Canad. Math. Bull. Vol. 36 (4), 1993 pp. 398–406

SOME PROPERTIES OF HANKEL CONVOLUTION OPERATORS

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ABSTRACT. Let $\mathcal{H}_\mu'$ be the Zemanian space of Hankel transformable generalized functions and let $O_{\mu,*}'$ be the space of Hankel convolution operators for $\mathcal{H}_\mu'$. This $\mathcal{H}_\mu'$ is the dual of a subspace $\mathcal{H}_\mu$ of $O_{\mu,*}'$ for which $O_{\mu,*}'$ is also the space of Hankel convolutors. In this paper the elements of $O_{\mu,*}'$ are characterized as those in $\mathcal{L}(\mathcal{H}_\mu)$ and in $\mathcal{L}(\mathcal{H}_\mu')$ that commute with Hankel translations. Moreover, necessary and sufficient conditions on the generalized Hankel transform $\mathfrak{H}_\mu' S$ of $S \in O_{\mu,*}'$ are established in order that every $T \in O_{\mu,*}'$ such that $S * T \in \mathcal{H}_\mu$ lie in $\mathcal{H}_\mu$.

1. **Introduction.** Let $\mu \in \mathbb{R}$, and let $\mathcal{H}_\mu$ be the space of Hankel transformable functions, as introduced by A. H. Zemanian [5]. We recall that $\mathcal{H}_\mu$ consists of all those infinitely differentiable functions $\phi = \phi(x)$ defined on $I =]0, \infty[$ such that the quantities

$$\gamma_{m,k}^\mu(\phi) = \sup_{x\in I} \left|(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x)\right| \quad (m, k \in \mathbb{N})$$

are finite. When endowed with the topology generated by the family of seminorms $\{\gamma_{m,k}^\mu\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$, $\mathcal{H}_\mu$ becomes a Fréchet space. The Hankel transformation

$$(\mathfrak{H}_\mu\phi)(t) = \int_0^\infty \phi(x)\sqrt{xt}J_\mu(xt)\,dx$$

is an automorphism of $\mathcal{H}_\mu$, provided that $\mu \geq -1/2$ (here, as usual, J_μ denotes the Bessel function of the first kind and order μ). If $\mu \geq -1/2$, the generalized Hankel transformation $\mathfrak{H}_\mu'$ is defined on $\mathcal{H}_\mu'$, the dual space of $\mathcal{H}_\mu$, as the adjoint of $\mathfrak{H}_\mu$. Then $\mathfrak{H}_\mu'$ is an automorphism of $\mathcal{H}_\mu'$.

In previous papers [2] and [3], for $\mu \geq -1/2$, the authors have introduced and studied the subspace $O_{\mu,*}'$ of $\mathcal{H}_\mu'$ formed by all those $T \in \mathcal{H}_\mu'$ such that $\theta(x) = x^{-\mu-1/2}(\mathfrak{H}_\mu' T)(x)$ is a smooth function on I with the property that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ satisfying

$$\sup_{x\in I} \left|(1+x^2)^{-n_k}(x^{-1}D)^k\theta(x)\right| < +\infty.$$

Clearly, $\mathcal{H}_\mu$ is a subspace of $O_{\mu,*}'$. The space O of all those smooth functions $\theta = \theta(x)$ on I possessing the above property turns out to be the space of multiplication operators on

Received by the editors March 3, 1992.
AMS subject classification: 46F12.
Key words and phrases: generalized functions, Hankel transformation, Hankel translation, Hankel convolution.

https://doi.org/10.4153/CMB-1993-054-8 Published online by Cambridge University Press

$\mathcal{H}_\mu$ and on $\mathcal{H}'_\mu$ ($\mu \in \mathbb{R}$), whereas $O'_{\mu,*}$ is the space of convolution operators on $\mathcal{H}_\mu$ and on $\mathcal{H}'_\mu$ ($\mu \geq -1/2$).

