. Thus we form the category C^t of t-algebras by means of the pullback

$$\begin{tikzcd} C^t \arrow[r] \arrow[d, "U^t"'] & {[T,V]} \arrow[d, "{[t,1]}"] \\ C \arrow[r] & {[A^{op},V]} \end{tikzcd} ,$$

where the horizontal functors are fully faithful.

By Corollary 1.9, C^T is a full reflective subcategory of $[T, V]$ and it lies in C^t. This gives a reflection $S : C^t \to C^T$:

$$\begin{tikzcd} C^T \arrow[rr, shift left, "H"'] \arrow[dr, shift left, "U"] & & C^t \arrow[ll, shift left, "S"'] \arrow[dl, "U^t"] \\ & C \arrow[ul, shift left, "F"] & \end{tikzcd}$$

THEOREM 2.1. *If* T *is strictly linear then* C^T *is category equivalent to* C^t .

Proof. Because $C(NA, C) \circ TNA \cong TC$ we have that T preserves N-absolute colimits. Thus the hypotheses of Diers [5], Theorem 5.1, are satisfied by $F \dashv U$. //

Now suppose that C has canonical factorisations for the system {strong epimorphisms and monomorphisms} (*cf.* Freyd and Kelly [7]).

PROPOSITION 2.2. *If the transformation*

$$\int^A C(NA, C) \otimes C(NA', TNA) \to C(NA', TC) \tag{2.1}$$

is a strong epimorphism in V *and* $C(NA, C) \circ TNA$ *exists in* C *and* $C(NA, -) : C \to V$ *preserves strong epimorphisms for all* $A \in A$, *then the* *unit* η *of the reflection* $S \dashv H$ *is a strong epimorphism.*

Proof. On applying $- \circ NA'$ to both sides of (2.1) we see that $C(NA,\ C) \circ TNA \to TC$ is a strong epimorphism. Thus Theorem 1.3 applies and also T preserves strong epimorphisms since, if $e : C \to D$ is a strong epimorphism in $\mathcal{C}$, we have that

$$\begin{tikzcd} C(NA,C)\circ TNA \arrow[r] \arrow[d, "C(1,e)\circ 1"'] & TC \arrow[d, "Te"] \\ C(NA,D)\circ TNA \arrow[r] & TD \end{tikzcd}$$

commutes. Thus the diagonal is a strong epimorphism so Te is a strong epimorphism. Now consider the factorisation $\eta_C : C \twoheadrightarrow D \rightarrowtail SC$ in $\mathcal{C}$. It is required to show that D has a T-algebra structure. This structure is derived from the following diagram:

$$\begin{tikzcd} \int^{A} C(NA,C)\otimes C(NA',TNA) \arrow[r, "\zeta_C"] \arrow[d, "a"'] & C(NA',C) \arrow[dd] \\ C(NA',TC) \arrow[d, "b"'] & \\ C(NA',TD) \arrow[r, dashed] \arrow[d] & C(NA',D) \arrow[d, tail] \\ C(NA',TSC) \arrow[r] & C(NA',SC) \end{tikzcd}$$,

where a and b are both strong epimorphisms and the top morphism ζ_C is derived from the $\mathcal{C}^t$-structure on C in the following manner. An algebra $(C,\ \zeta) \in \mathcal{C}^t$ comprises $C \in \mathcal{C}$ together with

$$\zeta_C : \int^{A} C(NA',\ TNA) \otimes C(NA,\ C) \to C(NA',\ C)\ ;$$

that is, a structure for the monad $[t,\ 1]\bar{t}$ on $[A^{op},\ V]$ where $\bar{t} \dashv [t,\ 1]$. Then, by factorisation, the dashed arrow provides a T-algebra structure on D, using the density of N. //

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PROPOSITION 2.3. *Suppose the unit* η *of the reflection* $S \dashv H$ *is a strong epimorphism and* C *has kernel pairs. If* $U : C^{\top} \to C$ *reflects kernel pairs then* $C^{\top}$ *is closed in* C^{t} *under coequalisers.*

