ON THEOREMS OF KAWADA AND WENDEL

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1. Introduction

Let G be a locally compact topological group, with left-invariant Haar measure. If $L_1(G)$ is the usual class of complex functions which are integrable with respect to this measure, and μ is any bounded Borel measure on G, then the convolution-product $\mu\star f$, defined for any f in L_1 by

$$\mu\star f(y)=\int_G f(x^{-1}y)d\mu(x)$$

is again in L_1, and

$$\| \mu\star f \| \leqslant \| \mu \| \, \| f \|.$$

Y. Kawada ((**1**), Theorem 2) has proved essentially the following result:

Theorem K. *If L_1 is mapped onto itself by the correspondence $f\rightarrow\mu\star f$, and $\mu\star f\geqslant 0$ p.p. if, and only if, $f\geqslant 0$ p.p., then μ has one-point support.*

J. G. Wendel ((**3**), Theorem 3) has proved essentially the following:

Theorem W. *If $\| \mu\star f \| = \| \mu \| \, \| f \|$ for all $f \in L_1$, then μ has one-point support.*

There is clearly a close connection between order-preserving and norm-preserving measures μ. Wendel ((**3**), footnote 4) appears to assert that the two classes are substantially identical (that is, up to scalar factors) and that Theorem K would continue to be valid if the condition that L_1 should be mapped onto itself were dropped. We shall refer to this modified version as the Kawada-into theorem, in distinction to the Kawada-onto theorem, which is the original Theorem K.

The principal aim of this note is to give a counter-example to the Kawada-into theorem in its general setting. It turns out, however, that the theorem is true in many cases; some of these are discussed in §**3**. Although it has not been possible to obtain definitive conditions for the validity of the theorem, a conjecture about this is advanced in the last section.

2. The Counter-Example

Let G be the group of matrices of the form

$$x=\begin{bmatrix} x_1 & x_2 \\ 0 & 1 \end{bmatrix} \quad (0<x_1<\infty\,;\ -\infty<x_2<\infty),$$

with ordinary matrix multiplication as the group operation. The topology of G is the ordinary topology of the Euclidean half-plane. Left-invariant Haar measure dx is here equal to $x_1^{-2}dx_1dx_2$. The modular function $\Delta(x)$ ((**2**), p. 117) is x_1^{-1}.

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For each positive integer n, let N_n be the neighbourhood of the identity e defined by

$$1-\tfrac{1}{2}n^{-\frac{1}{4}} \leqslant x_1 \leqslant 1+\tfrac{1}{2}n^{-\frac{1}{4}}\,; \quad -\tfrac{1}{2}n^{-\frac{1}{4}} \leqslant x_2 \leqslant \tfrac{1}{2}n^{-\frac{1}{4}}.$$

It is easy to verify that the measure of N_n, $m(N_n)$ (with the Haar measure indicated) is $(n^{\frac{1}{2}}-\frac{1}{4})^{-1}$. Thus $m(N_n) \sim n^{-\frac{1}{2}}$ as $n \to \infty$.

If $a_n = \begin{bmatrix} (n!)^{-3} & 0 \\ 0 & 1 \end{bmatrix}$, then $\Delta(a_n) = (n!)^3$. Let the function f_0 be defined as follows:

$$\begin{aligned} f_0(x) &= (n!)^{-1} \text{ if } x \in a_n N_n \ (n=1,\,2,\,3,\,\ldots) \\ &= 0 \text{ otherwise.} \end{aligned}$$

The sets $a_n N_n$ $(n=1, 2, 3, \ldots)$ are mutually disjoint, so that $f_0(x)$ consists of an infinity of separate pieces. Further, the function is clearly in L_1: in fact

$$\| f_0 \| = \sum_{n=1}^{\infty} (n!)^{-1} m(N_n)\,;$$

the series is certainly convergent.

