THE k-NORMAL COMPLETION OF FUNCTION LATTICES

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1. Introduction. A subset G of a non-empty partially ordered set C is called *normal* if it coincides with the set of all upper bounds of the set of lower bounds of G. This is equivalent to stipulating that G be the set of all upper bounds of some subset of C called a *set of generators* for G. When ordered by inclusion, the family of all normal subsets of C forms a complete lattice with maximum C and minimum empty or singleton. The meet operation is simply point set intersection; whence, the meet of a family G_i of normal subsets is the set of upper bounds of $\cup F_i$ where F_i generates G_i for each i. A normal subset is called *proper* if it is neither void nor C, and the proper normal subsets of C form a boundedly complete lattice.

Throughout this paper, k denotes a fixed infinite cardinal number, and a k-*set* (k-*family*) is a set (family) with k or less members. A normal subset of C which has a k-set of generators will be called k-*normal*, and the family of all k-normal subsets of C will be called the k-*normal completion* of C. The k-normal completion of a partially ordered set is k-complete from below; that is, the intersection of a k-family of k-normal subsets is k-normal.

From now on, S denotes a compact Hausdorff topological space, $C(S)$ denotes the lattice of all continuous real-valued functions on S, $B(S)$ the lattice of all bounded real-valued functions on S, and $\mathbf{N}(S)$ the lattice of all proper normal subsets of $C(S)$. The latter two of these are always boundedly complete. Given a subset F of $B(S)$ bounded from above (below), sup F(inf F) denotes the function obtained by taking suprema (infima) pointwise.

In **(3)**, Dilworth proved that the map $h: \mathbf{N}(S) \to B(S)$ given by $h(G) = \inf G$ is bi-order reversing (hence, 1–1) and that the functions in the range of h are precisely the normal upper semicontinuous functions; i.e., they are the functions f in $B(S)$ such that $(f_*)^* = f$. For the definitions of g^* and g_*, the upper and lower semicontinuous envelopes of a function g in $B(S)$, see **(3)**. We call the normal upper semicontinuous functions in $B(S)$ simply the *normal* functions on S, and we let $N(S)$ denote the lattice of all of these.

Dilworth gives the following formulas for the supremum (infimum) in $N(S)$ of a subset F of $N(S)$ bounded from above (below):

$$\sup(N(S))F = (\sup F)^*, \tag{1}$$

$$\inf(N(S))F = ((\inf F)_*)^*. \tag{2}$$

The set of all upper bounds in $C(S)$ of a subset E of $B(S)$ will be denoted by

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E^*. Given G in $N(S)$ with generating set F; viz., $G = F^*$, the image of G under the Dilworth map h can be described in terms of the generators. For since $h^{-1}(f) = \{f\}^*$ for each f in $C(S)$, we have

$$h^{-1}(\sup(N(S))F) = \{\sup(N(S))F\}^* = \{(\sup F)^*\}^* = F^*;$$

whence,

$$h(F^*) = \sup(N(S))F. \tag{3}$$

In this paper, we define the k-normal functions and prove that under the Dilworth map h, the image of the set $\mathbf{N}(k, S)$ of proper k-normal subsets of $C(S)$ is precisely the set $N(k, S)$ of all k-normal functions. We show that $\mathbf{N}(k, S)$ is a boundedly k-complete sublattice of $N(S)$ if and only if S is a k-space **(2)** and that, when this is the case, $N(k, S)$ is isomorphic to the lattice of continuous real-valued functions on the k-extremally disconnected space **(2)** determined by S.

2. The k-normal functions. In **(2)**, we introduced the notion of a *k-regular closed* set; namely, a subset of S of the form $\text{cl}(V)$ where V is k-open. We say that V is *k-open* if it is the union of a k-family of co-zero sets. A *co-zero* subset of S is a subset of the form $f^{-1}[U]$ where U is an open subset of the reals and f is in $C(S)$. A co-zero set can always be expressed in the form

$$S(0 < f < 1) = \{x \in S: 0 < f(x) < 1\},$$

where f is in $C(S)$ and $0 \leqslant f < 1$.