Proof. Both U and U^{t} create kernel pairs and we omit them from the notation. Let $q : HC \to D$ be a coequaliser in C^{t} and let $p = \eta_D \cdot q$. Let (π_1, π_2) be the kernel pair of q in C^{t} and let (ϕ_1, ϕ_2) be the kernel pair of p in $C^{\top}$. This gives

$$\begin{tikzcd}
HSP \arrow[rr, "H\lambda"] & & HQ \arrow[dr, shift left, "H\phi_i \ (i = 1,\, 2)"] \arrow[dr, shift right] & & \\
& P \arrow[ul, "\eta_P", "\cong"'] \arrow[ur, "\theta"] \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] & & HC \arrow[r, "q"] \arrow[dr, "p"'] & D \arrow[d, "\eta_D"] \\
& & & & HSD
\end{tikzcd},$$

where θ is monic, so η_P is monic and thus is an isomorphism. This implies that p is the coequaliser of $(H\phi_1 . H\lambda,\ H\phi_2 . H\lambda)$ in $C^{\top}$ and that this latter pair is a kernel pair in C^{t}. Thus $(\phi_1\lambda,\ \phi_2\lambda)$ is a kernel pair in $C^{\top}$, so λ is an isomorphism, so θ is an isomorphism, so η_D is an isomorphism, as required to show that D lies in $C^{\top}$. //

COROLLARY 2.4. *If* $\int^{A} C(NA, C) \otimes C(NA', TNA) \to C(NA', TC)$ *is a strong epimorphism in* V *and* $C(NA, C) \circ TNA$ *exists in* C *and* $C(NA, -)$ *preserves strong epimorphisms for all* $A \in A$ *and* C *has kernel pairs reflected by* $U : C^{\top} \to C$, *then* $C^{\top}$ *is a Birkhoff reflective subcategory of* C^{t}. //

PROPOSITION 2.5. *If* C^{t} *is cocomplete and* $C(NA, -)$ *preserves coequalisers of reflective pairs, then* C^{t} *is monadic over* C *iff*

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(a) f *is a coequaliser in* C^t *iff* $U^t f$ *is a coequaliser in* C *, and*

(b) U^t *reflects kernel pairs.*

Proof. If C^t is cocomplete then $F^t \dashv U^t$ exists and $U^t F^t$ preserves coequalisers of reflective pairs since $U^t : C^t \to C$ creates coequalisers because $C(NA', -)$ preserves them (*cf.* Diers [5], Proposition 1.1). Thus the result follows from Borceux and Day [1], Corollary 6.2. //

PROPOSITION 2.6. *Suppose* C *and* C^t *are cocomplete and let* $K : [T, V] \to C^t$ *denote the canonical reflection. If* U^t *preserves epimorphisms and those unit components of the form* $T(tA, -) \to KT(tA, -)$ *are epimorphisms, then* $U^t F^t$ *generates a linear monad.*

Proof. We have

$$U^t F^t NA = U^t K\left(\int^{A'} C(NA', NA) \otimes T(tA', -)\right) \cong U^t K\big(T(tA, -)\big)$$

by the representation theorem because N is fully faithful. Also

$$U^t F^t C = U^t K\left(\int^{A} C(NA, C) \otimes T(tA, -)\right) .$$

Thus, to show that

$$\int^A C(NA, C) \ . \ U^t F^t NA \to U^t F^t C \tag{2.2}$$

is an epimorphism consider the following diagram

$$\begin{tikzcd}
\int^A C(NA,C).U^t T(tA,-) \arrow[r] \arrow[dd, "\wr"'] & \int^A C(NA,C).U^t KT(tA,-) \arrow[d, "(2.2)"] \\
& U^t K\left(\int^A C(NA,C) \otimes T(tA,-)\right) \\
U^t\left(\int^A C(NA,C).T(tA,-)\right) \arrow[r] & U^t\left(\int^A C(NA,C).KT(tA,-)\right) \arrow[u, "\wr"']
\end{tikzcd} .$$

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The bottom arrow is an epimorphism by hypothesis, so (2.2) is an epimorphism, as required. //

3. Example

Suppose the ground category V has canonical $E - M$ factorisations for the system $E - M = \{$strong epimorphisms and monomorphisms$\}$ (see Freyd and Kelly [7]). Suppose also that V has arbitrary cointersections of E-quotients and that finite powers preserve strong epimorphisms.

DEFINITION 3.1. A functor $G : A \to V$ from a category A with finite products to V is said to *E-preserve* finite products if the canonical morphism $G(A \times A') \to GA \times GA'$ is a strong epimorphism for all $A, A' \in A$.