Let μ be the measure associated with f_0:

$$\mu(E) = \int_E f_0(x)dx.$$

For the convolution-product $\mu \star f$, where $f \in L_1$, we have

$$\begin{aligned} \mu \star f\,(y) &= \int_G f(x^{-1}y) f_0(x) dx \\ &= \int_G f(x^{-1}) f_0(yx) dx. \end{aligned}$$

We shall show that if $f \in L_1$, and $\mu \star f \geqslant 0$ p.p., then $f \geqslant 0$ p.p. Since the support of μ is not a single point, this will provide the required counter-example. .

First, we note that for the present purpose it is permissible to suppose that a given real function $f \in L_1$, not $\geqslant 0$ p.p., has the form

$$\begin{aligned} f(x) &= -1 \text{ if } x \in N_{n'} \\ &= 0 \text{ if } x \in N_{n'}^{-1} N_{n'}^{2} \cap \mathscr{C} N_{n'} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ &\geqslant 0 \text{ if } x \in \mathscr{C} N_{n'}^{-1} N_{n'}^{2} \end{aligned}$$

for some $n' \geqslant 16$ ($\mathscr{C}E$ is the complement of E).

Let f' be any real function in L_1, not $\geqslant 0$ p.p. There is a bounded non-negative function $g \in L_1$ such that $f' \star g$ is not $\geqslant 0$ p.p. (g could be any bounded non-negative function vanishing outside a sufficiently small neighbourhood of the identity). Since g is bounded, $f' \star g$ is continuous. Hence there is a positive real δ such that the set $\{x : f' \star g(x) < -\delta\}$ is open (and not empty). Let a be any point in this set, and write, for any function ϕ, $\phi_a(x) = \phi(xa)$: then $(f' \star g)_a$ is negative (in fact $< -\delta$) in some neighbourhood N of e.

Since the sets N_n as defined above form a base of neighbourhoods of e,

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we can find $n' \geqslant 16$ so that $N_{n'}^{-1}N_{n'}^{2} \subset N$. Let a be a positive real number such that $a(f' \star g)_a \leqslant -1$ in $N_{n'}$. If f'' is defined by

$$\begin{aligned} f''(x) &= -1 \text{ if } x \in N_{n'} \\ &= 0 \text{ if } x \in N_{n'}^{-1}N_{n'}^{2} \cap \mathscr{C}N_{n'} \\ &= \sup\{a(f' \star g)_a(x),\, 0\} \text{ if } x \in \mathscr{C}N_{n'}^{-1}N_{n'}^{2}, \end{aligned}$$

then evidently $f''(x) \geqslant a(f' \star g)_a(x)$ for all x. It is clear that $f'' \in L_1$.

The implications

$$\mu \star f' \geqslant 0 \Rightarrow \mu \star f' \star g \geqslant 0 \Rightarrow \mu \star (f' \star g)_a = (\mu \star f' \star g)_a \geqslant 0 \Rightarrow a\mu \star (f' \star g)_a = \mu \star a(f' \star g)_a \geqslant 0 \Rightarrow \mu \star f'' \geqslant 0$$

are immediate. So if it can be proved that $\mu \star f'' \geqslant 0$ is impossible when f'' has the form (1), it will follow that $\mu \star f \geqslant 0$ is impossible for real $f \in L_1$, unless $f \geqslant 0$ p.p.

It will follow at once from this that $\mu \star f \geqslant 0$ $(f \in L_1)$ implies $f \geqslant 0$; for let $f = \phi + i\psi$, where ϕ and ψ are real. Since μ is real, $\mu \star f \geqslant 0$ implies that $\mu \star \phi \geqslant 0$ and $\mu \star \psi = 0$, whence $\phi \geqslant 0$, $\psi \geqslant 0$ and $\psi \leqslant 0$, that is, $\psi = 0$ and $f = \phi \geqslant 0$.