Let Q denote the set of rational numbers.

THEOREM 1. *For a normal function ϕ, the following conditions are equivalent:*

(1) $\text{cl}\, S(\phi > \lambda)$ *is k-regular for each real number* λ.

(2) *For each real number λ, $S(\phi > \lambda)$ is the union of a countable family of k-regular closed sets.*

(3) *For each real number λ, $S(\phi > \lambda)$ is the union of a k-family of k-regular closed sets.*

Proof. The third condition is a trivial consequence of the second. Assuming the third condition for λ, let $\{V_i\}$ be a k-family of k-open sets such that $\bigcup_i \text{cl}(V_i) = S(\phi > \lambda)$. Then $V \equiv \bigcup_i V_i$ is k-open and $\text{cl}\, S(\phi > \lambda) = \text{cl}(V)$. For the remaining implication, we note from the first paragraph of the proof of **(3**, Theorem 3.2**)** that for each real number λ,

$$S(\phi > \lambda) = \bigcup\, (q \in Q, q > \lambda)\, \text{cl}\, S(\phi_* > q), \tag{4}$$

so that, taking closures and simplifying, we have

$$\text{cl}\, S(\phi > \lambda) = \text{cl}\, S(\phi_* > \lambda). \tag{5}$$

Using (5) in (4) yields

$$S(\phi > \lambda) = \bigcup\, (q \in Q, q > \lambda)\, \text{cl}\, S(\phi > q). \tag{6}$$

Definition. A *k-normal* function is a function satisfying the conditions of Theorem 1.

THEOREM 2. *The supremum in $N(S)$ of a k-family (bounded from above) of k-normal functions is k-normal.*

Proof. Let F denote a k-family, bounded from above, of k-normal functions; according to (1), it must be shown that $(\sup F)^*$ is k-normal. But this follows from the fact that for each real number λ, $\text{cl}\, S((\sup F)^* > \lambda)$ is equal to the k-regular closed set $\text{cl}(\bigcup\, (f \in F)S(f > \lambda))$.

THEOREM 3. *If V is a k-open subset of S, there exists a k-subset F of $C(S)$ such that* $\sup F = \chi(V$.

Proof. First suppose V is a co-zero set and therefore of the form $S(0 < g < 1)$ where g is in $C(S$ and $0 \leqslant g < 1$. For each positive integer n, let g_n map the reals into the reals continuously as follows:

$$\begin{aligned} g_n(r) &= r && \text{if } 1 \leqslant r,\\ 1 - 1/n < g_n(r) &< 1 && \text{if } 1/n < r < 1,\\ 0 < g_n(r) &\leqslant 1 - 1/n && \text{if } 0 < r \leqslant 1/n,\\ g_n(r) &= 0 && \text{if } r \leqslant 0. \end{aligned}$$

Then $f_n \equiv g_n \circ g$ is continuous and $\sup f_n = \chi(V)$.

Now let V be a k-open set, the union of a k-family $\{V_i\}$ of co-zero sets. If, for each i, F_i is a countable subset of $C(S)$ such that $\sup F_i = \chi(V_i)$, then the union F of the F_i is a k-subset of $C(S)$ such that $\sup F = \chi(V)$.

THEOREM 4. *The Dilworth map h: $\mathbf{N}(S) \to N(S)$ carries $\mathbf{N}(k, S)$ onto $N(k, S)$.*

Proof. If G is a proper k-normal subset of $C(S)$, then $G = F^*$ for some non-empty k-subset F of $C(S)$. From (1) and (3), $h(G) = \sup(N(S))F$, a k-normal function by Theorem 2.