DEFINITION 3.2. Let $M : A \to B$ be a functor between categories with finite products. Then V is said to satisfy *axiom* $E(\pi)$ if the left Kan extension of a functor $G : A \to V$ which E-preserves finite products along M again E-preserves finite products.

One then obtains results precisely analogous to those obtained for axiom π in Borceux and Day [1], Sections 1 and 2.

DEFINITION 3.3. If T is a finitary algebraic theory (see Borceux and Day [1], Definition 3.1) then a functor $G : T \to V$ which E-preserves finite products is called an *E-algebra* (of T).

Now let T^q denote the category of E-algebras for T, regarded as a full subcategory of $[T, V]$. Let T^b denote the ordinary category of algebras of T; namely, the full subcategory of $[T, V]$ defined by the finite-product-preserving functors. Then there are inclusions $T^b \subset T^q \subset [T, V]$. The second embedding is *coreflective* and the coreflection maps G to the union of the E-algebras which are M-subfunctors of G; the coreflection counit lies in M. The first embedding is reflective and the reflection maps $A \in T^q$ to the largest T^b E-quotient of A; the reflection unit is in E. Thus we have

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$$\begin{array}{ccccc} T^b & \leftrightarrows & T^q & \leftrightarrows & [T, V] \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ V & \leftrightarrows & \left\{V_f^{\mathrm{op}}\right\}^q & \leftrightarrows & \left[V_f^{\mathrm{op}}, V\right] \end{array},$$

where V_f is the initial finitary theory and where the centre adjunction is a strictly linear monadic situation.

THEOREM 3.4. *If V satisfies the hypotheses of this section (and satisfies axiom $E(\pi)$) then a monad T on V generates a Birkhoff sub-category of an algebraic category T^b iff*

(a) V^{T} *has coequalisers,*

(b) $\int^m [m, X] \otimes [n, Tm] \to [n, TX]$ *is a strong epimorphism, and*

(c) $U : V^{\mathsf{T}} \to V$ *reflects kernel pairs.*

Proof. Because V^{T} has coequalisers iff V^{T} is cocomplete (see Linton [8]) the conditions are sufficient by Corollary 2.4. Necessity of *(a)* is clear since T^b is always cocomplete. Moreover, if V^{T} is a Birkhoff subcategory of T^b then the unit of the composite reflection $T^q \to T^b \to V^{\mathsf{T}}$ is a strong epimorphism. This implies that *(b)* is necessary. Finally, the functor $U^t : T^b \to V$ reflects kernel pairs and the Birkhoff property implies that the embedding $V^{\mathsf{T}} \subset T^b$ reflects kernel pairs, so *(c)* is necessary. //

An example of a monad which satisfies *(a)* and *(b)* but not *(c)* is the reflection to Hausdorff k-spaces from non-Hausdorff k-spaces.

References

[1] Francis Borceux and Brian Day, "Universal algebra in a closed category", *J. Pure Appl. Algebra* (to appear).

[2] Brian Day, "On closed categories of functors II", *Category seminar*, 20-54 (Proc. Sydney Category Seminar 1972/1973. Lecture Notes in Mathematics, 420. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[3] B.J. Day, "Density presentations of functors", *Bull. Austral. Math. Soc.* 16 (1977), 427-448.

[4] B.J. Day and G.M. Kelly, "Enriched functor categories", *Report of the Midwest Category Seminar III*, 178-191 (Lecture Notes in Mathematics, 106. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[5] Y. Diers, "Foncteur pleinement fidèle dense classant les algèbres", (Publications Internes de l'U.E.R. de Mathématiques Pures et Appliquées, 58. Université des Science et Techniques de Lille I, 1975).

[6] Samuel Eilenberg and G. Max Kelly, "Closed categories", *Proc. Conf. Categorical Algebra*, La Jolla, California, 1965, 421-562 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[7] P.J. Freyd and G.M. Kelly, "Categories of continuous functors, I", *J. Pure Appl. Algebra* 2 (1972), 169-191.

[8] F.E.J. Linton, "Coequalisers in categories of algebras", *Seminar on Triples and Categorical Homology Theory*, 75-90 (Lecture Notes in Mathematics, 80. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[9] S. Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics, 5. Springer-Verlag, New York, Heidelberg, Berlin, 1971).

Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales.

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