Suppose then that f is of the form (1); write

$$\begin{aligned} f_1(x) &= -1 \text{ if } x \in N_{n'} \\ &= 0 \text{ otherwise}; \end{aligned}$$

and $f_2(x) = f(x) - f_1(x)$. What we show is that if $\mu \star f \geqslant 0$ p.p. then $\| f_2 \|$ is arbitrarily large, which provides the required contradiction.

Let

$$g_n(y) = \int_G (n!)^{-1} \chi_{a_n N_n}(yx) f_1(x^{-1})\, dx, \quad \text{......(2)}$$

$$h_n(y) = \int_G (n!)^{-1} \chi_{a_n N_n}(yx) f_2(x^{-1})\, dx, \quad \text{......(3)}$$

(where as usual $\chi_E(x) = 1$ if $x \in E$, $= 0$ otherwise). Since also

$$g_n(y) = \int_G (n!)^{-1} \chi_{a_n N_n}(x) f_1(x^{-1}y)\, dx,$$

it is clear that

$$\| g_n \| = (n!)^{-1} m(a_n N_n) \| f_1 \| = (n!)^{-1} m(N_n) \| f_1 \| \quad \text{......(4)}$$

(where in fact $\| f_1 \| = m(N_{n'})$). It is also clear that if $g_n(y) \neq 0$, then there is a point x such that $yx \in a_n N_n$ and $x^{-1} \in N_{n'}$; that is, $y \in a_n N_n N_{n'}$. Since $a_n N_n N_{n'}$ is a closed set, it contains the support of g_n. It is easy to see that the support of g_n is disjoint from that of g_m if $m \neq n$, since $n' \geqslant 16$.

Next we show that if $n \geqslant n'$ then $h_n(y) = 0$ for $y \in a_n N_n N_{n'}$. For, if also $yx \in a_n N_n$ then $x^{-1} \in N_n^{-1} a_n^{-1} a_n N_n N_{n'} = N_n^{-1} N_n N_{n'} \subset N_{n'}^{-1} N_{n'}^{2}$, since $N_n \subset N_{n'}$ if $n \geqslant n'$. But $f_2(x^{-1}) = 0$ if $x^{-1} \in N_{n'}^{-1} N_{n'}^{2}$, so that $h_n(y) = 0$ if $y \in a_n N_n N_{n'}$, from (3).

Write $h_{m,n}(x) = h_m(x)$ if $x \in a_n N_n N_{n'}$, $= 0$ otherwise; that is, $h_{m,n}$ is the restriction of h_m to $a_n N_n N_{n'}$. Now, for each n, $\mu \star f \geqslant 0$ throughout $a_n N_n N_{n'}$, if, and only if

$$\sum_{m=1}^{\infty} h_{m,n}(x) + g_n(x) \geqslant 0.$$

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It is thus necessary that

$$\| \sum_{m=1}^{\infty} h_{m,n} \| \geqslant \| g_n \|.$$

Since $f_2 \geqslant 0$, it follows that $h_{m,n} \geqslant 0$ for all m, n; and so

$$\| \sum_{m=1}^{\infty} h_{m,n} \| = \sum_{m=1}^{\infty} \| h_{m,n} \|.$$

Thus a necessary condition that $\mu \star f \geqslant 0$ in $a_n N_n N_{n'}$ is

$$\| g_n \| \leqslant \sum_{m=1}^{\infty} \| h_{m,n} \|. \qquad \text{.................................(5)}$$

In view of (4), and the fact that $h_{n,n} = 0$ for $n \geqslant n'$, we have, for $n \geqslant n'$, the inequality

$$(n!)^{-1} m(N_n) \| f_1 \| \leqslant \sum_{m=1}^{n-1} \| h_{m,n} \| + \sum_{m=n+1}^{\infty} \| h_{m,n} \| \qquad \text{..................(6)}$$

We estimate the terms on the right-hand side of (6) as follows. If $r > n$ then

$$\| h_{r,n} \| \leqslant \| h_r \| = (r!)^{-1} m(N_r) \| f_2 \|, \qquad \text{........................(7)}$$

while if $r < n$ then

$$\| h_{r,n} \| \leqslant m(a_n N_n N_{n'}) \sup_{y \in a_n N_n N_{n'}} h_r(y) \qquad \text{..........................(8)}$$