In what follows we shall always assume that μ is a real number not inferior to $-1/2$ and, unless otherwise stated, that $\mathcal{H}'_\mu$ is endowed with its weak* topology.

In Section 2 of this paper the elements of $O'_{\mu,*}$ are characterized as those in $\mathcal{L}(\mathcal{H}_\mu)$ and in $\mathcal{L}(\mathcal{H}'_\mu)$ that commute with Hankel translations. Here, as customary, $\mathcal{L}(\mathcal{H}_\mu)$ (respectively, $\mathcal{L}(\mathcal{H}'_\mu)$) denotes the space of all linear continuous operators from $\mathcal{H}_\mu$ (respectively, $\mathcal{H}'_\mu$) into itself. Furthermore, necessary and sufficient conditions on the generalized Hankel transform $\mathfrak{H}'_\mu S$ of $S \in O'_{\mu,*}$ are established in order that every distribution $T \in O'_{\mu,*}$ such that $S * T \in \mathcal{H}_\mu$ lie in $\mathcal{H}_\mu$. This is done in Section 3.

2. **Characterizing $O'_{\mu,*}$ in $\mathcal{L}(\mathcal{H}_\mu)$ and in $\mathcal{L}(\mathcal{H}'_\mu)$.** Let $\mathcal{L}(\mathcal{H}_\mu)$ (respectively, $\mathcal{L}(\mathcal{H}'_\mu)$) denote the space of all linear continuous operators from $\mathcal{H}_\mu$ (respectively, $\mathcal{H}'_\mu$) into itself. The characterization of the elements in $\mathcal{L}(\mathcal{H}_\mu)$ and in $\mathcal{L}(\mathcal{H}'_\mu)$ that commute with Hankel translations is our first objective.

We recall that the Hankel translation $\tau_x\phi$ of $\phi \in \mathcal{H}_\mu$ by $x \in I$ is defined as

$$(\tau_x\phi)(y) = \int_0^\infty \phi(z) D_\mu(x,y,z)\,dz \quad (y \in I),$$

where,

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-1/2} \mathcal{J}_\mu(xt)\mathcal{J}_\mu(yt)\mathcal{J}_\mu(zt)\,dt \quad (x,y,z \in I)$$

and $\mathcal{J}_\mu(z) = \sqrt{z}J_\mu(z)$ $(z \in I)$. The map $\phi \longmapsto \tau_x\phi$ is a continuous endomorphism of $\mathcal{H}_\mu$. Further

$$(\mathfrak{H}_\mu\tau_x\phi)(t) = t^{-\mu-1/2}\mathcal{J}_\mu(xt)(\mathfrak{H}_\mu\phi)(t) \quad (t \in I) \tag{2.1}$$

whenever $\phi \in \mathcal{H}_\mu$ and $x \in I$.

If $u \in \mathcal{H}'_\mu$ and $x \in I$, we define $\tau_x u \in \mathcal{H}'_\mu$ by transposition:

$$\langle \tau_x u, \phi\rangle = \langle u, \tau_x\phi\rangle \quad (\phi \in \mathcal{H}_\mu). \tag{2.2}$$

The following analogue of (2.1) holds for the generalized translation (2.2).

LEMMA 2.1. *Let $u \in \mathcal{H}'_\mu$ and $x \in I$. Then:*

$$(\mathfrak{H}'_\mu\tau_x u)(t) = t^{-\mu-1/2}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu u)(t) \quad (t \in I).$$