Now let ϕ denote a positive-valued k-normal function. For each rational number q, let V_q be a k-open set such that $\text{cl}\, S(\phi > q) = \text{cl}\, V_q$. Let F_q be a k-subset of $C(S)$ such that $\sup F_q = \chi(V_q)$ so that $(\sup F_q)^* = \chi(\text{cl}\, V_q)$; and let qF_q denote $\{qf\colon f \in F_q\}$. Given x in S, using the upper semicontinuity of ϕ,

$$\phi(x) = \sup\{q \in Q\colon x \in \text{cl}\, S(\phi > q)\} = \sup\{q \in Q\colon x \in \text{cl}\, V_q\};$$

hence, using the positiveness, $\phi = \sup(q \in Q)q\chi(\text{cl}\, V_q)$. Therefore,

$$\begin{aligned} \phi = \phi^* = (\sup(q \in Q)q\chi(\text{cl}\, V_q))^* &= (\sup(q \in Q)q(\sup F_q)^*)^* \\ &= (\sup(q \in Q) \sup(N(S))qF_q)^* = \sup(N(S))F, \end{aligned}$$

where F denotes the union of the qF_q, a k-subset of $C(S)$.

If ϕ is an arbitrary k-normal function, then $a + \phi$ is positive-valued and k-normal for a suitable scalar a and the result follows.

Corollary to the Proof. *Every positive-valued k-normal function is the supremum in $N(S)$ of a k-family of scalar multiples of characteristic functions of k-regular closed subsets of S.*

In **(5)**, M. H. Stone proved that $C(S)$ forms a boundedly k-complete lattice if and only if the cl–open sets of S form a base for the open sets and a k-complete Boolean algebra. An equivalent condition **(2)** is that every k-regular closed set be open. These conditions have yet another characterization, this time in terms of k-normal functions.

Definition. We shall say that S is *k-extremally disconnected* when cl V is open for each k-open subset V of S.

Theorem 5. *The following conditions are equivalent:*

(a) *S is k-extremally disconnected.*

(b) *Every k-normal function is continuous.*

(c) *$C(S)$ is boundedly k-complete.*

Proof. Assume S is k-extremally disconnected and let ϕ be a k-normal function. Then for each real number λ, $S(\phi > \lambda)$ is the union of k-regular closed sets each of which, by hypothesis, is open; hence, $S(\phi > \lambda)$ is open. Thus ϕ is lower semicontinuous as well as upper semicontinuous, and is therefore continuous.

Assume that each k-normal function is continuous and let F be a k-subset of $C(S)$ bounded from above. Then $\sup(N(S))F$, being k-normal, is continuous. Since $C(S) \subset N(S)$, we must have $\sup(N(S))F = \sup(C(S))F$.

Assume that $C(S)$ is boundedly k-complete and let V be k-open. Using Theorem 3, let F be a k-subset of $C(S)$ such that $\sup F = \chi(V)$. Then

$$\chi(\text{cl } V) = (\sup F)^* = \sup(N(S))F = h(F^*) = \inf F^*$$
$$\geqslant \sup(C(S))F \geqslant \sup(N(S))F = \chi(\text{cl } V).$$

In particular, $\chi(\text{cl } V) = \sup(C(S))F$, a continuous function, so cl V is open.

3. Characterization of the k-spaces.

The k-spaces were introduced in **(2)** to describe the class of (compact Hausdorff) spaces whose k-regular closed sets form a k-complete subalgebra of the Boolean algebra of regular closed subsets of S. The defining condition on S is that $\text{cl}(S \backslash M)$ be a k-regular closed set whenever M is; or, equivalently, given V k-open, there is a k-open subset W of S such that $W \cup V$ is dense and $W \cap V$ is void. In this section, the k-spaces are characterized in terms of their k-normal functions. We first introduce an operation in $N(S)$ analogous to $f \to -f$ in $C(S)$.

Definition. For each f in $N(S)$, set $\sim f = -(f_*) = (-f)^*$.

It is immediate that $\sim f$ is normal and $\sim(\sim f) = f$. Since $f \leqslant g$ implies that $\sim g \leqslant \sim f$, DeMorgan's laws hold; e.g., if $\vee f_i$ exists in $N(S)$, then $\wedge \sim f_i$ exists in $N(S)$ and is equal to $\sim \vee f_i$.