Now,

$$\sup_{y \in a_n N_n N_{n'}} h_r(y) = (r!)^{-1} \sup_{y \in a_n N_n N_{n'}} \int_G \chi_{a_r N_r}(yx) \Delta(x) \Delta(x^{-1}) f_2(x^{-1}) dx$$

$$\leqslant (r!)^{-1} \sup_{\substack{y \in a_n N_n N_{n'} \\ yx \in a_r N_r}} \chi_{a_r N_r}(yx) \Delta(x) \int_G \Delta(x^{-1}) f_2(x^{-1}) dx$$

$$= (r!)^{-1} \sup_{\substack{y \in a_n N_n N_n \\ yx \in a_r N_r}} \Delta(x) \| f_2 \|.$$

If $y \in a_n N_n N_{n'}$ and $yx \in a_r N_r$ then $x \in N_{n'}^{-1} N_n^{-1} a_n^{-1} a_r N_r$, so that

$$x_1 \geqslant (1 + \tfrac{1}{2} n'^{-\frac{1}{4}})^{-1} (1 + \tfrac{1}{2} n^{-\frac{1}{4}})^{-1} (1 - \tfrac{1}{2} r^{-\frac{1}{4}}) (n!/r!)^3,$$

and hence, for such x,

$$\Delta(x) \leqslant C (r!/n!)^3, \qquad \text{...................................(9)}$$

where C is a constant, independent of n, n' and r (it could for example be taken to be 8).

Then, using (9), the inequality (8) gives

$$\| h_{r,n} \| \leqslant m(a_n N_n N_{n'}) (r!)^{-1} C (r!/n!)^3 \| f_2 \|.$$

If $n \geqslant n'$ then $m(a_n N_n N_{n'}) \leqslant m(a_n N_{n'}^2) = m(N_{n'}^2)$; if $C' = C m(N_{n'}^2)$ then

$$\| h_{r,n} \| \leqslant C' (r!)^2 (n!)^{-3} \| f_2 \| \text{ for } r < n,\ n \geqslant n'. \qquad \text{..................(10)}$$

The inequality (6) now gives, using (7) and (10),

$$(n!)^{-1} m(N_n) \| f_1 \| \leqslant \{ C' (n!)^{-3} \sum_{r=1}^{n-1} (r!)^2 + \sum_{r=n+1}^{\infty} (r!)^{-1} m(N_r) \} \| f_2 \|,$$

and, using the trivial inequalities

$$\sum_{r=1}^{n-1}(r!)^2 < n\{(n-1)!\}^2, \quad m(N_r) < K, \quad \sum_{r=n+1}^{\infty}(r!)^{-1} < 2\{(n+1)!\}^{-1}$$

we have

$$(n!)^{-1}m(N_n)\,\|f_1\| \leqslant \{C'(n!)^{-3}n((n-1)!)^2 + 2K((n+1)!)^{-1}\}\,\|f_2\|,$$

which gives at once

$$m(N_n)\,\|f_1\| \leqslant C''n^{-1}\,\|f_2\|. \quad \text{.............................(11)}$$

But $m(N_n) = (n^{\frac{1}{4}} - \frac{1}{4})^{-1}$, so it is evident that (11) cannot hold for all $n \geqslant n'$; $\|f_2\|$ cannot be finite if $\|f_1\| \neq 0$ (which we have assumed).

The required contradiction has thus been produced.

It seems clear that the above construction could be carried out in any metrisable, non-unimodular, locally compact topological group.

3. Some Positive Results

We now turn to one or two cases in which the Kawada-into theorem is true.

In the following, S_μ denotes the support of the measure μ, which is assumed to be positive.