PROOF. For $u \in \mathcal{H}'_\mu$, $x \in I$, and $\phi \in \mathcal{H}_\mu$, a combination of (2.1) and (2.2) yields:

$$\begin{aligned}\langle \mathfrak{H}'_\mu\tau_x u, \mathfrak{H}_\mu\phi\rangle &= \langle \tau_x u, \phi\rangle = \langle u, \tau_x\phi\rangle = \langle \mathfrak{H}'_\mu u, \mathfrak{H}_\mu\tau_x\phi\rangle \\ &= \langle (\mathfrak{H}'_\mu u)(t), t^{-\mu-1/2}\mathcal{J}_\mu(xt)(\mathfrak{H}_\mu\phi)(t)\rangle \\ &= \langle t^{-\mu-1/2}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu u)(t), (\mathfrak{H}_\mu\phi)(t)\rangle. \end{aligned}$$

■

The classical Hankel convolution $\phi * \varphi$ of $\phi, \varphi \in \mathcal{H}_\mu$ is the function

$$\phi * \varphi(x) = \int_0^\infty \phi(y)(\tau_x\varphi)(y)\,dy \quad (x \in I).$$

The map $(\phi, \varphi) \longmapsto \phi * \varphi$ is continuous from $\mathcal{H}_\mu \times \mathcal{H}_\mu$ into $\mathcal{H}_\mu$. The generalized Hankel convolution $u * \phi$ of $u \in \mathcal{H}'_\mu$ and $\phi \in \mathcal{H}_\mu$ is the distribution given by

$$\langle u * \phi, \varphi\rangle = \langle u, \phi * \varphi\rangle \quad (\varphi \in \mathcal{H}_\mu).$$

The map $(u, \phi) \longmapsto u * \phi$ is separately continuous from $\mathcal{H}'_\mu \times \mathcal{H}_\mu$ into $\mathcal{H}'_\mu$, when $\mathcal{H}'_\mu$ is endowed either with its weak* or its strong topology. Finally, for $u \in \mathcal{H}'_\mu$ and $T \in O'_{\mu,*}$, the generalized function $u * T \in \mathcal{H}'_\mu$ is defined as

$$\langle u * T, \phi\rangle = \langle u, T * \phi\rangle \quad (\phi \in \mathcal{H}_\mu). \tag{2.3}$$

Note that each of these definitions extends the previous one. Moreover,

$$(\mathfrak{H}'_\mu u * T)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t) \quad (t \in I) \tag{2.4}$$

whenever $u \in \mathcal{H}'_\mu$ and $T \in O'_{\mu,*}$.

If $c_\mu = 2^\mu\Gamma(\mu + 1)$ then the element δ_μ of $O'_{\mu,*}$ given by

$$\langle \delta_\mu, \phi\rangle = c_\mu \lim_{x\to 0+} x^{-\mu-1/2}\phi(x) \quad (\phi \in \mathcal{H}_\mu)$$

is an identity for (2.3).

The generalized $*$-convolution commutes with Hankel translations:

LEMMA 2.2. *Assume that $u \in \mathcal{H}'_\mu$ and $x \in I$. If $T \in O'_{\mu,*}$, then*

$$\tau_x(u * T) = (\tau_x u) * T = u * (\tau_x T).$$

PROOF. Since $\mathfrak{H}'_\mu$ is an automorphism of $\mathcal{H}'_\mu$, we establish the lemma by fixing $t \in I$ and using Lemma 2.1, along with (2.4), to write:

$$\big(\mathfrak{H}'_\mu\tau_x(u * T)\big)(t) = t^{-\mu-1/2}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu u * T)(t) = t^{-2\mu-1}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t),$$
$$\big(\mathfrak{H}'_\mu(\tau_x u) * T\big)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu\tau_x u)(t) = t^{-2\mu-1}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t),$$
$$\big(\mathfrak{H}'_\mu u * (\tau_x T)\big)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu\tau_x T)(t)(\mathfrak{H}'_\mu u)(t) = t^{-2\mu-1}\mathcal{J}_\mu(xt)(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t). \quad ■$$