THEOREM 6. *The following conditions are equivalent:*

(a) *$N(k, S)$ is a boundedly k-complete sublattice of $N(S)$.*

(b) *If f is in $N(k, S)$, so is $\sim f$.*

(c) *S is a k-space.*

Proof. Assume (a) and let V be a given k-open subset of S. By Theorem 3, let F be a k-subset of $C(S)$ such that $\sup F = \chi(V)$. Set $G = \{1 - f : f \in F\}$ so that $\inf G = \chi(S \backslash V)$. Therefore,

$$\begin{aligned}\chi(\operatorname{cl}(S\backslash \operatorname{cl} V)) = \chi(\operatorname{cl}\operatorname{int}(S\backslash V)) = (\chi \operatorname{int}(S\backslash V))^* &= ((\chi(S\backslash V)_*)^* \\ &= ((\inf G)_*)^* = \inf(N(S))G,\end{aligned}$$

which is k-normal by hypothesis. This implies that $\operatorname{cl}(S\backslash \operatorname{cl} V)$ is k-regular. Therefore, S is a k-space.

Assume S is a k-space, and let f be k-normal. Let λ be a given real number. For each rational number $q < \lambda$, $\operatorname{cl} S(f > q)$ is k-regular: let V_q be a k-open subset of S such that $\operatorname{cl}(S\backslash \operatorname{cl} S(f > q)) = \operatorname{cl} V_q$. Then V, defined as $\bigcup_{(q<\lambda)} V_q$ is k-open and

$$\begin{aligned}\operatorname{cl} S(\sim f > \lambda) = \operatorname{cl} S(f_* < \lambda) &= \operatorname{cl} S(f < \lambda) \\ &= \operatorname{cl}[\textstyle\bigcup_{(q<\lambda)} (S\backslash \operatorname{cl} S(f > q))] = \operatorname{cl} V,\end{aligned}$$

a k-regular closed set. Thus $\sim f$ is k-normal.

That (b) implies (a) follows from Theorem 2 and DeMorgan's laws.

4. A second representation of N(k, S) when S is a k-space. The complement in S of a k-open set will be called a *k-closed* set; such a set is the intersection of a k-family of zero sets **(4)**. A *regular* (*k-regular*) *open* set is the interior of a closed (k-closed) set. The map $M \to \operatorname{int} M$ carries the (complete) Boolean algebra of regular closed subsets of S isomorphically onto the Boolean algebra $\mathbf{R}$ of regular open subsets of S. When S is a k-space, it is easy to prove that this isomorphism carries the k-regular closed sets onto the k-regular open sets. Thus, the collection $\mathbf{R}(k)$ of k-regular open subsets of S forms a k-complete subalgebra of $\mathbf{R}$. The meet of any two elements of $\mathbf{R}$ is simply their intersection. In general, the operations of $\mathbf{R}$ (of $\mathbf{R}(k)$) are:

(7) $$\bigwedge_i \operatorname{int} M_i = \operatorname{int}(\textstyle\bigcap_i M_i),$$

(8) $$\bigvee_i \operatorname{int} M_i = \operatorname{int}\operatorname{cl}(\textstyle\bigcup_i \operatorname{int} M_i),$$

(9) $$-\operatorname{int} M = \operatorname{int}(S\backslash \operatorname{int} M),$$

where M_i is a family (k-family) of closed (k-closed) sets.

Throughout the remainder of this section, S is a k-space. Let $\mathbf{S}$ denote the Stone space of $\mathbf{R}$, the elements of $\mathbf{S}$ being ultra-filters $\mathbf{x}$ of $\mathbf{R}$. Let $\tau: C(\mathbf{S}) \to N(S)$ and $\sigma: N(S) \to C(\mathbf{S})$ be the mutually inverse order-preserving functions given by Dilworth:

(10) $$\tau(z)(x) = \inf(V \in \mathbf{V}_x)\sup(\mathbf{x} \ni V)z(\mathbf{x}),$$

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$$\sigma(f)(\mathbf{x}) = \inf(V \in \mathbf{x})\sup(x \in V\ f(x). \tag{11}$$

Here, $\mathbf{V}_x$ stands for any fundamental system of neighbourhoods of x. For each V in $\mathbf{R}$, $\rho(V)$ denotes $\{\mathbf{x} \in \mathbf{S}:\mathbf{x} \ni V\}$, the cl–open set determined by V.