Proposition 1. *If G is abelian, and S_μ contains two distinct points, there exists $f \in L_1$, not $\geqslant 0$ p.p., such that $\mu \star f \geqslant 0$ p.p.*

Proof. There is clearly no loss of generality in assuming that $e \in S_\mu$. Suppose that h is another point in S_μ. Choose a compact symmetric neighbourhood N of e so that $h \notin N^4$. Let μ_1 be the restriction of μ to N, μ_2 its restriction to Nh. Write $\mu = \mu_1 + \mu_2 + \mu_3$; it is clear that $\mu_3 \geqslant 0$.

Let f_1 be the characteristic function of N^3 ($=1$ in N^3, $=0$ outside N^3). Let f_2 be equal to $\mu \star f_1$ throughout $\mathscr{C}N$, and zero in N. Write, for $k \geqslant 0$,

$$f_3(t) = f_2(t) - (\mu_1 \star f_1)(t) + kf_1(ht);$$

then f_3 is negative throughout N.

Also, $\mu \star f_3 = \mu_1 \star f_2 - \mu \star \mu_1 \star f_1 + k\mu_2 \star (f_1)_h +$ positive terms. Since $\mu_1 \star f_2 = \mu_1 \star \mu \star f_1$ throughout $\mathscr{C}N^2$, it follows that $\mu \star f_3 \geqslant 0$ throughout $\mathscr{C}N^2$. Since $(\mu_2 \star (f_1)_h)(t) = \int_G f_1(hu^{-1}t)\,d\mu_2(u)$, and $t \in N^2$, $u \in Nh$ implies $hu^{-1}t \in N^3$, it follows that $f_1(hu^{-1}t) = 1$ throughout S_{μ_2}, and $\int_G f_1(hu^{-1}t)\,d\mu_2(u) = \|\mu_2\|$ for all $t \in N^2$.

So, by taking k large enough, $\mu \star f_3$ can be made $\geqslant 0$ p.p.

Proposition 2. *If μ contains a point-mass, and S_μ contains two distinct points, then there is a function $f \in L_1$, not $\geqslant 0$ p.p., such that $\mu \star f \geqslant 0$ p.p.*

Proof. There is clearly no loss of generality in supposing that the point-mass is at e. Let h be another point of S_μ, and let N be a compact symmetric neighbourhood of e such that $h \notin N^3$. Let μ_2 be the restriction of μ to Nh; let the mass at e be m_1, and let $f_1 = 1$ in N, $=0$ outside N. Let f_2 be the restriction of $\mu \star f_1$ to $\mathscr{C}N$. Write $f_4 = 1$ in $h^{-1}N^2$, $=0$ elsewhere, and $f_3 = f_2 - m_1 f_1 + kf_4$ ($k \geqslant 0$). Then $f_3 < 0$ in N, and $\mu \star f_3 = m_1 f_2 - m_1 \mu \star f_1 + \mu_2 \star kf_4 +$ positive terms, so that $\mu \star f_3 \geqslant 0$ throughout $\mathscr{C}N$.

Also, $(\mu_2 \star f_4)(t) = \int_G f_4(u^{-1}t)d\mu_2(u)$; and if $t \in N$ and $u \in Nh$ then $u^{-1}t \in h^{-1}N^2$, so that $f_4(u^{-1}t) = 1$. So $(\mu \star f_4)(t) = \| \mu_2 \|$ throughout N, and by taking k large enough we have $\mu \star f_3 \geqslant 0$ p.p.

If μ is a measure, and M a suitable set (e.g., open or closed), denote by $(\mu)_M$ the restriction of μ to M.