We are now in a position to prove

THEOREM 2.3. *If $T \in O'_{\mu,*}$ and L is the element of $\mathcal{L}(\mathcal{H}_\mu)$ defined by*

$$L\phi = T * \phi \quad (\phi \in \mathcal{H}_\mu), \tag{2.5}$$

then

$$\tau_x L = L\tau_x \quad (x \in I). \tag{2.6}$$

Conversely, if $L \in \mathcal{L}(\mathcal{H}_\mu)$ *satisfies (2.6) then there exists a unique* $T \in O'_{\mu,*}$ *for which (2.5) holds.*

PROOF. Let $T \in O'_{\mu,*}$. The fact that $L \in \mathcal{L}(\mathcal{H}_\mu)$ defined by (2.5) satisfies (2.6) is contained in Lemma 2.2. On the other hand, assume that $L \in \mathcal{L}(\mathcal{H}_\mu)$ is such that (2.6) holds, and define $T \in \mathcal{H}'_\mu$ by

$$\langle T, \phi \rangle = \langle \delta_\mu, L\phi \rangle \quad (\phi \in \mathcal{H}_\mu).$$

Then

$$(T * \phi)(x) = \langle T, \tau_x \phi \rangle = \langle \delta_\mu, L\tau_x \phi \rangle = \langle \delta_\mu, \tau_x L\phi \rangle = (\delta_\mu * L\phi)(x) = (L\phi)(x) \quad (x \in I)$$

whenever $\phi \in \mathcal{H}_\mu$, which proves (2.5). Since $O'_{\mu,*}$ is the space of convolution operators of $\mathcal{H}_\mu$, it follows from (2.5) that $T \in O'_{\mu,*}$. As to the uniqueness assertion, note that if $S \in O'_{\mu,*}$ is such that $S * \phi = 0$ for every $\phi \in \mathcal{H}_\mu$, then $S = 0$. In fact, $S * \phi = 0$ ($\phi \in \mathcal{H}_\mu$) and (2.4) imply $t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)\varphi(t) = 0$ ($\varphi \in \mathcal{H}_\mu$, $t \in I$). By particularizing $\varphi(t) = t^{\mu+1/2}e^{-t^2}$ ($t \in I$) we find that $t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t) = 0$, whence $\mathfrak{H}'_\mu S = 0$ and $S = 0$. ■

The following result will help in characterizing the elements of $O'_{\mu,*}$ as those in $\mathcal{L}(\mathcal{H}'_\mu)$ that commute with Hankel translations.

LEMMA 2.4. *The linear hull of the set of generalized functions of the form* $\tau_x\delta_\mu$ ($x \in I$) *is weakly* dense in* $\mathcal{H}'_\mu$.

PROOF. Since $(\mathfrak{H}'_\mu \delta_\mu)(t) = t^{\mu+1/2}$ ($t \in I$), by Lemma 2.1 we have

$$(\mathfrak{H}'_\mu \tau_x \delta_\mu)(t) = \mathcal{J}_\mu(xt) \quad (x, t \in I).$$

If $\phi \in \mathcal{H}_\mu$ does not vanish identically then there exists $x \in I$ such that $\phi(x) \neq 0$, and hence

$$\langle \tau_x\delta_\mu, \phi \rangle = \langle \mathfrak{H}'_\mu \tau_x \delta_\mu, \mathfrak{H}_\mu \phi \rangle = \int_0^\infty (\mathfrak{H}_\mu \phi)(t) \mathcal{J}_\mu(xt)\, dt = \phi(x) \neq 0.$$

This means that the subset $\{\tau_x\delta_\mu\}_{x\in I}$ of $\mathcal{H}'_\mu$ separates points in $\mathcal{H}_\mu$. By [1], Problem W(b), this family is total in $\mathcal{H}'_\mu$ with respect to the weak* topology. ■

THEOREM 2.5. *If* $T \in O'_{\mu,*}$ *and* $L \in \mathcal{L}(\mathcal{H}'_\mu)$ *is defined by*

(2.7) $$Lu = u * T \quad (u \in \mathcal{H}'_\mu),$$

then

(2.8) $$\tau_x L = L\tau_x \quad (x \in I),$$

and also

(2.9) $$L\delta_\mu \in O'_{\mu,*}.$$

https://doi.org/10.4153/CMB-1993-054-8 Published online by Cambridge University Press

Conversely, given $L \in \mathcal{L}(\mathcal{H}_\mu')$ satisfying (2.8) and (2.9), a unique $T \in O_{\mu,*}'$ may be found so that (2.7) holds.