Now let $\mathbf{S}(k)$ denote the Stone space of ultra-filters of $\mathbf{R}(k)$ and, for each V in $\mathbf{R}(k)$, $\rho(k, V)$ the cl–open set $\{\mathbf{y} \in \mathbf{S}(k)\colon \mathbf{y} \ni V\}$. Let $i\colon \mathbf{R}(k) \to \mathbf{R}$ denote the inclusion so that $I\colon\mathbf{S} \to \mathbf{S}(k)$, the map dual to i, is onto and given by

$$I(\mathbf{x}) = \mathbf{R}(k) \cap \mathbf{x}. \tag{12}$$

For $\mathbf{R}(k) \cap \mathbf{x}$ is an element of $\mathbf{S}(k)$, and evidently a member of

$$\bigcap \{\rho(k, V)\colon V \in \mathbf{R}(k), \rho(V) \ni \mathbf{x}\} = \{I(\mathbf{x})\}.$$

The map $I^*\colon C(\mathbf{S}(k)) \to C(\mathbf{S})$ conjugate to I and given by $I^*(z)(\mathbf{x}) = z(I(\mathbf{x}))$ is a lattice isomorphism into. Define $t\colon C(\mathbf{S}(k)) \to N(S)$ by

$$t = \tau \circ I^*. \tag{13}$$

THEOREM 7. *The map t carries $C(\mathbf{S}(k))$ onto $N(k, S)$; hence, these lattices are isomorphic.*

Proof. We first show that the k-regular open sets form a base for the open subsets of S. Let U be open in S and x an element of U. Let V be a co-zero subset of S such that $x \in V \subset \text{cl } V \subset U$. Let W be a k-open set such that $V \cap W$ is void and $V \cup W$ dense in S. Then $S\backslash W$ is k-closed and $x \in \text{int}(S\backslash W) \subset U$. Thus, for each x is S, the collection $\mathbf{O}_x$ of k-regular open neighbourhoods of x forms a fundamental system. We assert that

$$t(z)(x) = \inf(V \in \mathbf{O}_x)\sup(V \in \mathbf{y} \in \mathbf{S}(k))z(\mathbf{y}). \tag{14}$$

For

$$\begin{aligned} t(z)(x) &= (\tau \circ I^*)(z)(x) = \tau(z \circ I)(x) = \inf(V \in \mathbf{V}_x)\sup(V \in \mathbf{x} \in \mathbf{S})z(I(\mathbf{x})) \\ &= \inf(V \in \mathbf{O}_x)\sup(V \in \mathbf{x} \in \mathbf{S})z(I(\mathbf{x})) = \inf(V \in \mathbf{O}_x)\sup(V \in \mathbf{y} \in \mathbf{S}(k))z(\mathbf{y}), \end{aligned}$$

using (12) in the last equality and the fact that I is onto.

Next we compute the action of t on characteristic functions. Let N be a cl–open subset of $\mathbf{S}(k)$. Then there is a k-closed subset M of S such that $N = \rho(k, \text{int } M)$. Given x in S and W in $\mathbf{O}_x$, $\sup(W \in \mathbf{y} \in \mathbf{S}(k))\chi(N)(\mathbf{y})$ is either 0 or 1: 0 if W is disjoint from int M and 1 if they meet. Therefore, $t(\chi(N))(x)$ is either 0 or 1, and it is 1 if and only if x is in cl int M. Therefore,

$$t(\chi(\rho(k, \text{int } M))) = \chi(\text{cl int } M) \tag{15}$$

for each k-closed subset M of S. Since every k-regular closed subset of S is of the form cl int M for some k-closed subset M, t carries the family of characteristic functions of cl–open subsets of $\mathbf{S}(k)$ onto the family of characteristic functions of k-regular closed subsets of S. Our result now follows from Theorem 5, the Corollary to Theorem 4, and the fact that t is an isomorphism.

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