Lemma. *If μ is a bounded measure on G which does not have one-point support, then there exist a compact set M and a positive real number $k < 1$, such that $\| (\mu)_{aM} \| \leqslant k \| \mu \|$ for all $a \in G$.*

Proof. Let x, y be two distinct points in S_μ, N a neighbourhood of e such that $y^{-1}x \notin N$, and N' a compact symmetric neighbourhood of e such that $N'^4 \subset N$. Then for any $a \in G$, aN' cannot intersect both xN' and yN'. For, if $an_1 = xn_2$, $an_3 = yn_4$ $(n_i \in N',\ 1 \leqslant i \leqslant 4)$ then $x = an_1n_2^{-1}$, $y = an_3n_4^{-1}$ and $y^{-1}x = n_4n_3^{-1}n_1n_2^{-1} \in N'^4 \subset N$, which is a contradiction.

Hence if p denotes the lesser of $\| (\mu)_{xN'} \|$, $\| (\mu)_{yN'} \|$ then $p > 0$ and $\| (\mu)_{aN'} \| \leqslant \| \mu \| - p = k \| \mu \|$ $(k < 1)$ for all $a \in G$.

Proposition 3. *If S_μ is compact, and not one-point, then there exists $f \in L_1$, not $\geqslant 0$ p.p., such that $\mu \star f \geqslant 0$ p.p.*

Proof. Let M and k be as in the above Lemma. Let q be positive, and let f be a function which is equal to -1 in M, and equal to q in $S_\mu^{-1}S_\mu M \cap \mathscr{C}M$. Then

$$\mu \star f(t) = \int_{tM^{-1}} f(u^{-1}t)d\mu(u) + \int_{\mathscr{C}tM^{-1}} f(u^{-1}t)d\mu(u).$$

The first integral is less in absolute value than $k \| \mu \|$. If $t \notin S_\mu M$, then $\mu \star f(t) = \int f(u^{-1}t)d\mu(u) \geqslant 0$, since $u \in S_\mu$ implies $u^{-1}t \notin M$ in this case. If on the other hand $t \in S_\mu M$ then

$$\begin{aligned} \mu \star f(t) &\geqslant \int_{\mathscr{C}tM^{-1}} f(u^{-1}t)d\mu(u) - \left| \int_{tM^{-1}} f(u^{-1}t)d\mu(u) \right| \\ &\geqslant q(1-k) \| \mu \| - k \| \mu \| \\ &> 0 \text{ if } q \text{ is large enough.} \end{aligned}$$

So $\mu \star f(t) \geqslant 0$ p.p. for suitable choice of q.

Theorem 1. *The Kawada-into theorem is true if G is (a) abelian or (b) discrete or (c) compact.*

Proof. The three cases follow at once from Propositions 1, 2 and 3.

It is possible to ensure the truth of the Kawada-into theorem by imposing on the measures μ considered, restrictions similar to, but more complicated than, those of Propositions 2 and 3. Since it is unlikely that these conditions are the best possible results in this direction, we have refrained from writing them down here.

4. Miscellaneous Remarks

The truth or falsity of the Kawada-into theorem is connected in an essential way with the function-class L_1. If a slightly different class of functions is taken, the results are completely altered. Thus if the class L of continuous functions on G with compact support is considered, it is soon apparent that there exist in general measures μ such that $\mu \star f \geqslant 0$ implies $f \geqslant 0$, for $f \in L$, but such that S_μ is not compact. For example, let G be the additive real numbers, and μ the measure consisting of point-masses n^{-2} at n^2 ($n=1$, 2, 3, ...).

Further, if L_1 is replaced by the class of all bounded functions on G, or the class of all bounded continuous functions, it is easy to see, by using the Lemma given above, that the analogue of the Kawada-into theorem is true, whatever G may be. In the counter-example of §2, of course, the properties of L_1 were involved in an essential way.

It is possible to produce a proof of Theorem W on the lines of the constructions of Propositions 1, 2 or 3, which appears to be shorter, and certainly involves simpler ideas than Wendel's original proof. Kawada's original proof of Theorem K can also be simplified.

In view of the counter-example of §2, and Theorem 1, it is tempting to conjecture that the Kawada-into theorem is true if, and only if, G is unimodular. But there is really no substantial evidence in support of this, and the role of metrisability in the counter-example certainly requires clarification.

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