PROOF. That L given by (2.7) satisfies (2.8) is a consequence of Lemma 2.2. Obviously, it also satisfies (2.9).

Conversely, let $L \in \mathcal{L}(\mathcal{H}_\mu')$ be such that both (2.8) and (2.9) hold. Then

$$L(u * \delta_\mu) = u * (L\delta_\mu) \quad (u \in \mathcal{H}_\mu'). \tag{2.10}$$

To demonstrate (2.10), define from $\mathcal{H}_\mu'$ into $\mathcal{H}_\mu'$ the linear map

$$\Lambda u = L(u * \delta_\mu) - u * (L\delta_\mu) \quad (u \in \mathcal{H}_\mu').$$

The definition of Λ is consistent by virtue of (2.9). Since $\Lambda \in \mathcal{L}(\mathcal{H}_\mu')$, its kernel is a closed subspace of $\mathcal{H}_\mu'$. In view of (2.8) this kernel contains $\tau_x\delta_\mu$ $(x \in I)$, and hence (Lemma 2.4) it is also dense in $\mathcal{H}_\mu'$. Therefore (2.10) holds.

Now, letting $T = L\delta_\mu$ we have

$$u * T = u * (L\delta_\mu) = L(u * \delta_\mu) = Lu,$$

which proves (2.7).

As to the uniqueness assertion, assume that $S \in O_{\mu,*}'$ is not the zero distribution, so that $\phi \in \mathcal{H}_\mu$ exists for which $S * \phi \neq 0$. Since $\mathcal{H}_\mu'$ separates points in $\mathcal{H}_\mu$ we may find $u \in \mathcal{H}_\mu'$ such that

$$\langle u * S, \phi \rangle = \langle u, S * \phi \rangle \neq 0.$$

This completes the proof. ∎

3. **A property of convolution operators.** Motivated by Theorem 2 in [4], the purpose of this section is to establish:

THEOREM 3.1. *Let $\mu \geq -1/2$. For $S \in O_{\mu,*}'$, the following are equivalent:*

(i) To every $k \in \mathbb{N}$ there correspond $m, n \in \mathbb{N}$ and a positive constant M, such that

$$\max_{0 \leq \ell \leq m} \sup\{|(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}_\mu' S)(t)| : t \in I, |x - t| \leq (1 + x^2)^{-k}\} \geq (1 + x^2)^{-n}$$

whenever $x \in I$, $x \geq M$.

(ii) If $T \in O_{\mu,*}'$ and $S * T \in \mathcal{H}_\mu$, then $T \in \mathcal{H}_\mu$.

PROOF. Suppose that condition (ii) is not satisfied. Then there exists $T \in O_{\mu,*}'$ such that $S * T \in \mathcal{H}_\mu$, but $T \notin \mathcal{H}_\mu$. This means that $t^{-\mu-1/2}(\mathfrak{H}_\mu' T)(t) \in O$, $t^{-\mu-1/2}(\mathfrak{H}_\mu' S)(t)(\mathfrak{H}_\mu' T)(t) \in \mathcal{H}_\mu$, and $\mathfrak{H}_\mu' T \notin \mathcal{H}_\mu$.

Since both $t^{-\mu-1/2}(\mathfrak{H}_\mu' S)(t)$ and $t^{-\mu-1/2}(\mathfrak{H}_\mu' T)(t)$ lie in O, to every $\ell \in \mathbb{N}$ there correspond $r_\ell \in \mathbb{N}$, $M_\ell > 0$ satisfying

$$|(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}_\mu' S)(t)| \leq M_\ell(1 + t^2)^{r_\ell} \quad (t \in I), \tag{3.1}$$

and $s_\ell \in \mathbb{N}$, $N_\ell > 0$ satisfying

$$|(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)| \leq N_\ell(1+t^2)^{s_\ell} \quad (t \in I). \tag{3.2}$$

Moreover, as $\mathfrak{H}'_\mu T \notin \mathcal{H}_\mu$, there are $\ell_0, n_0 \in \mathbb{N}$ and a sequence $\{t_j\}_{j\in\mathbb{N}}$ in I, such that $t_j \underset{j\to\infty}{\longrightarrow} \infty$ and

$$\left|(t^{-1}D)^{\ell_0} t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)|_{t=t_j}\right| \geq (1+t_j^2)^{-n_0} \quad (j \in \mathbb{N}). \tag{3.3}$$

Set $k = s_{\ell_0+1} + n_0 + 2$, and define

$$B_{j,k} = \{t \in I : |t - t_j| \leq (1+t_j^2)^{-k}\} \quad (j \in \mathbb{N}). \tag{3.4}$$

From (3.2) and (3.3) we infer that, for sufficiently large j,

$$\inf_{t\in B_{j,k}} |(t^{-1}D)^{\ell_0} t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)| \geq \frac{1}{2}(1+t_j^2)^{-n_0} > 0. \tag{3.5}$$

In fact, if j is large enough and if $t \in B_{j,k}$, then

$$\begin{aligned}
&|(t^{-1}D)^{\ell_0} t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)| \\
&\quad \geq \left|(y^{-1}D)^{\ell_0} y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)|_{y=t_j}\right| \\
&\qquad - \left(t_j + (1+t_j^2)^{-k}\right)(1+t_j^2)^{-k} \sup_{y\in B_{j,k}} |(y^{-1}D)^{\ell_0+1} y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)| \\
&\quad \geq (1+t_j^2)^{-n_0} - C(1+t_j^2)^{s_{\ell_0+1}-k+1} \\
&\quad = (1+t_j^2)^{-n_0} - C(1+t_j^2)^{-n_0-1},
\end{aligned}$$

where $C > 0$ is a constant independent from j. This proves (3.5).

Now $t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t) \in \mathcal{H}_\mu$, and therefore

$$\sup_{t\in B_{j,k}} |(t^{-1}D)^\ell t^{-2\mu-1}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t)| = O\left((1+t_j^2)^{-n}\right) \quad (\ell, n \in \mathbb{N},\ j \longrightarrow \infty). \tag{3.6}$$

Certainly, for fixed $\ell, n \in \mathbb{N}$ we may write

$$\begin{aligned}
&\sup_{t\in B_{j,k}} |(t^{-1}D)^\ell t^{-2\mu-1}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t)| \\
&\qquad = \sup_{|t|\leq(1+t_j^2)^{-k}} \left|(y^{-1}D)^\ell y^{-2\mu-1}(\mathfrak{H}'_\mu S)(y)(\mathfrak{H}'_\mu T)(y)|_{y=t+t_j}\right| \\
&\qquad \leq C_{n,\ell} \sup_{|t|\leq(1+t_j^2)^{-k}} \left(1+(t+t_j)^2\right)^{-n} \leq C_{n,\ell}\left(1+t_j^2-(1+t_j^2)^{-k}\right)^{-n},
\end{aligned}$$

where $C_{n,\ell} > 0$ is a constant, and the right-hand side of this inequality is clearly $O\left((1+t_j^2)^{-n}\right)$ as $j \longrightarrow \infty$.

Next we aim to prove that

$$\max_{0\leq\ell\leq m} \sup_{t\in B_{j,k}} |(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)| = O\left((1+t_j^2)^{-n}\right) \quad (m, n \in \mathbb{N},\ j \longrightarrow \infty), \tag{3.7}$$

a contradiction to (i). In the sequel, n will denote an arbitrary positive integer.

We first assume that $\ell_0 = 0$ and proceed by induction on m.

In view of (3.5) and (3.6), we have

$$\sup_{t\in B_{j,k}} |t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)| \leq 2(1+t_j^2)^{n_0} \sup_{t\in B_{j,k}} |t^{-2\mu-1}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t)|$$
$$= O\big((1+t_j^2)^{-n}\big) \quad (j \longrightarrow \infty).$$

Thus, condition (3.7) is satisfied for $m = 0$.

Now suppose that (3.7) holds for some m. We must prove that it also holds for $m+1$. By Leibniz's rule,

$$t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(t^{-1}D)^{m+1}t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)$$
$$= \sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(t^{-1}D)^{m+1-i}\big(t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(t^{-1}D)^i t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)\big) \quad (t \in I).$$

Bearing in mind (3.2), (3.6) and the induction hypotheses, we find that

$$\sup_{t\in B_{j,k}}\Big|(t^{-1}D)^{m+1-i}\big(t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(t^{-1}D)^i t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)\big)\Big| = O\big((1+t_j^2)^{-n}\big)$$

as $j \longrightarrow \infty$, whenever $0 \leq i \leq m+1$. Consequently

$$t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(t^{-1}D)^{m+1}t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)$$

satisfies this very estimate, and from (3.5) we conclude

$$\sup_{t\in B_{j,k}} |(t^{-1}D)^{m+1}t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)|$$
$$\leq 2(1+t_j^2)^{n_0} \sup_{t\in B_{j,k}} |t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(t^{-1}D)^{m+1}t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)|$$
$$= O\big((1+t_j^2)^{-n}\big) \quad (j \longrightarrow \infty).$$

This shows that (3.7) holds when $\ell_0 = 0$.

Next, assume that $\ell_0 \neq 0$ and ℓ_0 is the smallest positive integer for which $n_0 \in \mathbb{N}$ and a sequence $\{t_j\}_{j\in\mathbb{N}}$ in I may be found so that (3.3) (and hence, (3.5), with large enough j) is satisfied. This means that

$$(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t) = O\big((1+t^2)^{-n}\big) \quad (\ell < \ell_0,\ t \longrightarrow \infty).$$

Arguing as in the proof of (3.6) we are led to

$$\sup_{t\in B_{j,k}} |(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)| = O\big((1+t_j^2)^{-n}\big) \quad (\ell < \ell_0,\ j \longrightarrow \infty). \tag{3.8}$$

By virtue of Leibniz's rule,

$$t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(t^{-1}D)^{\ell_0}t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)$$
$$= \sum_{\ell=0}^{\ell_0}(-1)^\ell\binom{\ell_0}{\ell}(t^{-1}D)^{\ell_0-\ell}\big(t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu S)\big)(t) \quad (t \in I).$$

Then, from (3.1), (3.6) and (3.8) it follows that

$$\sup_{t\in B_{j,k}} |t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(t^{-1}D)^{\ell_0}t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)| = O\big((1+t_j^2)^{-n}\big) \quad (j\longrightarrow\infty). \tag{3.9}$$

Finally, using (3.5), (3.6) and (3.9) we obtain (3.7) by an argument similar to that employed in the case $\ell_0 = 0$. This completes the proof that (i) implies (ii).

Conversely, suppose that (i) does not hold. Then there exist $k \in \mathbb{N}$ and a sequence $\{t_j\}_{j\in\mathbb{N}}$ in I, with $t_j \underset{j\to\infty}{\longrightarrow} \infty$, such that

$$\max_{0\le\ell\le j}\sup_{t\in B_{j,k}} |(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)| < (1+t_j^2)^{-j} \quad (j\in\mathbb{N}), \tag{3.10}$$

where the sets $B_{j,k}$ are given by (3.4). There is no loss of generality in assuming that $t_0 > 1$ and $t_{j+1} > t_j + 1$. Let $\alpha \in \mathcal{D}(I)$ be such that $0 \le \alpha \le 1$, $\operatorname{supp}\alpha = [1/2, 3/2]$, and $\alpha(1) = 1$, and set

$$\theta_j(t) = \alpha\Big(1 + \frac{1}{2}(t-t_j)(1+t_j^2)^k\Big), \quad \theta(t) = \sum_{j=0}^{\infty}\theta_j(t) \quad (t\in I).$$

The sum defining θ is finite, because $\operatorname{supp}\theta_j = B_{j,k}$ $(j\in\mathbb{N})$ and $B_{i,k}\cap B_{j,k} = \emptyset$ $(i,j\in\mathbb{N}$, $i\neq j)$. If $\ell, j \in \mathbb{N}$ and $t\in B_{j,k}$ then, for some $a_m \in \mathbb{R}$ $(0\le m\le \ell)$, we have

$$\begin{aligned}
|(t^{-1}D)^\ell\theta(t)| = |(t^{-1}D)^\ell\theta_j(t)| &= \sum_{m=0}^{\ell} |a_m t^{-\ell-m}D^m\theta_j(t)| \\
&\le 2^{\ell+m}\sum_{m=0}^{\ell}|a_m D^m\theta_j(t)| \\
&\le C_\ell 2^{-k\ell}(1+t_j^2)^{k\ell}\sum_{m=0}^{\ell}\Big|D^m\theta_j(y)|_{y=1+\frac{1}{2}(t-t_j)(1+t_j^2)^k}\Big| \\
&\le C_\ell(1+t_j^2)^{k\ell} \le C_\ell(1+t^2)^{k\ell},
\end{aligned}$$

where $C_\ell > 0$ denotes an appropriate constant (not necessarily the same in each occurrence). Then

$$|(t^{-1}D)^\ell\theta(t)| \le C_\ell(1+t^2)^{k\ell} \quad (t\in I), \tag{3.11}$$

thus proving that $\theta \in \mathcal{O}$. Hence, there exists $T \in \mathcal{O}'_{\mu,*}$ such that $(\mathfrak{H}'_\mu T)(t) = t^{\mu+1/2}\theta(t)$ $(t\in I)$. Let $n, \ell \in \mathbb{N}$. The function

$$(1+t^2)^n(t^{-1}D)^\ell t^{-2\mu-1}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t) \quad (t\in I)$$

is bounded on the interval $0 < t < t_{n+k\ell} - (1+t_{n+k\ell}^2)^{-k}$. Letting $j = n + k\ell + r$ $(r\in\mathbb{N})$ and $t\in B_{j,k}$, Leibniz's rule, along with (3.10) and (3.11), implies

$$\begin{aligned}
|(1+t^2)^n(t^{-1}D)^\ell t^{-2\mu-1}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t)| &= |(1+t^2)^n(t^{-1}D)^\ell t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)\theta(t)| \\
&\le C(1+t^2)^{n+k\ell}(1+t_j^2)^{-n-k\ell} \le C,
\end{aligned}$$

where $C > 0$ is a suitable constant (concerning the value of C, we make the same convention as before). This shows that $t^{-\mu-1/2}(\mathfrak{H}'_\mu S)(t)(\mathfrak{H}'_\mu T)(t) \in \mathcal{H}_\mu$. But $\mathfrak{H}'_\mu T \notin \mathcal{H}_\mu$, since

$$t_j^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t_j) = \alpha(1) = 1$$

as $t_j \underset{j\to\infty}{\longrightarrow} \infty$. We conclude that $T \in O'_{\mu,*}$ and that $S * T \in \mathcal{H}_\mu$ although $T \notin \mathcal{H}_\mu$, which contradicts (ii) and completes the proof. ∎

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